From Morse Triangular Form of ODE Control Systems to Feedback Canonical Form of DAE Control Systems

Yahao CHEN

In collaboration with Prof. Witold Respondek

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Preliminary: geometric subspaces of linear ODE control systems

Consider a linear ordinary differential equation control system ODECS:

$$\Lambda : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du, \end{cases} \tag{1}$$

- where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, denoted $\Lambda_{n,m,p} = (A, B, C, D)$.
- Recall the following geometric subspaces (Molinari1974):

$\begin{cases} \mathcal{V}_0 := \mathbb{R}^n, \\ \mathcal{V}_{i+1} := \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{V}_i + \operatorname{Im} \begin{bmatrix} B \\ D \end{bmatrix} \right) \end{cases}$ $\mathcal{U}_i := \begin{bmatrix} B \\ D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_i \\ 0 \end{bmatrix}.$	$\mathcal{V}^* = \mathcal{V}_{k^*}$ is the largest s.t. $\exists F$, $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ and $(C+DF)\mathcal{V} = 0$; $\mathcal{U}^* = \mathcal{U}_{k^*}$ is the corresponding input subspace
$\begin{cases} \mathcal{W}_0 = \{0\}, \\ \mathcal{W}_{i+1} = [A, B] \left(\begin{bmatrix} \mathcal{W}_i \\ \mathscr{U} \end{bmatrix} \cap \ker [C, D] \right). \end{cases}$ $\mathcal{Y}_i = [C \quad D] \begin{bmatrix} \mathcal{W}_i \\ \mathscr{U} \end{bmatrix}.$	$\mathcal{W}^* = \mathcal{W}_{l^*}$ is the smallest s.t. $\exists K$, $(A+KC)\mathcal{W} + (B+KD)\mathscr{U} = \mathcal{W}$; $\mathcal{Y}^* = \mathcal{Y}_{l^*}$ is the corresponding output subspace

Preliminary: Morse normal form and Morse canonical form

Definition 1 (Morse equivalence (Morse1973,Molinari1978))

Two ODECSs $\Lambda \overset{M}{\sim} \tilde{\Lambda}$, if $\exists$ invertible matrices T_s, T_i, T_o and matrices F, K s.t.

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} T_s & T_s K \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ F T_s^{-1} & T_i^{-1} \end{bmatrix} \tag{2}$$

Morse transformation: $M_{tran} = (T_s, T_o, T_i, F, K)$

Morse normal form **MNF** (Morse1973)(Molinari1978): Any control system $\Lambda = (A, B, C, D) \overset{M}{\sim} \tilde{\Lambda} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, where

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \left[\begin{array}{cccc|cc} A^c & 0 & 0 & 0 & B^c & 0 \\ 0 & A^{nn} & 0 & 0 & 0 & 0 \\ 0 & 0 & A^p & 0 & 0 & B^p \\ 0 & 0 & 0 & A^o & 0 & 0 \\ \hline 0 & 0 & C^p & 0 & 0 & D^p \\ 0 & 0 & 0 & C^o & 0 & 0 \end{array}\right],$$

- where (A^c, B^c) is controllable, (A^o, C^o) is observable;
- (A^p, B^p, C^p, D^p) is called prime and it is controllable and observable;
- Via extra Morse transformations, we can get the Morse canonical form **MCF** from the **MNF**.
- The Mores indices of can be calculated with the help of the sequences of subspaces $\mathcal{V}_i$, $\mathcal{W}_i$, $\mathcal{U}_i$, $\mathcal{Y}_i$.

Outline

1. Explicitation with driving variables for linear DAE control systems
2. Morse triangular form, Morse normal form and their extensions
3. From extended Morse canonical form to the feedback canonical form
4. An algorithm and conclusions

We consider a linear DAE control system DAECS:

$$\Delta^u : E\dot{x} = Hx + Lu, \tag{3}$$

- where $x \in \mathbb{R}^n$ is called the "generalized" state, $u \in \mathbb{R}^m$ is a vector of predefined control variables,
- where $E \in \mathbb{R}^{l\times n}$, $H \in \mathbb{R}^{l\times n}$, $L \in \mathbb{R}^{l\times m}$,
- denoted by $\Delta^u_{l,n,m} = (E, H, L)$?

Definition 2 (External feedback equivalence)

Two DAECSs $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$ if $\exists$ F and invertible Q, P, G s.t.

$$\tilde{E} = QEP^{-1}, \quad \tilde{H} = Q(H + LF)P^{-1}, \quad \tilde{L} = QLG. \tag{4}$$

Feedback canonical form of linear DAE control systems

Set $N_\beta = \mathrm{diag}\left\{N_{\beta_1}, \dots, N_{\beta_k}\right\}$, $K_\beta = \mathrm{diag}\left\{K_{\beta_1}, \dots, K_{\beta_k}\right\}$,$L_\beta = \mathrm{diag}\left\{L_{\beta_1}, \dots, L_{\beta_k}\right\}$.

$$K_i = \begin{bmatrix} 0 & I_{i-1} \end{bmatrix} \in \mathbb{R}^{(i-1)\times i}, \quad L_i = \begin{bmatrix} I_{i-1} & 0 \end{bmatrix} \in \mathbb{R}^{(i-1)\times i}, \quad N_i = \begin{bmatrix} 0 & 0 \\ I_{i-1} & 0 \end{bmatrix} \in \mathbb{R}^{i\times i}.$$

Any $\Delta^u = (E, H, L)$ is ex-fb-equivalent to the following feedback canonical form **FBCF** (Loiseau et al 1991):

$$\left(\begin{bmatrix} I_{|\epsilon'|} & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{\bar{\epsilon}'} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & K^T_{\sigma'} & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{\bar{\sigma}'} & 0 \\ 0 & 0 & 0 & 0 & 0 & L^T_{\eta'} \end{bmatrix}, \begin{bmatrix} N^T_{\epsilon'} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{\bar{\epsilon}'} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & L^T_{\sigma'} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\bar{\sigma}'|} & 0 \\ 0 & 0 & 0 & 0 & 0 & K^T_{\eta'} \end{bmatrix}, \begin{bmatrix} \mathcal{E}_{\epsilon'} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathcal{E}_{\sigma'} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right),$$

where $\epsilon'_i (1 \le i \le a'), \bar{\epsilon}'_i (1 \le i \le b'), \sigma'_i (1 \le i \le c'), \bar{\sigma}'_i (1 \le i \le d'), \eta'_i (1 \le i \le e')$ and the Jordan structure of A_ρ are its invariants.

- Is there a simpler and geometrical way to get the **FBCF** ?
- The **FBCF** seems to have some simularities with the **MCF**, do they have connections ?
- In general, can we connect DAECSs with ODECSs ?

Explicitation with driving variables

Explicitation procedure:

- Given Δ^u, let rank $E = r$. Let $s = n - r$ and $m = l - r$. Choose Q s.t.

$$QE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \tag{5}$$

where E_1 is of full row rank, denote $QF = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ and $QL = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$. .

- Solutions $\dot{x}$ of $E_1\dot{x} = H_1 x + L_1 u$ satisfy

$$\dot{x} \in Ax + B^u u + \ker E_1 = Ax + B^u u + \ker E. \tag{6}$$

where $A = E_1^{\dagger} H_1$, $B^u = E_1^{\dagger} L_1$.

- Choose $\operatorname{Im} B^v = \ker E$ and v to parametrize $\ker E$ and let

$$y = Cx + D^u u = H_2 x + L_2 u.$$

- Attach to Ξ^u the following control system $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$,

$$\Lambda^{uv} : \begin{cases} \dot{x} = Ax + B^u u + B^v v \\ y = Cx + D^u u, \end{cases} \tag{7}$$

where v is called the vector of driving variables.

Analysis of the above procedure:

- The choices of Q, B^v and $E_1^\dagger$ are not unique !
- If $\begin{cases} Q, E_1^\dagger, B^v \Rightarrow \Lambda^{uv} \\ Q, \tilde{E}_1^\dagger, \tilde{B}^{\tilde{v}} \Rightarrow \tilde{\Lambda}^{u\tilde{v}} \end{cases}$, then $\Lambda^{uv} \sim \tilde{\Lambda}^{u\tilde{v}}$ via $v = F_v x + Ru + \tilde{v}$;
- If $\begin{cases} Q, E_1^\dagger, B^v \Rightarrow \Lambda^{uv} \\ \tilde{Q}, E_1^\dagger, B^v \Rightarrow \tilde{\Lambda}^{uv} \end{cases}$, then $\Lambda^{uv} \sim \tilde{\Lambda}^{uv}$ via $Ky = K(Cx + D^u u)$ and $\tilde{y} = T_y y$;
- We attach a class of ODECSs to Δ^u, given by all choices of K, F_v, R, and invertible T_v, T_y:

$$\begin{cases} \dot{x} = Ax + B^u u + Ky + B^v(F_v x + Ru + T_v^{-1}\tilde{v}) \\ y = T_y(Cx + Du). \end{cases}$$

Definition 3 (Explicitation with driving variables)

We will call a control system Λ^{uv} given by the above procedure a (Q, v)-explicitation of Δ^u. The class of all (Q, v)-explicitations of Δ^u is denoted by $\mathbf{Expl}(\Delta^u)$.

Definition 4 (Extended Morse equivalence)

$\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$, if $\exists$ invertible matrices T_x,T_y,T_u,T_v and matrices F_u, F_v,R, K s.t.

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{bmatrix} = \begin{bmatrix} T_x & T_x K \\ 0 & T_y \end{bmatrix} \begin{bmatrix} A & B^u & B^v \\ C & D^u & 0 \end{bmatrix} \begin{bmatrix} T_x^{-1} & 0 & 0 \\ F_u T_x^{-1} & T_u^{-1} & 0 \\ (F_v + RF_u)T_x^{-1} & RT_u^{-1} & T_v^{-1} \end{bmatrix},$$

Extended Morse transformation: $EM_{tran} = (T_x, T_y, T_u, T_v, F_u, F_v, R, K)$.

- Two kinds of feedback transformations (v is more powerful than u !):

$$v = F_v x + Ru + T_v^{-1}\tilde{v} \quad \text{and} \quad u = F_u x + T_u^{-1}\tilde{u}.$$

- If we write $w = (u, v)$, $[B^u \quad B^v] = B^w$, $D^w = [D^u \quad 0]$, then $\Delta^{uv} = \Delta^w = (A, B^w, C, D^w)$.

- EM_{tran} can be represented as M_{tran} with a triangular input coordinates transformation $T_w^{-1} = \begin{bmatrix} T_u^{-1} & 0 \\ RT_u^{-1} & T_v^{-1} \end{bmatrix}$.

Theorem 1

*Given $\Lambda^{uv} \in$ **Expl**(Δ^u) and $\tilde{\Lambda}^{\tilde{u}\tilde{v}} \in$ **Expl**$(\tilde{\Delta}^{\tilde{u}})$, locally $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$ iff $\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$.*

Augmented Wong sequences of DAECSs and invariant subspaces of ODECSs

$\Delta^u = (E, H, L)$	$\Lambda^{uv} = (A, B^u, B^v, C, D^u)$ or $\Lambda^w = (A, B^w, C, D^w)$
The augmented Wong sequences: $\begin{cases} \mathscr{V}_0 = \mathbb{R}^n, \ \mathscr{V}_{i+1} = H^{-1}(E\mathscr{V}_i + \operatorname{Im} L) \\ \mathscr{W}_0 = 0, \ \mathscr{W}_{i+1} = E^{-1}(H\mathscr{W}_i + \operatorname{Im} L) \end{cases}$	$\begin{cases} \mathcal{V}_0 = \mathbb{R}^n, \ \mathcal{V}_{i+1} = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{V}_i + \operatorname{Im} \begin{bmatrix} B^w \\ D^w \end{bmatrix} \right) \\ \mathcal{W}_0 = \{0\}, \ \mathcal{W}_{i+1} = [A \ \ B^w] \left(\begin{bmatrix} \mathcal{W}_i \\ \mathscr{U}_w \end{bmatrix} \cap \ker \, [C \ \ D^w] \right) \end{cases}$
$\hat{\mathscr{W}}_1 = \ker E, \ \hat{\mathscr{W}}_{i+1} = E^{-1}(H\hat{\mathscr{W}}_i + \operatorname{Im} L)$	$\hat{\mathcal{W}}_1 = \operatorname{Im} B^v, \ \hat{\mathcal{W}}_{i+1} = [A \ \ B^w] \left(\begin{bmatrix} \hat{\mathcal{W}}_i \\ \mathscr{U}_w \end{bmatrix} \cap \ker \, [C \ \ D^w] \right)$
$\mathscr{V}^* = \mathscr{V}_{k^*}$ is the largest s.t. $\mathscr{V} = H^{-1}(E\mathscr{V} + \operatorname{Im} L)$; $\mathscr{W}^* = \mathscr{W}_{l^*} = \hat{\mathscr{W}}_{l^*\pm 1}$ is the smallest s.t. $\mathscr{W} = E^{-1}(H\mathscr{V} + \operatorname{Im} L)$	$\mathcal{V}^* = \mathcal{V}_{k^*}$; $\mathcal{W}^* = \mathcal{W}_{l^*} = \hat{\mathcal{W}}_{l^*\pm 1}$.

Proposition 1 (Subspaces relations)

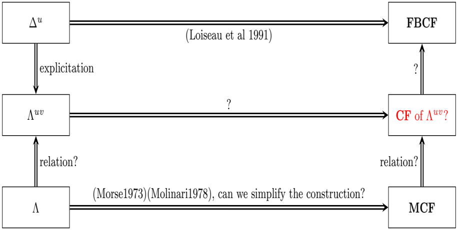
Assume that $\Lambda^{uv} \in$ **Expl**(Δ^u). *Then we have for* $i \in \mathbb{N}$,

$$\mathscr{V}_i(\Delta^u) = \mathcal{V}_i(\Lambda^w), \qquad \mathscr{W}_i(\Delta^u) = \mathcal{W}_i(\Lambda^w),$$

and for $i \in \mathbb{N}^+$,

$$\hat{\mathscr{W}}_i(\Delta^u) = \hat{\mathcal{W}}_i(\Lambda^w).$$

Our plan of getting the **FBCF** of Δ^u:



Plan

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Proposition 2 (Morse triangular form MTF)

For an ODECS $\Lambda^u_{n,m,p} = (A, B^u, C, D^u)$, *choose full rank matrices* $T^j_s, j = 1,2,3,4$, $T^j_i, j = 1,2$, $T^j_o, j = 1,2$, *s.t.*

$$\begin{array}{ll}
\operatorname{Im} T^1_s = \mathcal{V}^* \cap \mathcal{W}^*, & \mathcal{V}^* \cap \mathcal{W}^* \oplus \operatorname{Im} T^2_s = \mathcal{V}^*, \\
\mathcal{V}^* \cap \mathcal{W}^* \oplus \operatorname{Im} T^3_s = \mathcal{W}^*, & (\mathcal{V}^* + \mathcal{W}^*) \oplus \operatorname{Im} T^4_s = \mathscr{X} = \mathbb{R}^n, \\
\operatorname{Im} T^1_i = \mathcal{U}^*_u, & \operatorname{Im} T^2_i \oplus \operatorname{Im} T^1_i = \mathscr{U}_u = \mathbb{R}^m, \\
\operatorname{Im} T^1_o = \mathcal{Y}^*, & \operatorname{Im} T^2_o \oplus \operatorname{Im} T^1_o = \mathscr{Y} = \mathbb{R}^p.
\end{array}$$

Then $T_s = [T^1_s \ T^2_s \ T^3_s \ T^4_s]^{-1}$, $T_i = [T^1_i \ T^2_i]^{-1}$, $T_o = [T^1_o \ T^2_o]^{-1}$, *are invertible and* $\exists F_{MT}, K_{MT}$ *s.t.* $M_{tran} = (T_s, T_i, T_o, F_{MT}, K_{MT})$ *brings* Λ^u *into* $\tilde{\Lambda}^{\tilde{u}} = M_{tran}(\Lambda^u)$, *represented in the Morse triangular form* ***MTF***,

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} \end{bmatrix} = \left[\begin{array}{cccc|cc}
\tilde{A}_1 & \tilde{A}^2_1 & \tilde{A}^3_1 & \tilde{A}^4_1 & \tilde{B}_1 & \tilde{B}^2_1 \\
0 & \tilde{A}_2 & 0 & \tilde{A}^4_2 & 0 & 0 \\
0 & 0 & \tilde{A}_3 & \tilde{A}^4_3 & 0 & \tilde{B}_3 \\
0 & 0 & 0 & \tilde{A}_4 & 0 & 0 \\
\hline
0 & 0 & \tilde{C}_3 & \tilde{C}^4_3 & 0 & \tilde{D}_3 \\
0 & 0 & 0 & \tilde{C}_4 & 0 & 0
\end{array}\right]. \tag{8}$$

In the above ***MTF***, *the pair* $(\tilde{A}_1, \tilde{B}_1)$ *is controllable, the pair* $(\tilde{C}_4, \tilde{A}_4)$ *is observable and the 4-tuple* $(\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$ *is prime.*

Proposition 3 (Morse normal form MNF)

There exists F_{MN}, K_{MN} *and* T_{MN}, *which can be chosen by Algorithm 1 below, s.t.* $M_{tran} = (T_{MN}, I_u, I_y, F_{MN}, K_{MN})$ *brings* $\tilde{\Lambda}^{\tilde{u}}$ *into* $\bar{\Lambda}^{\bar{u}} = M_{tran}(\tilde{\Lambda}^{\tilde{u}})$, *represented in the Morse normal form* ***MNF***,

$$\begin{bmatrix} \bar{A} & \bar{B}^{\bar{u}} \\ \bar{C} & \bar{D}^{\bar{u}} \end{bmatrix} = \left[\begin{array}{cccc|cc} \bar{A}_1 & 0 & 0 & 0 & \bar{B}_1 & 0 \\ 0 & \bar{A}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{A}_3 & 0 & 0 & \bar{B}_3 \\ 0 & 0 & 0 & \bar{A}_4 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_3 & 0 & 0 & \bar{D}_3 \\ 0 & 0 & 0 & \bar{C}_4 & 0 & 0 \end{array}\right]. \tag{9}$$

In the above ***MNF***, *the pair* $(\bar{A}_1, \bar{B}_1)$ *is controllable, the pair* $(\bar{C}_4, \bar{A}_4)$ *is observable, and the 4-tuple* $(\bar{A}_3, \bar{B}_3, \bar{C}_3, \bar{D}_3)$ *is prime.*

Algorithm 1

Step 1: Choose $F_{MN} = \begin{bmatrix} F^1_{MN} & 0 & 0 & 0 \\ 0 & 0 & F^2_{MN} & F^3_{MN} \end{bmatrix}$, $K_{MN} = \begin{bmatrix} K^1_{MN} & 0 \\ 0 & 0 \\ K^2_{MN} & 0 \\ 0 & K^3_{MN} \end{bmatrix}$ *s.t. the eigenvalues of* $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ *and* $\bar{A}_4$ *of the equation below are pairwise disjoint:*

$$\begin{bmatrix} I_n & K_{MN} \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ F_{MN} & I_m \end{bmatrix} = \left[\begin{array}{cccc|cc} \bar{A}_1 & \bar{A}_1^2 & \bar{A}_1^3 & \bar{A}_1^4 & \bar{B}_1 & \bar{B}_1^2 \\ 0 & \bar{A}_2 & 0 & \bar{A}_2^4 & 0 & 0 \\ 0 & 0 & \bar{A}_3 & \bar{A}_3^4 & 0 & \bar{B}_3 \\ 0 & 0 & 0 & \bar{A}_4 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_3 & \bar{C}_3^4 & 0 & \bar{D}_3 \\ 0 & 0 & 0 & \bar{C}_4 & 0 & 0 \end{array}\right].$$

Step 2: Find matrices T^1_{MN}, T^2_{MN}, T^3_{MN}, T^4_{MN}, T^5_{MN} *via the following (constrained) Sylvester equations:*

$$\begin{gathered} \bar{A}_1 T^1_{MN} - T^1_{MN} \bar{A}_2 = -\bar{A}_1^2, \quad \bar{A}_2 T^4_{MN} - T^4_{MN} \bar{A}_4 = -\bar{A}_2^4, \\ \bar{A}_1 T^3_{MN} - T^3_{MN} \bar{A}_4 = -\bar{A}_1^4 - \bar{A}_1^2 T^4_{MN} - \bar{A}_1^3 T^5_{MN}; \end{gathered} \tag{10}$$

$$\begin{aligned} \bar{A}_1 T^2_{MN} - T^2_{MN} \bar{A}_3 &= -\bar{A}_1^3, \quad & T^2_{MN} \bar{B}_3 &= -\bar{B}_1^2, \\ \bar{A}_3 T^5_{MN} - T^5_{MN} \bar{A}_4 &= -\bar{A}_3^4, \quad & \bar{C}_3 T^5_{MN} &= -\bar{C}_4. \end{aligned} \tag{11}$$

Step 3: Set

$$T_{MN} = \begin{bmatrix} I & T^1_{MN} & T^2_{MN} & T^3_{MN} \\ 0 & I & 0 & T^4_{MN} \\ 0 & 0 & I & T^5_{MN} \\ 0 & 0 & 0 & I \end{bmatrix}^{-1}.$$

For ODECS $\Lambda^{uv} = \Lambda^{w}$ with two kinds of inputs (u, v), we propose a similar procedure.

Theorem 2 (extended Morse triangular form EMTF)

$\Lambda^{uv} = (A, B^u, B^v, C, D^u) \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^{\tilde{u}})$, *where*

$$\textbf{EMTF}: \begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{bmatrix} = \left[\begin{array}{cccc|cc|cc} \tilde{A}_1 & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} & \tilde{B}^{\tilde{u}}_1 & \tilde{B}^{\tilde{u}}_{12} & \tilde{B}^{\tilde{v}}_1 & \tilde{B}^{\tilde{v}}_{12} \\ 0 & \tilde{A}_2 & 0 & \tilde{A}_{24} & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{A}_3 & \tilde{A}_{34} & 0 & \tilde{B}^{\tilde{u}}_3 & 0 & \tilde{B}^{\tilde{v}}_3 \\ 0 & 0 & 0 & \tilde{A}_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{C}_3 & \tilde{C}_{34} & 0 & \tilde{D}^{\tilde{u}}_3 & 0 & 0 \\ 0 & 0 & 0 & \tilde{C}_4 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{12}$$

For ODECS $\Lambda^{uv} = \Lambda^{w}$ with two kinds of inputs (u, v), we propose a similar procedure.

Theorem 3 (extended Morse normal form EMNF)

$\tilde{\Lambda}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^{\tilde{u}}) \overset{EM}{\sim} \bar{\Lambda}^{\bar{u}\bar{v}} = (\bar{A}, \bar{B}^{\bar{u}}, \bar{B}^{\bar{v}}, \bar{C}, \bar{D}^{\bar{u}})$, *where*

$$\textbf{EMNF}: \begin{bmatrix} \bar{A} & \bar{B}^{\bar{u}} & \bar{B}^{\bar{v}} \\ \bar{C} & \bar{D}^{\bar{u}} & 0 \end{bmatrix} = \left[\begin{array}{cccc|cc|cc} \bar{A}_1 & 0 & 0 & 0 & \bar{B}_1^{\bar{u}} & 0 & \bar{B}_1^{\bar{v}} & 0 \\ 0 & \bar{A}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{A}_3 & 0 & 0 & \bar{B}_3^{\bar{u}} & 0 & \bar{B}_3^{\bar{v}} \\ 0 & 0 & 0 & \bar{A}_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_3 & 0 & 0 & \bar{D}_3^{\bar{u}} & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_4 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{13}$$

Plan

Extended Morse canonical form and its indices

For ODECSs with two kinds of inputs (u, v), we propose a similar procedure.

Theorem 4

For an ODECS Λ^{uv}, we can explicitly construct an extended Morse transformation bringing Λ^{uv} into its extended Morse canonical form ***EMCF****, passing through intermediate extended Morse triangular form* ***EMTF*** *and extended Morse normal form* ***EMNF****.*

$$\textbf{EMCF}: \begin{cases} \dot{z}^{cu} = A^{cu} z^{cu} + B^{cu} u & \\ \dot{z}^{cv} = A^{cv} z^{cv} + B^{cv} v & \\ \dot{z}^{nn} = A^{nn} z^{nn} & \\ \dot{z}^{pu} = A^{pu} z^{pu} + B^{pu} u, & y^{pu} = C^{pu} z^{pu} + D^{pu} u \\ \dot{z}^{pv} = A^{pv} z^{pv} + B^{pv} v, & y^{pv} = C^{pv} z^{pv} \\ \dot{z}^{o} = A^{o} z^{o} & y^{o} = C^{o} z^{o}, \end{cases}$$

- both the pairs (A^{cu}, B^{cu}) and (A^{cv}, B^{cv}) are controllable and in the Brunovský canonical forms with indices $\epsilon_1, \ldots, \epsilon_a$ and $\bar{\epsilon}_1, \ldots, \bar{\epsilon}_b$, resp.;
- A^{nn} is up to similarity;
- the 4-tuple $(A^{pu}, B^{pu}, C^{pu}, D^{pu})$ and the triple (A^{pv}, B^{pv}, C^{pv}) are prime with indices $\sigma_1, \ldots, \sigma_c$ and $\bar{\sigma}_1, \ldots \bar{\sigma}_d$, resp.;
- the pair (C^{o}, A^{o}) is observable and in the dual Brunovský canonical form with indices $\eta_1, \ldots, \eta_e$.

Proposition 4 (the EMCF indices)

*For an ODECS $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$, the **EMCF** indices in Theorem 4 can be calculated as follows and thus are invariant under EM-transformations.*

(i) *Set*

$$\begin{aligned} \hat{\epsilon}_i &= \dim(\mathcal{V}^* \cap \mathcal{W}_i) - \dim(\mathcal{V}^* \cap \hat{\mathcal{W}}_i),\ i \geq 1, \\ \hat{\hat{\epsilon}}_i &= \dim(\mathcal{V}^* \cap \hat{\mathcal{W}}_i) - \dim(\mathcal{V}^* \cap \mathcal{W}_{i-1}),\ i \geq 1, \\ \hat{\hat{\sigma}}_i &= \dim\, \hat{\mathcal{W}}_i - \dim\, \mathcal{W}_{i-1} - \hat{\hat{\epsilon}}_i,\ i \geq 1, \\ \hat{\eta}_i &= \dim(\mathcal{W}^* + \mathcal{V}_{i-1}) - \dim(\mathcal{W}^* + \mathcal{V}_i),\ i \geq 1. \end{aligned}$$

Then $a = \hat{\epsilon}_1$, $b = \hat{\hat{\epsilon}}_1$, $d = \hat{\hat{\sigma}}_1$, $e = \hat{\eta}_1$. *The indices* $(\epsilon_1, \ldots, \epsilon_a) = \mathfrak{d}(\hat{\epsilon})$, $(\bar{\epsilon}_1, \ldots, \bar{\epsilon}_b) = \mathfrak{d}(\hat{\hat{\epsilon}})$, $(\bar{\sigma}_1, \ldots, \bar{\sigma}_d) = \mathfrak{d}(\hat{\hat{\sigma}})$ *and* $(\eta_1, \ldots, \eta_e) = \mathfrak{d}(\hat{\eta})$.

(ii) *Set*

$$\hat{\sigma}_1 = m - \hat{\epsilon}_1, \quad \hat{\sigma}_i = \dim\, \mathcal{W}_{i-1} - \dim\, \hat{\mathcal{W}}_{i-1} - \hat{\epsilon}_{i-1},\ i \geq 2.$$

Then $c = \hat{\sigma}_2$ *and* $\delta = \hat{\sigma}_1 - c$. *The indices* $(\sigma_1, \ldots, \sigma_c) = \mathfrak{d}(\hat{\sigma}) - (1, \ldots, 1)$.

Example 1

Consider a prime subsystem $(A^{pv}_{\bar{\sigma}}, B^{pv}_{\bar{\sigma}}, C^{pv}_{\bar{\sigma}})$ of (A^{pv}, B^{pv}, C^{pv}), for which we get:

$$(A^{pv}_{\bar{\sigma}}, B^{pv}_{\bar{\sigma}}, C^{pv}_{\bar{\sigma}}) : \begin{cases} y & = x^1, \\ \dot{x}^1 & = x^2 \\ & \cdots \\ \dot{x}^{\bar{\sigma}-1} & = x^{\bar{\sigma}} \\ \dot{x}^{\bar{\sigma}} & = v, \end{cases} \rightarrow (N_{\bar{\sigma}}, I_{\bar{\sigma}}, 0) : \begin{cases} 0 & = x^1 \\ \dot{x}^1 & = x^2 \\ & \cdots \\ \dot{x}^{\bar{\sigma}-1} & = x^{\bar{\sigma}}. \end{cases}$$

- The **FBCF** is the implicitation and reduction of the **EMCF** of Δ^u. A crucial observation is that **EMCF** $\in$ **Expl(FBCF)**. Thus $\Delta^u \overset{ex-fb}{\sim}$ **FBCF** (since $\Lambda^{uv} \in$ **Expl**(Δ^u), $\Lambda^{uv} \overset{EM}{\sim}$ **EMCF**).
- With the help of the reduction and implicitation procedure, we can regard the **FBCF** (Loiseau et al 1991) as a corollary of Theorem 4 (**EMCF**).

Proposition 5 (Relations of the indices)

*Assume $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$, then the **EMCF** indices of Λ^{uv} and the **FBCF** indices of Δ^u are related by*

- $a = a'$ *and* $\epsilon_k = \epsilon'_k$ *for* $k = 1, \ldots, a$, $b = b'$ *and* $\bar{\epsilon}_k = \bar{\epsilon}'_k$ *for* $k = 1, \ldots, b$;
- $n_2 = n_\rho$ *and* $A^{nn} \approx A_\rho$;
- $c + \delta = c'$ *and* $\sigma'_1 = \sigma'_2 =, \cdots, = \sigma'_\delta = 1$, $\sigma'_{\delta+1} = \sigma_1 + 1$, $\sigma'_{\delta+2} = \sigma_2 + 1$, $\ldots$, $\sigma'_{\delta+c} = \sigma_c + 1$; *Moreover,* $d = d'$ *and* $\bar{\sigma}_k = \bar{\sigma}'_k$ *for* $k = 1, \ldots, d$;
- $e = e'$ *and* $\eta_k + 1 = \eta'_k$ *for* $k = 1, \ldots, e$.

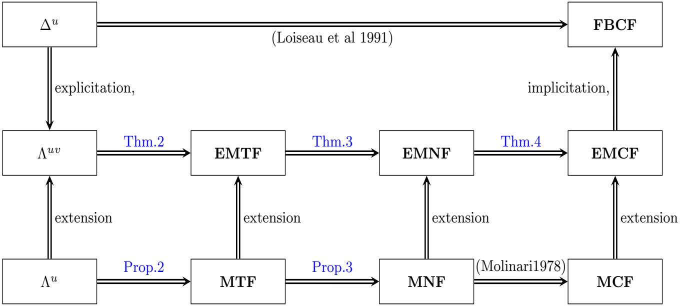
- With the help of Proposition 5, we can regard the results of calculating **FBCF** indices (Loiseau et al 1991) (Berger 2015) as a corollary of Proposition 4 (**EMCF** indices).
- There exists a perfect correspondence between the **EMCF** and of the **FBCF**:

$$\begin{array}{ll} (A^{cu}, B^{cu}) \leftrightarrow (I_{|\epsilon'|}, N^T_{\epsilon'}, \mathcal{E}_{\epsilon'}), & (A^{cu}, B^{cu}) \leftrightarrow (L_{\bar{\epsilon}'}, K_{\bar{\epsilon}'}, 0), \\ A_{n_2} \leftrightarrow (I_{n_\rho}, A_\rho), & (A^{pu}, B^{pu}, C^{pu}, D^{pu}) \leftrightarrow (K^T_{\sigma'}, L^T_{\sigma'}, \mathcal{E}_{\sigma'}), \\ (A^{pv}, B^{pv}, C^{pv}) \leftrightarrow (N_{\bar{\sigma}'}, I_{|\bar{\sigma}'|}, 0), & (C^o, A^o) \leftrightarrow (L^T_{\eta'}, K^T_{\eta'}, 0). \end{array}$$

Our algorithm of finding the **FBCF** of a linear DAECS

Algorithm 2

- *Step 1: For* Δ^u*, construct* Λ^{uv} *st.* $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$*.*
- *Step 2: Find an* EM_{tran} *s.t.* $\tilde{\Lambda}^{\tilde{u}\tilde{v}} = EM_{tran}(\Lambda^{uv})$ *is in the* ***EMTF****.*
- *Step 3: Find an* EM_{tran} *s.t.* $\bar{\Lambda}^{\bar{u}\bar{v}} = EM_{tran}(\tilde{\Lambda}^{\tilde{u}\tilde{v}})$ *is in the* ***EMNF****.*
- *Step 4: Bring* $\bar{\Lambda}^{\bar{u}\bar{v}}$ *into the* ***EMCF*** *by normalizing the subsystems in the* ***EMNF****.*
- *Step 5: Find the implicitation of* ***EMCF****, denoted by* $\bar{\Delta}^{\bar{u}}$*. Then* $\bar{\Delta}^{\bar{u}}$ *is in the* ***FBCF*** *and* $\Delta^u \overset{ex-fb}{\sim} \bar{\Delta}^{\bar{u}}$*.*



Conclusions

- Propose the explicitation with driving variables procedure.
- Show the role of the driving variables.
- Connect DAECSs and ODECSs via their equivalences and geometric subspaces.
- Simple way to transform an ODECS to its **MCF** or **EMCF**.
- Geometrical way to get the **FBCF** of a DAECS

Thank you for listening !!!