Internal and External Linearization of Semi-Explicit Differential Algebraic Equations

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Outline

Plan

We study differential-algebraic equations DAEs of semi-explicit SE form

$$\Xi^{se}: \begin{cases} \mathcal{R}(x)\dot{x} = a(x) \\ 0 \quad = c(x), \end{cases} \tag{1}$$

- $\mathcal{R}(x)$, $a(x)$, and $c(x)$ are smooth maps with values in $\mathbb{R}^{r\times n}$, $\mathbb{R}^r$, and $\mathbb{R}^p$, respectively;
- assume $\mathcal{R}(x)$ is locally of full row rank in a neighborhood X_0 of x^0;
- the "generalized" state $x \in X$ and X is an open subset of $\mathbb{R}^n$.

Why called semi-explicit?

A linear SE DAE is of the following form

$$\Delta^{se}: \begin{cases} R\dot{x} = Ax \\ 0 \;= Cx, \end{cases} \tag{2}$$

- where $R \in \mathbb{R}^{r\times n}$, $A \in \mathbb{R}^{r\times n}$, $C \in \mathbb{R}^{p\times n}$.

When is $\Xi^{se} \sim \Delta^{se}$? How to define "$\sim$"?

External equivalence of different levels

Definition 1 (external equivalence)

Consider two SE DAEs $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ and $\tilde{\Xi}^{se}_{n,r,p} = (\tilde{\mathcal{R}}, \tilde{a}, \tilde{c})$. If $\exists$ a diffeomorphism $\psi : X \to \tilde{X}$ and a smooth invertible $r \times r$-matrix $Q^a(x)$ s.t.

$$\tilde{\mathcal{R}}(\psi(x)) = Q^a(x)\mathcal{R}(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1}, \tag{3}$$

$$\tilde{a}(\psi(x)) = Q^a(x)a(x), \tag{4}$$

and if, additionally,

- (i) $\exists$ invertible $p \times p$-matrix $Q^c(x)$ s.t. $\tilde{c}(\psi(x)) = Q^c(x)c(x)$, we call Ξ^{se} and $\tilde{\Xi}^{se}$ externally equivalent, or shortly ex-equivalent, of level-1;
- (ii) $\exists$ a smooth invertible $p \times p$-matrix $Q^c(x)$ s.t. $\tilde{c}(\psi(x)) = Q^c(x)c(x)$ and $Q^c(x) = Q(c(x))$ for some invertible $Q(x)$, ...of level-2;
- (iii) $\exists$ a constant invertible $p \times p$-matrix T s.t. $\tilde{c}(\psi(x)) = Tc(x)$,....of level-3.

The level-i ($i = 1, 2, 3$) ex-equivalence of two SE DAEs will be denoted by $\Xi^{se} \overset{ex-i}{\sim} \tilde{\Xi}^{se}$.

A canonical form for linear semi-explicit DAE

Proposition 1

Any linear SE DAE $\Delta^{se}_{r,n,p} = (R, A, C)$ *is ex-equivalent to the following semi-explicit canonical form:*

$$SCF : \begin{cases} \dot z^1 = A^1 z^1 + B^1 w^1 & +K^1 y \\ \dot z^2 = A^2 z^2 & +K^2 y \\ \dot z^3 = A^3 z^3 + B^3 w^3 & +K^3 y \\ \dot z^4 = A^4 z^4 & +K^4 y \\ 0 = w^0 & \\ 0 = C^3 z^3 & \\ 0 = C^4 z^4, & \end{cases}$$

where $y = (y^0, y^3, y^4)$, $y^0 = w^0$, $y^3 = C^3 z^3$ *and* $y^4 = C^4 z^4$, *and the system matrices satisfy* $A^k = \mathrm{diag}\,[A^k_1, \ldots, A^k_e]$ *for* $k = 1, 3, 4$, $B^k = \mathrm{diag}\,[B^k_1, \ldots, B^k_e]$ *for* $k = 1, 3$ *and* B^k *is empty for* $k = 2, 4$, $C^k = \mathrm{diag}\,[C^k_1, \ldots, C^k_e]$ *for* $k = 3, 4$ *and* C^k *is empty for* $k = 1, 2$, *with*

$$A^k_i = \begin{bmatrix} 0 & I_{\mu_i - 1} \\ 0 & 0 \end{bmatrix}, \ B^k_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\mu_i \times 1}, \ C^k_i = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1 \times \mu_i},$$

for $i = 1, \ldots, e$, *where* e *depends on* k *and is equal to* a, b, c, d *for* $k = 1, 2, 3, 4$, *respectively;* A^2 *is in the Jordan canonical form for real matrices.*

- Compare the above canonical form SCF with the Kronecker canonical form (Kronecker 1890) for linear DAEs and the Morse canonical form (Morse 1973) for linear control systems.

Plan

Definition 2 (explicitation)

For $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$, set $m = n - r$. Then the *explicitation* of Ξ^{se}, denoted by **Expl**(Ξ^{se}), is a class of control systems of the following form:

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)v \\ y = h(x), \end{cases} \tag{5}$$

where $v \in \mathbb{R}^m$ is called the driving variable, $h(x)$ is a smooth $\mathbb{R}^p$-valued function on X_0, and where $f, g_1, \ldots, g_m$ are smooth vector fields on X_0 satisfying

$$f(x) = R^{\dagger}(x)a(x), \quad \operatorname{Im} g(x) = \ker \mathcal{R}(x), \quad h(x) = c(x).$$

- $R^{\dagger}(x)$ is the right inverse of $R(x)$, i.e., $R(x)R^{\dagger}(x) = I_r$.

System equivalence of control systems

Definition 3 (system equivalence)

Consider two control systems $\Sigma_{n,m,p} = (f, g, h)$ and $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$ defined on X and $\tilde{X}$, respectively. If there exists a diffeomorphism $\psi : X \to \tilde{X}$, an $\mathbb{R}^m$-valued function $\alpha(x)$, and an invertible $m \times m$-matrix-valued function $\beta(x)$ satisfying

$$\begin{aligned} \tilde{f}(\psi(x)) &= \frac{\partial \psi(x)}{\partial x} (f + g\alpha)(x), \\ \tilde{g}(\psi(x)) &= \frac{\partial \psi(x)}{\partial x} (g\beta)(x), \end{aligned} \tag{6}$$

and, additionally,

(i) either $\exists$ a constant invertible matrix T such that $\tilde{h}(\psi(x)) = Th(x)$, we call Σ and $\tilde{\Sigma}$ system equivalent, shortly sys-equivalent, of level-3,

(ii) or $\exists$ a diffeomorphism $\varphi : \mathbb{R}^p \to \mathbb{R}^p$ such that $\tilde{h}(\psi(x)) = \varphi(h(x))$, we call the two control systems sys-equivalent of level-2.

The sys-equivalence of level-i ($i = 2, 3$) of two control systems will be denoted by $\Sigma \overset{sys-i}{\sim} \tilde{\Sigma}$.

Coordinates transformations in the output space ?

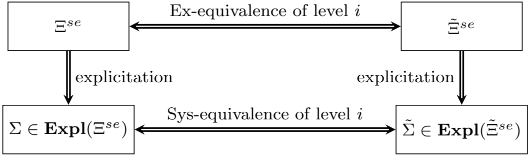
Proposition 2

(i) Consider two control systems $\Sigma_{n,m,p} = (f, g, h)$ *and* $\Sigma_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$, *that belong to the explicitation class of* $\Xi^{se}_{n,r,p}$, *i.e.* $\Sigma, \tilde{\Sigma} \in \mathbf{Expl}(\Xi^{se})$. *Then there exist* $\alpha(x)$, $\beta(x)$ *with values in* $\mathbb{R}^m$ *and invertible* $m \times m$*-matrices such that*

$$\tilde{f}(x) = f(x) + g(x)\alpha(x), \; \tilde{g}(x) = g(x)\beta(x). \tag{7}$$

(ii) Two SE DAEs $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ *and* $\tilde{\Xi}^{se}_{n,r,p} = (\tilde{\mathcal{R}}, \tilde{a}, \tilde{c})$ *are ex-equivalent of level-2 (respectively, level-3) if and only if two control systems* $(f, g, h) = \Sigma \in \mathbf{Expl}(\Xi^{se})$ *and* $(\tilde{f}, \tilde{g}, \tilde{h}) = \tilde{\Sigma} \in \mathbf{Expl}(\tilde{\Xi}^{se})$ *are sys-equivalent of level-2 (respectively, level-3).*

- The explicitation of SE DAEs is a control system defined up to feedback transformations.
- Sys-equivalence (of level-2, and, respectively, level-3) is for explicitation systems the same as ex-equivalence (of level-2, and respectively, level-3) for DAE's.



- A solution of Ξ^{se} is a curve $x(t) \in \mathscr{C}^1(I;X)$ with an open interval I such that for all $t \in I$, $x(t)$ solves (1).

- A submanifold M^* is called a *maximal invariant submanifold* (for details, see Chen and Respondek (2018)) of Ξ^{se} if M^* is the largest submanifold of X s.t. $\forall\ x^0 \in M^*$, $\exists\ x(t)$ such that $x(0) = x^0$ and $x(t) \in M^*$, $t \in I$. (M^* is where the solutions exist)

Definition 4 (internal equivalence)

Consider two SE DAEs $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ and $\tilde{\Xi}^{se}_{n,r,p} = (\tilde{\mathcal{R}}, \tilde{a}, \tilde{c})$. Let M^* and $\tilde{M}^*$ be their maximal invariant submanifolds. We call Ξ^{se} and $\tilde{\Xi}^{se}$ internally equivalent, shortly in-equivalent, if $\Xi^{se}|_{M^*}$ and $\tilde{\Xi}^{se}|_{\tilde{M}^*}$ are ex-equivalent.

Internal linearizability of SE DAEs

Theorem 1

For $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$*, the following are locally equivalent:*

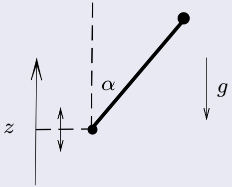
(i) Ξ^{se} *is in-equivalent to a linear SE DAE* Δ^{se} *with internal reachability;*

(ii) *A (and then any) control system* $(f^*, g^*) = \Sigma^* \in \mathbf{Expl}(\Xi^{se}|_{M^*})$ *is feedback linearizable;*

(iii) *The linearizability distributions* $G_i(\Sigma^*)$ *(given below) are involutive and of constant rank and* $G^*(\Sigma^*) = TM^*$*.*

- The equivalence of (ii) and (iii) is proved in Jakubczyk and Respondek (1980), Hunt and Su (1983), as classical results of feedback linearization of nonlinear control systems.

Example 1 (Fliess 1995)



The dynamics of the system:

$$\begin{cases} \dot{\alpha} = p + \frac{u_1}{l}\sin\alpha \\ \dot{p} = \left(\frac{g}{l} - \frac{(u_1)^2}{l^2}\cos\alpha - \frac{(u_2)^2}{2l^2}\cos\alpha\right)\sin\alpha - \frac{u_1}{l}p\cos\alpha \\ \dot{z} = u_1, \end{cases} \tag{8}$$

Case 1: under holonomic constraint $z + l\cos\alpha = c_{10}$, where c_{10} denotes a fixed constant. Our DAE is $\Xi_1^{se} = (\mathcal{R}_1, a_1, c_1)$ where

$$\mathcal{R}_1(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad c_1(x) = x_3 + l\cos x_1 - c_{10},$$

$$a_1(x) = \begin{bmatrix} x_2 + \frac{x_4}{l}\sin x_1 \\ \left(\frac{g}{l} - \frac{(x_4)^2}{l^2}\cos x_1 - \frac{(x_5)^2}{2l^2}\cos x_1\right)\sin x_1 - \frac{x_4}{l}x_2\cos x_1 \\ x_4 \end{bmatrix}.$$

Example 1

Case 1: Our DAE is $\Xi_1^{se} = (\mathcal{R}_1, a_1, c_1)$. A control system $\Sigma_1 = (f_1, g_1, h_1) \in \mathbf{Expl}(\Xi_1^{se})$ is

$$\Sigma_1 : \begin{cases} \dot{x} = \begin{bmatrix} a_1(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ y = x_3 + l\cos x_1 - c_{10}. \end{cases}$$

The maximal output zeroing submanifold M_1^* of Σ_1:

$$M_1^* = \left\{ x \mid x_3 + l\cos x_1 - c_{10} = x_4 \cos^2 x_1 - l x_2 \sin x_1 = 0 \right\}.$$

Thus

$$\Sigma_1|_{M_1^*} : \begin{cases} \dot{x}_1 = \frac{x_2}{\cos^2 x_1} \\ \dot{x}_2 = \left(\frac{g}{l} - \frac{(x_2)^2}{\cos^3 x_1} - \frac{(x_5)^2}{2l^2} \cos x_1 \right) \sin x_1 \\ \dot{x}_5 = v_2, \end{cases}$$

which is locally static feedback equivalent to

$$\dot{\tilde{x}}_1 = \tilde{x}_2, \quad \dot{\tilde{x}}_2 = \tilde{x}_5, \quad \dot{\tilde{x}}_5 = \tilde{v}_2,$$

Ξ_1^{se} is internally equivalent to the following linear DAE:

$$\begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 \\ \dot{\tilde{x}}_2 = \tilde{x}_5. \end{cases}$$

Example 1

Case 2: We change the holonomic constrains to

$$\begin{cases} 0 = z \\ 0 = \ln|\tan\frac{\alpha}{2}| + (k-1)z, \end{cases}.$$

- Our DAE is $\Xi_2^{se} = (\mathcal{R}_2, a_2, c_2)$, where $\mathcal{R}_2(x) = \mathcal{R}_1(x)$, $a_2(x) = a_1(x)$ and

$$c_2(x) = \begin{bmatrix} x_3 \\ \ln|\tan\frac{x_1}{2}| + (k-1)x_3 \end{bmatrix}.$$

- A control system $\Sigma_2 = (f_2, g_2, h_2) \in \mathbf{Expl}(\Xi_2^{se})$ is given by

$$\Sigma_2 : \begin{cases} \dot{x} = \begin{bmatrix} a_2(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_4 \\ \ln|\tan\frac{x_1}{2}|+(k-1)x_3 \end{bmatrix}. \end{cases} \tag{9}$$

- The maximal output zeroing submanifold M_2^* of Σ_2 is

$$M_2^* = \left\{ x \;\middle|\; \begin{array}{l} \ln|\tan\frac{x_1}{2}| + (k-1)x_3 = x_2 = x_4 = \\ 2lg - (x_5)^2 \cos x_1 = 0 \end{array} \right\}.$$

The zero dynamics of Σ_2 is $\dot{x}_1 = 0.$

For a nonlinear control system $\Sigma_{n,m,p} = (f, g, h)$, define sequences of distributions G_i, S_i and codistributions P_i by

$$\begin{aligned}
G_1 &:= G := \operatorname{span}\{g_1, \ldots, g_m\} \\
G_{i+1} &:= G_i + [f, G_i] \\
G^* &:= \sum_{i \geq 1} G_i.
\end{aligned}$$

$$\begin{aligned}
S_1 &:= G, \\
S_{i+1} &:= S_i + [f, S_i \cap \ker dh] + \textstyle\sum_{j=1}^{m} [g_j, S_i \cap \ker dh] \\
S^* &:= \sum_{i \geq 1} S_i.
\end{aligned}$$

$$\begin{aligned}
P_1 &:= \operatorname{span}\left\{dh_1, \ldots, dh_p\right\}, \\
P_{i+1} &:= P_i + L_f(P_i \cap G^{\perp}) + \textstyle\sum_{j=1}^{m} L_{g_j}(P_i \cap G^{\perp}) \\
P^* &:= \sum_{i \geq 1} P_i.
\end{aligned}$$

and $V_i := P_i^{\perp}$, $V^* := (P^*)^{\perp}$.

Theorem 2

Consider $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ around a point x^0. Then in a neighborhood X_0 of x^0, Ξ^{se} is level-3 ex-equivalent to a linear SE DAE Δ^{se} of the form

$$\left\{\begin{array}{llll} \dot z^2 = A^2 z^2 & & +K^2 y, & 0 = w^0, \\ \dot z^3 = A^3 z^3 + B^3 w^3 & & +K^3 y, & 0 = C^3 z^3, \\ \dot z^4 = A^4 z^4 & & +K^4 y, & 0 = C^4 z^4, \end{array}\right.$$

with constraint-free controllability if and only if a (and then any) control system $\Sigma = (f, g, h) \in \mathbf{Expl}(\Xi^{se})$ satisfies the following conditions in X_0:

(i) *The Toeplitz matrices* $M_k = \begin{bmatrix} T_0(x) & T_1(x) & \cdots & T_k(x) \\ 0 & T_0(x) & \cdots & T_{k-1}(x) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & T_0(x) \end{bmatrix}$ *satisfy* $r(M_k(x)) = r_{\mathbb{R}}(M_k(x))$ *for all* $k \le 2n-1$, *where* $T_k(x) = L_g L_f^k h(x)$;

(ii) $G^* = TX_0$;

(iii) $[ad^k_{\tilde f}\tilde g_i, ad^l_{\tilde f}\tilde g_j] = 0$ *for* $1 \le i, j \le m$, $0 \le l, k \le n$, *where* $\tilde f$ *and* $\tilde g_i$ *are vector fields modified by a feedback transformation resulting from the structure algorithm;*

(iv) $V^* \cap S^* = 0$.

Moreover, Δ^{se} is regular if and only if Ξ^{se} satisfies (i)-(iv) and, additionally, condition

(v) $V^* \oplus S^* = TX_0$.

Theorem 3

Consider $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ *around a point* x_0*. Then in a neighborhood* X_0 *of* x_0*,* Ξ^{se} *is level-3 ex-equivalent to a linear SE DAE* Δ^{se} *of the form*

$$\begin{cases} \dot{z}^1 = A^1 z^1 + B^1 w^1, & & 0 = w^0 \\ \dot{z}^3 = A^3 z^3 + B^3 w^3 & +K^3 y, & 0 = C^3 z^3, \end{cases} \tag{10}$$

where all matrices are as in the SCF, if and only if a (and then any) control system $\Sigma \in \mathbf{Expl}(\Xi^{se})$ *satisfies the following conditions in* X_0*:*

(i) *The Toeplitz matrices* $M_k = \begin{bmatrix} T_0(x) & T_1(x) & \cdots & T_k(x) \\ 0 & T_0(x) & \cdots & T_{k-1}(x) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & T_0(x) \end{bmatrix}$ *satisfy* $r(M_k(x)) = r_{\mathbb{R}}(M_k(x))$ *for all* $k \leq 2n-1$*, where* $T_k(x) = L_g L_f^k h(x)$*;*

(ii) S_i *and* G_i *are involutive and of constant rank;*

(iii) $S^* = TX_0$*;*

(iv) $S_i \cap V^* = G_i \cap V^*$*.*

Continuation of the Kapitsa pendulum example

Example 2 (continuation of Example 1)

Case 1: Σ_1 satisfies conditions (i)-(iv) of the above theorem and is level-3 sys-equivalent to $\tilde{\Sigma}_1$ below. Thus Ξ_1^{se} is level-3 ex-equivalent to the following Δ_1^{se}:

$$\tilde{\Sigma}_1 : \begin{cases} \dot{\tilde{x}}_3 &= \tilde{x}_4, \quad y = \tilde{x}_3 \\ \dot{\tilde{x}}_4 &= \tilde{v}_1 \\ \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= \tilde{x}_5 \\ \dot{\tilde{x}}_5 &= \tilde{v}_2 \end{cases} \Rightarrow \Delta_1^{se} : \begin{cases} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= \tilde{x}_5 \\ \dot{\tilde{x}}_3 &= \tilde{x}_4 \\ 0 &= \tilde{x}_3. \end{cases}$$

Case 2: Σ_2 satisfies conditions (i)-(v) of the above theorem and is level-3 sys-equivalent to the following $\tilde{\Sigma}_2$. Thus Ξ_2^{se} is level-3 ex-equivalent to the following Δ_2^{se}.

$$\tilde{\Sigma}_2 : \begin{cases} \dot{\tilde{x}}_3 &= \tilde{x}_4 \\ \dot{\tilde{x}}_4 &= \tilde{v}_1, \qquad y_1 = \tilde{x}_4 \\ \dot{\tilde{x}}_1 &= \tilde{x}_2 + ky_1, \quad y_2 = \tilde{x}_1 \\ \dot{\tilde{x}}_2 &= \tilde{x}_5 \\ \dot{\tilde{x}}_5 &= \tilde{v}_2 \end{cases} \Rightarrow \Delta_2^{se} : \begin{cases} \dot{\tilde{x}}_1 &= \tilde{x}_2 + k\tilde{x}_4 \\ \dot{\tilde{x}}_2 &= \tilde{x}_5 \\ \dot{\tilde{x}}_3 &= \tilde{x}_4 \\ 0 &= \tilde{x}_4 \\ 0 &= \tilde{x}_1. \end{cases}$$

- Although internally Ξ_2^{se} is equivalent to $\dot{x}_1 = 0$, it is level-3 ex-equivalent to the above linear SE DAE !!!

Example 3

Consider a SE DAE $\Xi_3^{se} = (\mathcal{R}_3, a_3, c_3)$, described by

$$\mathcal{R}_3(x) = \begin{bmatrix} 1 & 0 & -x_1 & 0 & 0 & 0 \\ 0 & 0 & e^{3x_3} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad a_3(x) = \begin{bmatrix} 2(x_1 e^{x_3})^{\frac{1}{2}} x_2 \\ -(x_5 + k e^{x_3}) \\ x_6 \end{bmatrix}, \quad c_3(x) = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}.$$

A control system $(f_3, g_3, h_3) = \Sigma_3 \in \mathbf{Expl}(\Xi_3^{se})$, given by

$$\Sigma_3 : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 2(x_1 e^{x_3})^{\frac{1}{2}} x_2 \\ 0 \\ 0 \\ x_5 + k e^{x_3} \\ x_6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & x_1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{3x_3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ y_1 = x_3 \\ y_2 = x_4. \end{cases}$$

- Σ_3 is not level-3 input-output linearizable (since the Toeplitz matrices $M_k(\Sigma_3)$ do not satisfy the rank condition).
- However, via $\tilde{y}_1 = e^{y_1}$, $\tilde{y}_2 = y_2 - \frac{1}{3}e^{3y_1}$, the system with the new outputs $\tilde{y}_1$, $\tilde{y}_2$ is level-3 input-output linearizable.

An example which is not level-3 externally linearizable but so is level-2

Example 3

- The system with the new outputs satisfies conditions (i)-(iv). In fact, Σ_3 is level-2 sys-equivalent to the linear control system $\tilde{\Sigma}_3$ below Ξ_3^{se} is level-2 ex-equivalent to the linear DAE Δ_3^{se} below

$$\tilde{\Sigma}_3 : \begin{cases} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= \tilde{v}_1 \\ \dot{\tilde{x}}_3 &= \tilde{v}_2, \qquad \tilde{y}_1 = \tilde{x}_3 \\ \dot{\tilde{x}}_4 &= \tilde{x}_5 + k\tilde{y}_1, \quad \tilde{y}_2 = \tilde{x}_4 \\ \dot{\tilde{x}}_5 &= \tilde{x}_6 \\ \dot{\tilde{x}}_6 &= \tilde{v}_3 \end{cases} \Rightarrow \Delta_3^{se} : \begin{cases} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_4 &= \tilde{x}_5 + k\tilde{y}_1 \\ \dot{\tilde{x}}_5 &= \tilde{x}_6 \\ 0 &= \tilde{x}_3 \\ 0 &= \tilde{x}_4. \end{cases}$$

- Even if an explicit control system is not level-3 input-output linearizable, it may be so under level-2 sys-equivalence.
- The future work should be focused on level-2 (and level-1) input-output linearizability of control systems and corresponding SE DAEs.

Plan

Conclusions

- Notion of level-i (i=1,2,3) ex-equivalence;
- Explicitation of SE DAEs;
- Difference between internal equivalence and external equivalence;
- Characterization of the internal linearizability of SE DAEs;
- Necessary and sufficient conditions for Level-3 external linearization;
- An example to show level-2 external linearization of SE DAEs.

Thank you for listening !!!