Geometric Analysis of Differential-Algebraic Equations: Linear, Nonlinear and Linearizable

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Linear	Nonlinear
$E\dot{x} = Hx$	$E(x)\dot{x} = F(x)$

- If E is square and invertible, then

$$E\dot{x} = Hx \Rightarrow \dot{x} = E^{-1}Hx.$$

Linear	Nonlinear
$E\dot{x} = Hx$	$E(x)\dot{x} = F(x)$

Example 1

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1}$$

Solutions exist only on $\{x_1 = 0\}$. (Existence)

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2}$$

There exist infinite solutions. (Uniqueness)

Differential-algebraic equation DAE systems

Linear	Nonlinear
$E\dot{x} = Hx + Lu$	$E(x)\dot{x} = F(x) + G(x)u$

Example 1

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1}$$

Solutions exist only on $\{x_1 = 0\}$. (Existence)

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2}$$

There exist infinite solutions. (Uniqueness)

Example 2

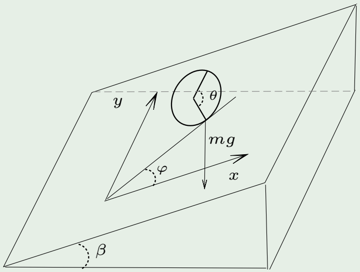


Figure: A rolling disk on a slope

The Lagrangian:

$$\mathcal{L} = mgx\sin\beta + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\varphi^2. \tag{3}$$

The Euler-Lagrange dynamical equations:

$$\begin{cases} \ddot{x} = \sin\beta - \lambda\sin\varphi \\ \ddot{y} = \lambda\cos\varphi \end{cases} \tag{4}$$

Nonholonomic constrains:

$$\begin{cases} 0 = -\sin\varphi dx + \cos\varphi dy \\ 0 = \cos\varphi dx + \sin\varphi dy - d\theta. \end{cases} \tag{5}$$

The "generalized" states:

$$\xi = (x, \dot{x}, y, \dot{y}, \varphi, \theta, \beta, \lambda),$$

and controls $u = (u_1, u_2) = (\dot{\theta}, \dot{\beta})$, we get a DAE control system DAECS

$$E(\xi)\dot{x} = F(\xi) + G(\xi)u.$$

A mechanical system modeled by DAEs

Example 2

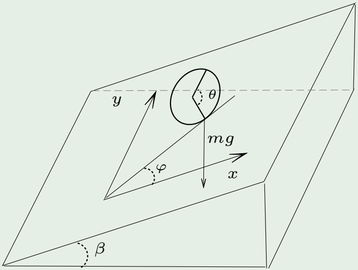


Figure: A rolling disk on a slope

Let the “generalized” states be

$$\xi = (x, \dot{x}, y, \dot{y}, \varphi, \theta, \beta, \lambda),$$

and $u = (u_1, u_2)$, we get a DAE control system

$$E(\xi)\dot{\xi} = F(\xi) + G(\xi)u.$$

$$E(\xi)\dot{x} = F(\xi) + G(\xi)u,$$

where

$$E(\xi) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 & 0 & 0 & 0 & 0 \\ \cos\varphi & 0 & \sin\varphi & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$F(\xi) = \begin{bmatrix} \dot{x} \\ \sin\beta - \lambda\sin\varphi \\ \dot{y} \\ \lambda\cos\varphi \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$G(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

A simple electrical circuit modeled by DAEs

Example 3 (Rabier2013)

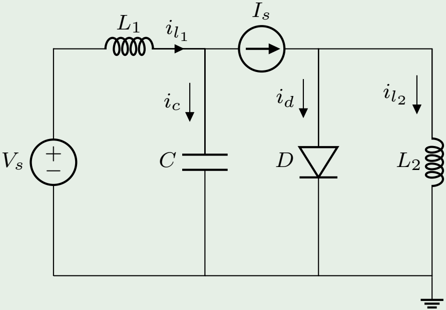


Figure: A simple electrical circuit

Characteristics of the capacitor and inductors:

$$\begin{cases} C(v_c)\dot{v}_c = i_c \\ L_1(i_{l_1})\dot{i}_{l_1} = v_{l_1} \\ L_2(i_{l_2})\dot{i}_{l_2} = v_{l_2}, \end{cases} \qquad (3)$$

The Kirchhoff's laws give

$$\begin{cases} Ai = 0 \\ v = A^T e \end{cases} \qquad (4)$$

where e are the node potentials and where

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}.$$

The characteristic of the diode:

$$i_d = f(u_d). \qquad (5)$$

The combination of (3), (4) and (5) gives a DAE of the form $\Xi^{se}: \begin{cases} R(x)\dot{x} = a(x) \\ 0 = c(x) \end{cases}$, or more general, $\Xi : E(x)\dot{x} = F(x)$.

Outline

Plan

Linear DAEs and ODE control systems

We consider a linear DAE:

$$\Delta : E\dot{x} = Hx, \tag{6}$$

- where $x \in \mathbb{R}^n$, $E \in \mathbb{R}^{l\times n}$, $H \in \mathbb{R}^{l\times n}$.
- Denoted by $\Delta_{l,n} = (E, H)$.
- External equivalence: $\Delta \overset{ex}{\sim} \tilde{\Delta}$ if $\exists$ invertible Q and P s.t.

$$\begin{aligned} \tilde{E} &= QEP^{-1} \\ \tilde{H} &= QHP^{-1}. \end{aligned} \tag{7}$$

- P changes coordinate and Q combines equations.

External equivalence of DAEs and Morse of ODECSs

- A linear control system

$$\Lambda : \begin{cases} \dot{z} = Az + Bu \\ y = Cz + Du, \end{cases} \tag{8}$$

where $z \in \mathbb{R}^q$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, denoted $\Lambda_{q,m,p} = (A, B, C, D)$.

- Morse equivalence (Morse1973,Molinari1978): $\Lambda \overset{M}{\sim} \tilde{\Lambda}$, if $\exists$ invertible matrices T_s,T_i,T_o and matrices F,K s.t.

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} T_s & T_sK \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} \tag{9}$$

- The prolongation of Λ:

$$\mathbf{\Lambda} : \begin{cases} \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}v \\ y = \mathbf{C}\mathbf{z}, \end{cases} \tag{10}$$

where $\dot{u} = v$,

$$\mathbf{z} = \begin{bmatrix} z \\ u \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} C & D \end{bmatrix}.$$

Implicitation of linear ODE control systems

From an ODE control system towards a DAE:

Definition 1 (Implicitation of linear ODE control systems)

For a linear control system $\Lambda_{q,m,p} = (A, B, C, D)$, set the output $y = Cx + Du$ to be zero, we define the following DAE with "generalized" states (z, u) in $\mathbb{R}^{q+m}$:

$$\Delta^{Impl} : \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}. \qquad (11)$$

The DAE given by (11) is called the implicitation of Λ and denoted by $\Delta^{Impl} = Impl(\Lambda)$.

- Can we also go the other way around: DAEs $\Rightarrow$ control systems ?
- Constructing a control system Λ requires to identify states, controls, and outputs. How?

Explicitation: from linear DAEs towards linear ODECSs

- Consider $\Delta_{l,n} = (E, H)$. Denote $\operatorname{rank} E = q$, define $p = l - q$ and $m = n - q$. Choose a map

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in Gl(n, \mathbb{R}),$$

where $P_1 \in \mathbb{R}^{q \times n}$, $P_2 \in \mathbb{R}^{m \times n}$ such that $\ker P_1 = \ker E$.

- Define coordinates transformation

$$\begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} P_1 x \\ P_2 x \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} x = Px.$$

- Choose Q such that $QEP^{-1} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}$ and denote $QHP^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$,

- Let $y = 0 = Cx + Du$, we attach an ODECS $\Lambda = (A, B, C, D)$ to Δ, that is, $\Delta \overset{ex}{\sim} \Delta^{Impl} = Impl(\Lambda)$.

Definition 2 (Explicitation of linear control systems)

We call the just defined ODECS $\Lambda_{q,m,p} = (A, B, C, D)$ the (Q, P)-explicitation of Δ.

Our construction is not unique: it depends on the choice of P and Q:

- Non-uniqueness of the choice of coordinates: choose other coordinates (z', u')

$$\begin{cases} z' = T_s z \\ u' = F'z + T_i u, \end{cases} \tag{12}$$

where T_s and T_i are invertible. Clearly, $z' = T_s z$ is another set of coordinates on the state space and $u' = F'z + T_i u$ is a state feedback transformation.

- Non-uniqueness of the choice of Q: a triangular transformation (output injection and multiplication) of the system

$$\begin{bmatrix} \dot{z}' \\ y' \end{bmatrix} = \begin{bmatrix} T_s & K' \\ 0 & T_o \end{bmatrix} \begin{bmatrix} \dot{z} \\ y \end{bmatrix} = \begin{bmatrix} T_s & K' \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} \tag{13}$$

where $K' \in \mathbb{R}^{n \times p}$, $T_o \in Gl(p, \mathbb{R})$.

- The explicitation is a class of control systems!!!

- The class of all (Q, P)-explicitations will be denoted by $Expl(\Delta)$ and we will write $\Lambda \in Expl(\Delta)$.

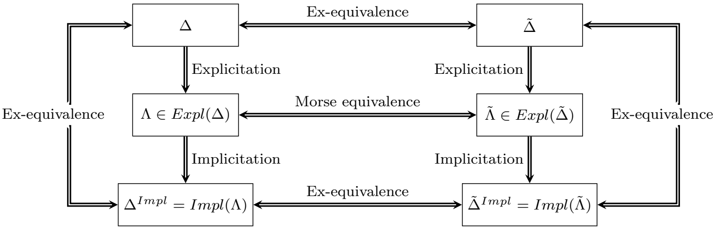
Theorem 1 (Theorem 2.3.4)

(i) *Given two DAEs* $\Delta = (E, H)$ *and* $\tilde{\Delta} = (\tilde{E}, \tilde{H})$, *choose two control systems* $\Lambda \in Expl(\Delta)$ *and* $\tilde{\Lambda} \in Expl(\tilde{\Delta})$. *Then* $\Delta \overset{ex}{\sim} \tilde{\Delta}$ *if and only if* $\Lambda \overset{M}{\sim} \tilde{\Lambda}$.

(ii) *Consider two control systems* Λ *and* $\tilde{\Lambda}$. *Then* $\Lambda \overset{M}{\sim} \tilde{\Lambda}$ *if and only if* $\Delta^{Impl} \overset{ex}{\sim} \tilde{\Delta}^{Impl}$, *where* $\Delta^{Impl} = Impl(\Lambda)$ *and* $\tilde{\Delta}^{Impl} = Impl(\tilde{\Lambda})$.

(iii) *Consider a DAE* $\Delta = (E, H)$ *and a control system* Λ. *Then* $\Lambda \in Expl(\Delta)$ *if and only if* $\Delta \overset{ex}{\sim} \Delta^{Impl}$, *where* $\Delta^{Impl} = Impl(\Lambda)$. *More specifically,* Λ *is the* (Q, P)*-explicitation of* Δ *if and only if* $\Delta \overset{ex}{\sim} \Delta^{Impl}$ *via* (Q, P).

- Morse equivalent control systems (and only such) give, via implicitation, ex-equivalent DAEs.
- Ex-equivalent DAEs produce Morse equivalent control systems.



Subspaces relations between DAEs and ODE control systems

$\Delta : E\dot{x} = Hx$	$\Lambda : \begin{cases} \dot{z} = Az + Bu \\ y = Cz + Du, \end{cases}$	$\boldsymbol{\Lambda} : \begin{cases} \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}v \\ y = \mathbf{C}\mathbf{z}, \end{cases}$
The Wong sequences (Wong1974): $\mathscr{V}_0 = \mathbb{R}^n, \mathscr{V}_{i+1} = H^{-1}E\mathscr{V}_i,$ $\mathscr{W}_0 = \{0\}, \mathscr{W}_{i+1} = E^{-1}H\mathscr{W}_i.$	$\mathcal{V}_0 = \mathbb{R}^q,$ $\mathcal{V}_{i+1} = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{V}_i + \operatorname{Im} \begin{bmatrix} B \\ D \end{bmatrix} \right)$ $\mathcal{W}_0 = \{0\},$ $\mathcal{W}_{i+1} = [A,\ B] \left(\begin{bmatrix} \mathcal{W}_i \\ \mathscr{U} \end{bmatrix} \cap \ker\ [C,\ D] \right).$	$\boldsymbol{\mathcal{V}}_0 = \mathbb{R}^n,$ $\boldsymbol{\mathcal{V}}_{i+1} = \ker \mathbf{C} \cap \mathbf{A}^{-1}(\boldsymbol{\mathcal{V}}_i + \operatorname{Im} \mathbf{B});$ $\boldsymbol{\mathcal{W}}_0 = 0,$ $\boldsymbol{\mathcal{W}}_{i+1} = \mathbf{A}(\boldsymbol{\mathcal{W}}_i \cap \ker \mathbf{C}) + \operatorname{Im} \mathbf{B}.$
$\mathscr{V}^* = \mathscr{V}_{k^*}$ is the largest s.t. $\mathscr{V} = H^{-1}E\mathscr{V}$; $\mathscr{W}^* = \mathscr{W}_{l^*}$ is the smallest s.t. $\mathscr{W} = E^{-1}H\mathscr{W}$.	$\mathcal{V}^* = \mathcal{V}_{k^*}$ is the largest s.t. $\exists F$, $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ and $(C+DF)\mathcal{V} = 0$; $\mathcal{W}^* = \mathcal{W}_{l^*}$ is the smallest s.t. $\exists K$, $(A+KC)\mathcal{W} + (B+KD)\mathscr{U} = \mathcal{W}$.	$\boldsymbol{\mathcal{V}}^* = \boldsymbol{\mathcal{V}}_{k^*}$ is the largest controlled invariant subspace i.e., $\mathbf{A}\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{V}} + \operatorname{Im} \mathbf{B}$ in $\ker \mathbf{C}$; $\boldsymbol{\mathcal{W}}^* = \boldsymbol{\mathcal{W}}_{l^*}$ is the smallest conditioned invariant subspace, i.e., $\mathbf{A}(\boldsymbol{\mathcal{W}} \cap \ker \mathbf{C}) \subseteq \boldsymbol{\mathcal{W}}$, containing $\operatorname{Im} \mathbf{B}$.

Proposition 1 (Subspaces relations Proposition 2.4.10)

Given a DAE $\Delta = (E, H)$, *the* (Q, P)*-explicitation* $\Lambda = (A, B, C, D) \in Expl(\Delta)$, *and the prolongation* $\mathbf{\Lambda} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ *of* Λ, *consider the limits of the Wong sequences* $\mathscr{V}^*$ *and* $\mathscr{W}^*$ *of* Δ *and of* $\Delta^{Impl} = Impl(\Lambda)$, *the invariant subspaces* $\mathcal{V}^*$ *and* $\mathcal{W}^*$ *of* Λ, *and the invariant subspaces* $\boldsymbol{\mathcal{V}}^*$ *and* $\boldsymbol{\mathcal{W}}^*$ *of* $\mathbf{\Lambda}$. *Then the following hold*

(i) $P\mathscr{V}^*(\Delta) = \mathscr{V}^*(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}^*(\mathbf{\Lambda}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}^*(\Lambda) \\ 0 \end{bmatrix}$,

(ii) $P\mathscr{W}^*(\Delta) = \mathscr{W}^*(\Delta^{Impl}) = \boldsymbol{\mathcal{W}}^*(\mathbf{\Lambda}) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}^*(\Lambda) \\ 0 \end{bmatrix}$.

- Similar relations hold for augmented Wong sequences of the linear DAE control systems and invariant subspaces of its explicitation. (see Proposition 3.2.10).

Kronecker canonical form versus Morse canonical form

Set $N_\beta = \operatorname{diag}\left\{N_{\beta_1}, \ldots, N_{\beta_k}\right\}$, $K_\beta = \operatorname{diag}\left\{K_{\beta_1}, \ldots, K_{\beta_k}\right\}$,$L_\beta = \operatorname{diag}\left\{L_{\beta_1}, \ldots, L_{\beta_k}\right\}$.

$$K_i = \begin{bmatrix} 0 & I_{i-1} \end{bmatrix} \in \mathbb{R}^{(i-1)\times i}, \quad L_i = \begin{bmatrix} I_{i-1} & 0 \end{bmatrix} \in \mathbb{R}^{(i-1)\times i}, \quad N_i = \begin{bmatrix} 0 & 0 \\ I_{i-1} & 0 \end{bmatrix} \in \mathbb{R}^{i\times i}.$$

Kronecker canonical form **KCF** (Kronecker1890): Any DAE $\Delta = (E, H) \overset{ex}{\sim} \tilde{\Delta} = (\tilde{E}, \tilde{H})$, where $(\tilde{E}, \tilde{H}) =$

$$\left(\begin{bmatrix} L_\varepsilon & 0 & 0 & 0 \\ 0 & I_{|\rho|} & 0 & 0 \\ 0 & 0 & N_\sigma & 0 \\ 0 & 0 & 0 & K_\eta^T \end{bmatrix}, \begin{bmatrix} K_\varepsilon & 0 & 0 & 0 \\ 0 & A_\rho & 0 & 0 \\ 0 & 0 & I_{|\sigma|} & 0 \\ 0 & 0 & 0 & L_\eta^T \end{bmatrix} \right) \begin{pmatrix} \text{under} - \text{determin} \\ \text{ODE} \\ \text{nilpotent} \\ \text{over} - \text{determin} \end{pmatrix},$$

where $\beta = (\beta_1, \ldots, \beta_k), |\beta| = \sum_{i=1}^{k} \beta_i$, A_ρ is in the real Jordan canonical form. The integers $(\varepsilon_1, \ldots, \varepsilon_a)$, $(\rho_1, \ldots, \rho_b)$, $(\sigma_1, \ldots, \sigma_c)$, $(\eta_1, \ldots, \eta_d)$ are called the Kronecker indices.

- The Kronecker indices can be calculated with the help of the Wong sequences $\mathscr{V}_i$, $\mathscr{W}_i$.

Morse canonical form **MCF** (Morse1973)(Molinari1978): Any control system $\Lambda = (A, B, C, D) \overset{M}{\sim} \tilde{\Lambda} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, where

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \left[\begin{array}{cccc|cc} A^1 & 0 & 0 & 0 & B^1 & 0 \\ 0 & A^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & A^3 & 0 & 0 & B^3 \\ 0 & 0 & 0 & A^4 & 0 & 0 \\ \hline 0 & 0 & C^3 & 0 & 0 & D^3 \\ 0 & 0 & 0 & C^4 & 0 & 0 \end{array}\right],$$

where (A^1, B^1) is controllable and in its Brunovský canonical form with indices $\varepsilon_i'(1 \le i \le a')$; (A^4, C^4) is observable and in its dual Brunovský canonical form with indices $\eta_i'(1 \le i \le d')$; (A^3, B^3, C^3, D^3) is controllable and observable and in its prime form with indices $\sigma_i'(1 \le i \le c')$, and the matrix A^2 is in its real Jordan canonical form with each block of $\rho_i' \times \rho_i'(1 \le i \le b')$ dimension.

- The Mores indices can be calculated with the help of the sequences of subspaces $\mathcal{V}_i$, $\mathcal{W}_i$.

Proposition 2 (Indices relations Proposition 2.5.3)

- *For a DAE Δ, the Kronecker indices of its* ***KCF*** *coincide with the Morse indices of the* ***MCF*** *of $\Lambda \in Expl(\Delta)$.*
- *$(N_\sigma, I_{|\sigma|})$ of the* ***KCF*** *is present iff the subsystem MCF^3 of the* ***MCF*** *is present.*
- *The invariant factors of A_ρ in the* ***KCF*** *of Δ coincide with that of A^2 in the* ***MCF*** *of Λ.*

- There exists a perfect correspondence between the **KCF** of a DAE and the **MCF** of its explicitation systems !!

Example 4

$$\begin{aligned}
(L_\varepsilon, K_\varepsilon) : \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1^1 \\ \dot{x}_2^1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} \quad &\leftrightarrow \quad (A^1, B^1) : \dot{z}^1 = u^1 \\
(I_{|\rho|}, A_\rho) : \dot{x}^2 = A_\rho x \quad &\leftrightarrow \quad A^2 : \dot{z}^2 = A^2 z^2 \\
(N_\sigma, I_{|\sigma|}) : \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1^3 \\ \dot{x}_2^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix} \quad &\leftrightarrow \quad (A^3, B^3, C^3, D^3) : \begin{cases} y^3 = z^3 \\ \dot{z}^3 = u^3 \end{cases} \\
(K_\eta^T, L_\eta^T) : \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{x}^4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x^4 \quad &\leftrightarrow \quad (A^4, C^4) : \begin{cases} y^4 = z^4 \\ \dot{z}^4 = 0 \end{cases}
\end{aligned}$$

- Consider a DAE control system DAECS (denoted by $\Delta^u_{l,n,m} = (E, H, L)$:)

$$\Delta^u : E\dot{x} = Hx + Lu, \tag{14}$$

- External feedback equivalence: $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$ if $\exists$ F and invertible Q, P, G s.t.

$$\tilde{E} = QEP^{-1}, \quad \tilde{H} = Q(H + LF)P^{-1}, \quad \tilde{L} = QLG. \tag{15}$$

Any $\Delta^u = (E, H, L)$ is ex-fb-equivalent to the following feedback canonical form **FBCF** (Loiseau et al 1991):

$$\left(\begin{bmatrix} I_{|\epsilon'|} & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{\bar{\epsilon}'} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & K^T_{\sigma'} & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{\bar{\sigma}'} & 0 \\ 0 & 0 & 0 & 0 & 0 & L^T_{\eta'} \end{bmatrix}, \begin{bmatrix} N^T_{\epsilon'} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{\bar{\epsilon}'} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & L^T_{\sigma'} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\bar{\sigma}'|} & 0 \\ 0 & 0 & 0 & 0 & 0 & K^T_{\eta'} \end{bmatrix}, \begin{bmatrix} \mathcal{E}_{\epsilon'} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathcal{E}_{\sigma'} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right),$$

where $\epsilon'_i (1 \le i \le a'), \bar{\epsilon}'_i (1 \le i \le b'), \sigma'_i (1 \le i \le c'), \bar{\sigma}'_i (1 \le i \le d'), \eta'_i (1 \le i \le e')$ and the Jordan structure of A_ρ are its invariants.

- Is there a simpler and geometrical way to get **FBCF**? Using explicitation ?

Explicitation with driving variables

- Given Δ^u, let $\operatorname{rank} E = r$. Let $s = n - r$ and $m = l - r$. Choose Q s.t.

$$QE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \tag{16}$$

where E_1 is of full row rank, denote $QF = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ and $QL = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$. .

- Solutions $\dot{x}$ of $E_1\dot{x} = H_1 x + L_1 u$ satisfy

$$\dot{x} \in Ax + B^u u + \ker E_1 = Ax + B^u u + \ker E. \tag{17}$$

where $A = E_1^{\dagger} H_1$, $B^u = E_1^{\dagger} L_1$. Then choose $\operatorname{Im} B^v = \ker E$ and v to parametrize $\ker E$ and let $y = Cx + D^u u = H_2 x + L_2 u$.

- Attach to Ξ^u the following control system $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$,

$$\Lambda^{uv} : \begin{cases} \dot{x} = Ax + B^u u + B^v v \\ y = Cx + D^u u, \end{cases} \tag{18}$$

where v is called the vector of driving variables.

Definition 3 (Explicitation with driving variables)

We will call the just defined control system Λ^{uv} a (Q, v)-explicitation of Δ^u. The class of all (Q, v)-explicitations of Δ^u is denoted by $\mathbf{Expl}(\Delta^u)$.

Equivalence of DAECSs and ODECSs with two kinds of inputs

- Extended Morse equivalence: $\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$, if $\exists$ invertible matrices T_x,T_y,T_u,T_v and matrices F_u, F_v,R, K s.t.

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{bmatrix} = \begin{bmatrix} T_x & T_xK \\ 0 & T_y \end{bmatrix} \begin{bmatrix} A & B^u & B^v \\ C & D^u & 0 \end{bmatrix} \begin{bmatrix} T_x^{-1} & 0 & 0 \\ F_uT_x^{-1} & T_u^{-1} & 0 \\ (F_v+RF_u)T_x^{-1} & RT_u^{-1} & T_v^{-1} \end{bmatrix},$$

  Extended Morse transformation: $EM_{tran} = (T_x, T_y, T_u, T_v, F_u, F_v, R, K)$.

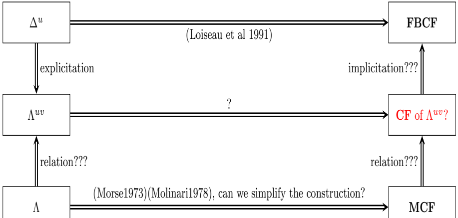
- Two kinds of feedback transformations (v is more powerful than u !):

$$v = F_vx + Ru + T_v^{-1}\tilde{v} \quad \text{and} \quad u = F_ux + T_u^{-1}\tilde{u}.$$

Theorem 2 (Theorem 3.2.9)

Given $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$ *and* $\tilde{\Lambda}^{\tilde{u}\tilde{v}} \in \mathbf{Expl}(\tilde{\Delta}^{\tilde{u}})$, *locally* $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$ *iff* $\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$.

Our plan of getting the **FBCF** of a DAECS Δ^u:



Morse triangular form and Morse normal form

Proposition 3 (Proposition 3.3.1 and 3.3.2)

*For an ODECS $\Lambda = (A, B, C, D)$, we can explicitly construct a Morse transformation bringing Λ into its Morse canonical form **MCF**, passing through intermediate Morse triangular form **MTF** ($\tilde{\Lambda} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$) and Morse normal form **MNF** ($\bar{\Lambda} = (\bar{A}, \bar{B}^{\bar{u}}, \bar{C}, \bar{D})$).*

$$\mathbf{MTF}: \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \left[\begin{array}{cccc|cc} \tilde{A}_1 & \tilde{A}_1^2 & \tilde{A}_1^3 & \tilde{A}_1^4 & \tilde{B}_1 & \tilde{B}_1^2 \\ 0 & \tilde{A}_2 & 0 & \tilde{A}_2^4 & 0 & 0 \\ 0 & 0 & \tilde{A}_3 & \tilde{A}_3^4 & 0 & \tilde{B}_3 \\ 0 & 0 & 0 & \tilde{A}_4 & 0 & 0 \\ \hline 0 & 0 & \tilde{C}_3 & \tilde{C}_3^4 & 0 & \tilde{D}_3 \\ 0 & 0 & 0 & \tilde{C}_4 & 0 & 0 \end{array}\right].$$

Morse triangular form and Morse normal form

Proposition 3 (Proposition 3.3.1 and 3.3.2)

*For an ODECS $\Lambda = (A, B, C, D)$, we can explicitly construct a Morse transformation bringing Λ into its Morse canonical form **MCF**, passing through intermediate Morse triangular form **MTF** ($\tilde{\Lambda} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}^{)}$) and Morse normal form **MNF** ($\bar{\Lambda}$=$(\bar{A}, \bar{B}, \bar{C}, \bar{D})$).*

$$\mathbf{MNF}: \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = \left[\begin{array}{cccc|cc} \bar{A}_1 & 0 & 0 & 0 & \bar{B}_1 & 0 \\ 0 & \bar{A}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{A}_3 & 0 & 0 & \bar{B}_3 \\ 0 & 0 & 0 & \bar{A}_4 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_3 & 0 & 0 & \bar{D}_3 \\ 0 & 0 & 0 & \bar{C}_4 & 0 & 0 \end{array}\right].$$

For ODECSs with two kinds of inputs (u, v), we propose a similar procedure.

Theorem 3 (Theorem 3.3.4, 3.3.5 and 3.4.2)

*For an ODECS Λ^{uv}, we can explicitly construct an extended Morse transformation bringing Λ^{uv} into its extended Morse canonical form **EMCF**, passing through intermediate extended Morse triangular form **EMTF** and extended Morse normal form **EMNF**.*

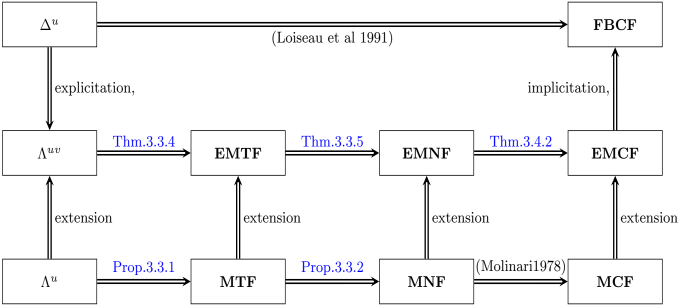
$$\textbf{EMCF}: \begin{cases} \dot z^{cu} = A^{cu} z^{cu} + B^{cu} u \\ \dot z^{cv} = A^{cv} z^{cv} + B^{cv} v \\ \dot z^{nn} = A^{nn} z^{nn} \\ \dot z^{pu} = A^{pu} z^{pu} + B^{pu} u, & y^{pu} = C^{pu} z^{pu} + D^{pu} u \\ \dot z^{pv} = A^{pv} z^{pv} + B^{pv} v, & y^{pv} = C^{pv} z^{pv} \\ \dot z^{o} = A^{o} z^{o} & y^{o} = C^{o} z^{o}, \end{cases}$$

- both the pairs (A^{cu}, B^{cu}) and (A^{cv}, B^{cv}) are controllable and in the Brunovský canonical forms;
- A^{nn} is up to similarity;
- the 4-tuple $(A^{pu}, B^{pu}, C^{pu}, D^{pu})$ and the triple (A^{pv}, B^{pv}, C^{pv}) are prime;
- the pair (C^{o}, A^{o}) is observable and in the dual Brunovský canonical form.

Our algorithm of finding the **FBCF** of a linear DAECS

Algorithm 1

- *Step 1: For* Δ^u*, construct* Λ^{uv} *st.* $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$.
- *Step 2: Find an* EM_{tran} *s.t.* $\tilde{\Lambda}^{\tilde{u}\tilde{v}} = EM_{tran}(\Lambda^{uv})$ *is in the* ***EMTF***.
- *Step 3: Find an* EM_{tran} *s.t.* $\bar{\Lambda}^{\bar{u}\bar{v}} = EM_{tran}(\tilde{\Lambda}^{\tilde{u}\tilde{v}})$ *is in the* ***EMNF***.
- *Step 4: Bring* $\bar{\Lambda}^{\bar{u}\bar{v}}$ *into the* ***EMCF*** *by normalizing the subsystems in the* ***EMNF***.
- *Step 5: Find the implicitation of* ***EMCF****, denoted by* $\bar{\Delta}^{\bar{u}}$*. Then* $\bar{\Delta}^{\bar{u}}$ *is in the* ***FBCF*** *and* $\Delta^u \overset{ex-fb}{\sim} \bar{\Delta}^{\bar{u}}$.



Consider a nonlinear DAE (DAECS):

$$\Xi^u : E(x)\dot{x} = F(x) + G(x)u, \tag{19}$$

- $x \in X$, an open subset of $\mathbb{R}^n$, $u \in \mathbb{R}^m$, functions $E(x)$, $F(x)$, $G(x)$ are $\mathcal{C}^\infty$ smooth, denote DAECS (19) by $\Xi^u_{l,n,m} = (E, F, G)$.
- A solution of a DAE $\Xi = (E, F)$ (DAECS $\Xi^u = (E, F, G)$) is a $\mathcal{C}^1$ curve $\gamma : I \to X$ defined on an open interval I (a curve $(\gamma, u) : I \to X \times \mathscr{U}$ with $\gamma(t) \in \mathcal{C}^1$ and $u(t) \in \mathcal{C}^0$) s.t. for all $t \in I$, the curve $\gamma(t)$ $(\gamma(t), u(t))$ satisfies $E\left(\gamma(t)\right)\dot{\gamma}(t) = F\left(\gamma(t)\right)$ $(E(\gamma(t))\dot{\gamma}(t) = F(\gamma(t)) + G(\gamma(t))u(t))$.
- γ_{x^0}: a solution $\gamma(t)$ satisfying $\gamma(0) = x^0$;
- I_{x^0}: the maximal time-interval on which γ_{x^0} exists.
- x_a: admissible point, through x_a there exist at least one solution.

- External feedback equivalence: $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$, $\exists$ a diffeomorphism $\psi : X \to \tilde{X}$ and smooth functions $Q : X \to Gl(l, \mathbb{R})$, $\alpha^u : X \to \mathbb{R}^m$, $\beta^u : X \to Gl(m, \mathbb{R})$ s.t.

$$\tilde{E}(\psi(x)) = Q(x)E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1}, \qquad \tilde{f}(\psi(x)) = Q(x)\left(F(x) + G(x)\alpha^u(x)\right),$$
$$\tilde{g}(\psi(x)) = Q(x)G(x)\beta^u(x).$$

- The ex-fb-equivalence preserves trajectories, but even if we can smoothly conjugate all trajectories of two DAEs, they are not necessarily ex-fb-equivalent.

Example 5

Consider two DAEs $\Xi_1 = (E_1, F_1)$ and $\Xi_2 = (E_2, F_2)$, where

$$E_1(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_1(x) = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}.$$

- Solutions of Ξ_1 exist on $\{x_{12} = x_{13} = 0\}$ only, while those of Ξ_2 on $\{x_{22} = x_{23} = 0\}$ only. Ex-fb-equivalence is defined on X or a neighborhood U of x^0.

- We need another equivalence defined only on where the solution exists !!

Locally invariant submanifold (existence of solutions)

Definition 4 (Locally (controlled) invariant submanifold)

Consider a DAE $\Xi_{l,n} = (E, F)$ (DAECS $\Xi^u_{l,n,m} = (E, F, G)$) defined on X . A smooth submanifold M s.t. $x_a \in M$ is called locally (controlled) invariant if $\exists$ a neighborhood $U \subseteq X$ of x_a s.t. for any point $x^0 \in M \cap U$, $\exists$ a solution $\gamma_{x^0} : I_{x^0} \to X$ (a $\mathcal{C}^0$-control $u(t)$) of Ξ (Ξ^u) s.t. $\gamma_{x^0}(0) = x^0$ and $\gamma_{x^0}(t) \in M \cap U$ for all $t \in I_{x^0}$.

Proposition 4 (Proposition 4.3.2 and 5.3.3 known results!!!)

For $\Xi = (E, F)$ $(\Xi^u = (E, F, G))$, *assume that locally on* M,

(Reg) *the dimension of* $E(x)T_xM$ *(and of* $E(x)T_xM + \mathrm{Im}\, G(x)$ *are) is constant*

then M *is a locally (controlled) invariant submanifold iff locally for all* $x \in M$,

$$F(x) \in E(x)T_xM + \mathrm{Im}\, G(x). \tag{20}$$

- For $\Delta = (E, H)$, a subspace $\mathscr{M}$ is invariant iff $H\mathscr{M} \subset E\mathscr{M}$.
- How to identify (locally) maximal invariant submanifold (where the solution exists)?

Proposition 5 (Proposition 4.3.3)

For a DAE DAECS $\Xi^u = (E, F, G)$, *assume that a point* x^0 *satisfies* $F(x^0) \in \operatorname{Im} E(x^0) + \operatorname{Im} G(x^0)$. *Set*

$$M_0 = \left\{x \in X : F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x)\right\};$$

Assume that M_{k-1} *is a smooth embedded submanifold and denote by* M^c_{k-1} *the connected component of* M_{k-1} *containing* $x^0 \in M^c_{k-1}$. *Set*

$$M_k = \left\{x \in M^c_{k-1} : F(x) \in E(x) T_x M^c_{k-1} + \operatorname{Im} G(x)\right\}.$$

Then there exists a smallest k, *denoted* $k^* < n$, *s.t.* $M_{k^*+1} = M^c_{k^*}$. *If* $M^c_{k^*}$ *satisfies the assumption* ***(Reg)*** *locally for all* $x \in M^c_{k^*}$, *then* x^0 *is an admissible point and* $M^* = M^c_{k^*}$ *is a locally (controlled) maximal invariant submanifold.*

- Identifying M^* was called the reduction procedure, e.g. (Reich1990)(Riaza2008).
- We propose an algorithm procedure (see Algorithm 4.3.4 and compare the zero dynamics algorithm (Isidori 1989)).
- Linear case: the maximal invariant subspace $\mathscr{M}^* = \mathscr{V}^*$.
- M_k can be seen as a nonlinear generalization of the Wong sequence $\mathscr{V}_i$ and thus M^* can be seen that of the limits $\mathscr{V}^*$.

- Restriction: If $M = \{z_2 = 0\}$, then

$$\Xi|_M : \tilde{E}(z_1, 0) \begin{bmatrix} \dot{z}_1 \\ 0 \end{bmatrix} = \tilde{F}(z_1, 0). \tag{21}$$

- We would like the restricted system as simple as possible!
- Reduction:
$\Xi_1 : \begin{cases} \dot{x} &= f(x) \\ q(x)\dot{x} &= q(x)f(x) \end{cases}, \quad \Xi_2 : \begin{cases} \dot{x} = f(x) \\ 0 = 0 \end{cases}, \quad \Xi_1^{red} = \Xi_2^{red} : \ \dot{x} = f(x)$.
- $\Xi^u|_{M^*}^{red}$ (a restricted system): a reduction of the restriction $\Xi^u|_{M^*}$. The order is important!!!

Definition 5 (Internal feedback equivalence)

Given two DAEs DAECSs $\Xi^u = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}} = (\tilde{E}, \tilde{F}, \tilde{G})$, let M^* and $\tilde{M}^*$ be two smooth submanifolds. Assume that

(A1) M^* and $\tilde{M}^*$ are locally maximal controlled invariant submanifolds of Ξ^u and $\tilde{\Xi}^{\tilde{u}}$, respectively.

(A2) Locally M^* and $\tilde{M}^*$ satisfy the assumption **(Reg)**.

Then, Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ are called locally internally (feedback) equivalent, shortly in-fb-equivalent , if $\Xi^u|_{M^*}^{red}$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}^{red}$ are ex-fb-equivalent, denoted by $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$.

Internal equivalence of linear DAEs

- The internal equivalence is a proper tool to study solutions of DAEs !!

Theorem 4 (Theorem 2.6.10)

Consider two control systems: $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{M}^*})$, $\tilde{\Lambda}^* \in Expl(\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*})$. Then the following are equivalent:

(i) $\Delta \overset{in}{\sim} \tilde{\Delta}$;

(ii) Λ^* *and* $\tilde{\Lambda}^*$ *are feedback equivalent;*

(iii) Δ *and* $\tilde{\Delta}$ *have isomorphic trajectories, i.e, there exists a linear and invertible map* $S : \mathscr{M}^* \to \tilde{\mathscr{M}}^*$ *transforming any trajectory* $x(t, x^0)$, *where* $x^0 \in \mathscr{M}^*$ *of* $\Delta|^{red}_{\mathscr{M}^*}$ *into a trajectory* $\tilde{x}(t, \tilde{x}^0)$, $\tilde{x}^0 \in \tilde{\mathscr{M}}^*$ *of* $\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*}$, *where* $\tilde{x}^0 = Sx^0$, *and vice versa.*

Internal regularity of DAEs (uniqueness of solutions)

Proposition 6 (Proposition 2.6.12)

For a DAE $\Delta_{l,n} = (E, H)$, *denote* $\operatorname{rank} E = q$. *The following statements are equivalent:*

(i) Δ *is internally regular i.e., through each point of* $\mathscr{M}^*$, $\exists$ *only one solution.*

(ii) *Any* $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{M}^*})$ *has no inputs;*

(iii) $\operatorname{rank} E = \dim\, E\mathscr{M}^*$.

Theorem 5 (Theorem 4.3.14)

Consider a DAE $\Xi_{l,n} = (E, F)$. *Let* M^* *be a locally maximal invariant submanifold. Assume that* $\dim\, E(x)T_xM^*$ *is constant locally for all* $x \in M^*$. *The following are locally equivalent:*

(i) Ξ *is internally regular i.e., locally through each point of* M^*, $\exists$ *only one solution.*

(ii) Ξ *is internally equivalent to*

$$\Xi^* : \dot{z}^* = F^*\left(z^*\right), \tag{22}$$

where z^* *is a local, around* x_a, *system of coordinates on* M^*.

(iii) $\dim\, M^* = \dim\, E(x)T_xM^*$ *for all* $x \in M^*$.

- Given a DAE DAECS $\Xi^{u}_{l,n,m} = (E, F, G)$, fix a point x^0. Assume rank $E(x) = const.$ around x^0. Choose $Q(x)$ s.t. $Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$ where $E_1(x)$ is locally full row rank, denote $QF(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$ and $QG(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix}$.

Definition 6 (Explicitation with driving variables)

We will call the control system

$$\Sigma^{uv}: \begin{cases} \dot{x} = f(x) + g^u(x)u + g^v(x)v \\ y = h(x) + l^u(x)u, \end{cases} \tag{23}$$

with two inputs (u, v), where $f(x) = E_1^{\dagger} F_1(x)$, $g^u(x) = E_1^{\dagger} G_1(x)$, $\operatorname{Im} g^v(x) = \ker E(x)$, $h(x) = F_2(x)$, $l^u(x) = G_2(x)$, a (Q, v)-explicitation of Ξ^u, denote the class of all (Q, v)-explicitation of Ξ^u by $\mathbf{Expl}(\Xi^u)$.

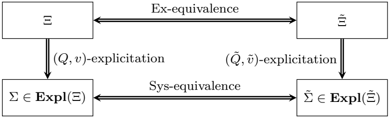
- System feedback equivalence $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$: if $\exists$ a diffeomorphism $\psi : X \to \tilde{X}$, smooth functions $\alpha^u(x)$, $\alpha^v(x)$, $\lambda(x)$ and $\gamma(x)$ and invertible smooth functions $\beta^u(x)$, $\beta^v(x)$ and $\eta(x)$ s.t. (compare EM-equivalence of Λ^{uv})

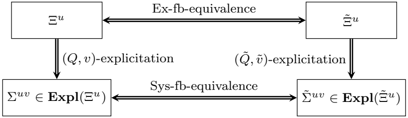
$$\begin{bmatrix} \tilde{f} \circ \psi & \tilde{g}^{\tilde{u}} \circ \psi & \tilde{g}^{\tilde{v}} \circ \psi \\ \tilde{h} \circ \psi & \tilde{l}^{\tilde{u}} \circ \psi & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial x}\gamma\eta \\ 0 & \eta \end{bmatrix} \begin{bmatrix} f & g^u & g^v \\ h & l^u & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ \alpha^u & \beta^u & 0 \\ \alpha^v + \lambda\alpha^u & \lambda\beta^u & \beta^v \end{bmatrix}.$$

External (feedback) equivalence of DAE systems vs. system (feedback) equivalence of ODE control systems

Theorem 6 (Theorem 5.2.9)

Locally, assume $\operatorname{rank} E(x) = const.$ *and* $\operatorname{rank} \tilde{E}(\tilde{x}) = const.$ *Then, given* $\Sigma^{uv} \in \mathbf{Expl}(\Xi^{u})$ *and* $\tilde{\Sigma}^{\tilde{u}\tilde{v}} \in \mathbf{Expl}(\Xi^{u})$ *, we have* $\Xi^{u} \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ *iff* $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$.





- System (feedback) equivalence for explicitation systems is a true counterpart of the external (feedback) equivalence for DAEs (DAECSs) !!!

Does $Expl(\Xi)$ (the class of (Q,P)-explicitations of nonlinear DAEs) exists?

- When is a nonlinear DAE $\Xi = (E, F)$ ex-equivalent to a *pure* semi-explicit PSE DAE?

$$\Xi^{pse} : \begin{cases} \dot{x}_1 = F_1(x_1, x_2) \\ \;0 \;= F_2(x_1, x_2), \end{cases} \tag{24}$$

- Can we get rid of all the driving variables v in a (Q, v)-expl of Ξ?

Example 6

Consider a DAE $\Xi = (E, F)$, given by $\begin{bmatrix} \sin x_3 & -\cos x_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} F_1(x) \\ x_1^2 + x_2^2 - 1 \end{bmatrix}$,

where $F_1 : X \to \mathbb{R}$. A control system $\Sigma \in \mathbf{Expl}(\Xi)$ is:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{bmatrix} F_1(x) + \begin{bmatrix} 0 & \cos x_3 \\ 0 & -\sin x_3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ \;\; y \;\; = x_1^2 + x_2^2 - 1, \end{cases}$$

where $\begin{bmatrix} \sin x_3 & -\cos x_3 & 0 \end{bmatrix}^T$ is a right inverse of $E_1(x) = \begin{bmatrix} \sin x_3 & -\cos x_3 & 0 \end{bmatrix}$.

- We may get rid of v_1 and regard x_3 as a new control, but not for v_2:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \sin x_3 \\ -\cos x_3 \end{bmatrix} F_1(x) + \begin{bmatrix} \cos x_3 \\ -\sin x_3 \end{bmatrix} v_2,$$

Explicitation without driving variables and pure semi-explicit DAEs

Theorem 7 (Theorem 4.3.27)

For a DAE $\Xi_{l,n} = (E, F)$, the following conditions are equivalent around a point x^0:

(i) $\operatorname{rank} E(x)$ *is constant and the distribution* $\mathscr{D} = \ker E(x)$ *is involutive.*

(ii) Ξ *is ex-equivalent to a pure semi-explicit DAE* Ξ^{PSE}*.*

(iii) *The driving variables v of any control system $\Sigma_{n,m,p} = (f, g, h) \in \mathbf{Expl}(\Xi)$ can be fully reduced.*

- (Q, P)- and (Q, v)- explicitations of a linear DAE $\Delta = (E, H)$ always exist. But $\mathbf{Expl}(\Xi)$ of of a nonlinear DAE $\Xi = (E, F)$ exits when $E(x)$ is const. rank and "$Expl(\Xi)$" exists when $\ker E(x)$ is const. rank and involutive.
- Claim: SE DAES of the form below are the right class to be considered.

$$\Xi^{se} : \begin{cases} E_1(x)\dot{x}_1 & = & F_1(x) \\ 0 & = & F_2(x), \end{cases}$$

- They are more general than pure semi-explicit DAE's, since $\ker E$ need not be integrable.

Plan

Nonlinear Weistrass form

If a linear DAE Δ is regular, i.e., E and H are square ($l = n$) and $|sE - H| \not\equiv 0$ for $s \in \mathbb{C}$, then Δ is ex-equivalent to the Weierstrass form **WF** (Weierstrass1868):

$$\mathbf{WF}: \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix},$$

Theorem 8 (Nonlinear Weistrass form Theorem 4.3.29)

Consider $\Xi_{l,n} = (E, F)$ *with* $l = n$*, assume that* $\operatorname{rank} E(x) = const. = q$ *around an admissible point* x_a*. Then under some constant rank assumptions in the reduction procedure and* $\dim E(x)T_x M^* = \dim M^*$*,* Ξ *is internally regular and* Ξ *is locally ex-equivalent to:*

$$\mathbf{NWF}: \begin{cases} 0 &= \xi_i^1, \quad 1 \le i \le m, \quad 1 \le j \le \rho_i - 1 \\ \dot{\xi}_i^j &= \xi_i^{j+1} + a_i^j + \sum\limits_{l=1}^{m} b_{i,l}^j \dot{\xi}_l^{\rho_l} + E_i^j(\xi, z, \dot{\xi}^{\rho}), \\ \vdots & \\ \dot{z} &= F^*(\xi, z) - G(\xi, z)\dot{\xi}, \end{cases}$$

where the scalar functions $a_i^k, b_{i,l}^k \in \mathbf{I}^k$, $1 \le k \le \rho_i - 1$, $\mathbf{I}^k$ *is the ideal generated by* ξ_i^j, $1 \le i \le m$, $1 \le j \le k$ *in the ring of smooth functions of* ξ_s^t *and* z_r*, and where*

$$E_i^j(\xi, z, \dot{\xi}^{\rho}) = \sum_{s=1}^{i-1} E_{i,s}^j(\xi, z)\dot{\xi}_s^{\rho_s}, \quad j \ge \rho_s.$$

- When is a nonlinear DAECS $\Xi^u = (E, F, G)$ is feedback equivalent to a linear DAECS $\Delta^u = (E, H, L)$?
- An ODECS $\Sigma_{n,m} : \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$, is feedback equivalent to a linear controllable ODECS $\Lambda_{n,m} : \dot{x} = Ax + Bu$ iff for all $i \geq 1$, the distributions G_i (defined by $G_1 := \text{span}\{g_1, \ldots, g_m\}$, $G_{i+1} = G_i + [f, G_i]$) are constant dimensional, involutive and $G_n = TX$ (Jakubczyk and Respondek1980) (Hunt and Su1981).

Lemma 1 (Berger2013)

A linear DAECS Δ^u is completely controllable(i.e., for any $x^0, x^f \in \mathbb{R}^n$, $\exists$ a solution (x, u) s.t. $x(0) = x^0$ and $x(t) = x^f$) iff $\mathscr{V}^* \cap \mathscr{W}^* = \mathbb{R}^n$.

- $\mathscr{V}^*$ and $\mathscr{W}^*$ are the limits of the augmented Wong sequences

$$\mathscr{V}_0 = \mathbb{R}^n, \ \mathscr{V}_{i+1} = H^{-1}(E\mathscr{V}_i + \operatorname{Im} L),$$
$$\mathscr{W}_0 = 0, \ \mathscr{W}_{i+1} = E^{-1}(H\mathscr{W}_i + \operatorname{Im} L).$$
$$\left(\hat{\mathscr{W}}_1 = \ker E, \ \hat{\mathscr{W}}_{i+1} = E^{-1}(H\hat{\mathscr{W}}_i + \operatorname{Im} L)\right).$$

- What are the linearizability distributions for a nonlinear DAECS?
- What is a nonlinear generalization of the augmented Wong sequences ($\mathscr{V}_i$ and $\mathscr{W}_i$) ?

Define the following two sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$, called the linearizability distributions of Σ^{uv}. ($\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ generalize $\mathcal{W}_i$ and $\hat{\mathcal{W}}_i$)

$$\begin{cases} \mathcal{D}_0 & := \{0\}, \\ \mathcal{D}_1 & := \operatorname{span}\left\{g_1^u, \ldots, g_m^u, g_1^v, \ldots, g_s^v\right\} \\ \mathcal{D}_{i+1} & := \mathcal{D}_i + [f, \mathcal{D}_i], \quad i = 1, 2, \ldots, \end{cases} \qquad \begin{cases} \\ \hat{\mathcal{D}}_1 & := \operatorname{span}\left\{g_1^v, \ldots, g_s^v\right\} \\ \hat{\mathcal{D}}_{i+1} & := \mathcal{D}_i + [f, \hat{\mathcal{D}}_i], \quad i = 1, 2, \ldots. \end{cases}$$

Theorem 9 (Internal feedback linearization Theorem 5.4.5)

Consider a DAECS $\Xi^u = \Xi^u_{l,n,m} = (E, F, G)$, fix an admissible point x_a. Let M^* be the n^*-dimensional maximal controlled invariant submanifold of Ξ^u around x_a. Assume that locally on M^*, we have

(A1) *M^* satisfies **(Reg)**,*

(A2) *the rank of $G(x)$ is m.*

*Ξ^u is locally in-fb-equivalent to a linear completely controllable DAECS iff the distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ of one (and thus any) ODECS $\Sigma^{uv} \in$ **Expl**$(\Xi^u|^{red}_{M^*})$ locally satisfy :*

(FL1) *$\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ are of constant rank for $1 \le i \le n^*$.*

(FL2) *$\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ are involutive for $1 \le i \le n^* - 1$.*

(FL3) *$\mathcal{D}_{n^*} = \hat{\mathcal{D}}_{n^*} = TM^*$.*

Theorem 10 (External feedback linearization Theorem 5.4.6)

Consider a DAECS $\Xi^u_{l,n,m} = (E, F, G)$, fix a point x^0. Then Ξ^u is locally ex-fb-equivalent to a linear complete controllable DAECS, locally around x^0, if and only if there exists a neighborhood $U \subseteq X$ of x^0 in which the following conditions are satisfied.

(EFL1) $\operatorname{rank} E(x)$ *and* $\operatorname{rank} [E(x), G(x)]$ *are constant.*

(EFL2) $F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x)$.

(EFL3) *For one (and thus any) control system $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u|_{M^*})$, which is a system with no outputs on $M^* = U$, a neighborhood of x^0, the distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ satisfy conditions (FL1)-(FL3) of Theorem 9.*

- By (EFL1)-(EFL2), $M^* = U$, which is a neighborhood of x^0.
- Note that condition $F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x)$ and the condition $\hat{\mathcal{D}}_{n^*} = \mathcal{D}_{n^*} = TM^*$ are nonlinear counterparts of the condition $\mathscr{V}^* \cap \mathscr{W}^* = \mathbb{R}^n$ of Lemma 1.

Example 7

Consider the following academic example borrowed from (Berger2016zero):

$$\Xi^u : \begin{bmatrix} x_2 & x_1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_2^2 - x_1^3 + x_3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

We consider an admissible point $x_a = (x_{1a}, x_{2a}, x_{3a}) = (1, 1, 0)$. Clearly, $\exists U$ ($x_1 \neq 0$ for $x \in U$) of x_a s.t. (EFL1)-(EFL2) of Theorem 10 are satisfied. The reduction of the restriction of Ξ^u to $M^* = U$ is

$$\Xi^u|_{M^*}^{red} : \begin{bmatrix} x_2 & x_1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2^2 - x_1^3 + x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1.$$

Now an ODECS $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u|_{M^*}^{red})$ can be taken as

$$\Sigma^{uv} : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_2^2 - x_1^3 + x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2/x_1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} x_1 \\ -x_2 \\ -x_1 \end{bmatrix} v,$$

where v is a driving variable. It is not hard to verify $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ satisfy (EFL3). The original DAECS Ξ^u is ex-fb-equivalent to the following completely controllable linear DAECS:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}.$$

External equivalence of Semi-explicit DAEs

A semi-explicit DAE is

$$\Xi^{se} : \begin{cases} \mathcal{R}(x)\dot{x} = a(x) \\ \quad 0 \quad = c(x) \end{cases} \tag{25}$$

- $\mathcal{R}(x)$ is locally of full row rank.
- For the ex-equivalence of Ξ^{se}, we use $Q(x) = \begin{bmatrix} Q^a & 0 \\ 0 & Q^c \end{bmatrix}$ to preserve the decoupling into differential and algebraic parts.
- Three levels of equivalence (on the algebraic parts), Q^c always invertible: Level-1 : $Q^c(x)$-any; Level-2 : $Q^c(x) = S(c(x))$; Level-3: $Q^c(x) = Tc(x)$, T-const.;

Level-3 external linearization of SE DAEs

Theorem 11 (Theorem 6.4.4)

Consider $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ around a point x_0. Then in a neighborhood X_0 of x_0, Ξ^{se} is level-3 ex-equivalent to a linear SE DAE Δ^{se} of the form

$$\begin{cases} \dot z^1 = A^1 z^1 + B^1 w^1, & \\ \dot z^3 = A^3 z^3 + B^3 w^3 & +K^3 y, \qquad 0 = C^3 z^3 + D^3 w^3, \end{cases} \tag{26}$$

*where (A^1, B^1) is controllable and in its Brunovsky form, (A^3, B^3, C^3, D^3) is in prime form, iff a (and then any) control system $\Sigma \in$ **Expl**(Ξ^{se}) satisfies the following conditions in X_0:*

(i) *Σ is level-3 input-output linearizable;*

(ii) *S_i and G_i are involutive and of constant rank;*

(iii) *$S^* = TX_0$;*

(iv) *$S_i \cap V^* = G_i \cap V^*$.*

- Standard G_i V_i, S_i distribution in nonlinear control theory. (see e.g. (Isidori 1989) (Nijmeijer & Van der Schaft 1990)) V^* is the largest controlled invariant distributions in $\ker dh$ and S^* is smallest conditioned distributions containing Img.
- The distributions V^* and S^* are, obviously, the nonlinear generalizations of the limits of Wong sequences $\mathscr{V}^*$ and $\mathscr{W}^*$, respectively.

Conclusions

- Existence and uniqueness of solutions of DAE systems.
- Internal and external (feedback) equivalence of DAE systems.
- Two kinds of explicitation procedures.
- Connections between DAE and ODE systems: equivalences, invariant subspaces, canonical forms.
- Nonlinear generalizations of the notions in linear DAEs theory.
- Linearization and feedback linearization of nonlinear DAE systems.

Thank you for listening !!!