On geometric and differentiation index of nonlinear differential-algebraic equations

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Linear	Nonlinear
$E\dot{x} = Ax$	$E(x)\dot{x} = F(x)$

- If E is square and invertible, then

$$E\dot{x} = Hx \Rightarrow \dot{x} = E^{-1}Hx.$$

Linear	Nonlinear
$E\dot{x} = Hx$	$E(x)\dot{x} = F(x)$

Example 1

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1}$$

Solutions exist only on $\{x_1 = 0\}$. (Existence)

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2}$$

There exist infinite solutions. (Uniqueness)

Outline

Consider a nonlinear DAE:

$$\Xi : E(x)\dot{x} = F(x), \tag{3}$$

- the generalized state $x \in X$, X is an open subset of $\mathbb{R}^n$;
- $E(x)$, $F(x)$ are $\mathcal{C}^\infty$-smooth functions with values in $\mathbb{R}^{l\times n}$ and $\mathbb{R}^l$, respectively;
- denote DAE (3) by $\Xi_{l,n} = (E, F)$;
- a special case of general form:

$$\Xi^{gen} : G(t, x, x') = 0, \tag{4}$$

where $G : I \times TX \to \mathbb{R}^l$.

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Index of DAEs

- For linear DAE $\Xi^{lin}_{l,n} : E\dot{x} = Ax$, if $l = n$ and $|sE - A| \not\equiv 0$ (i.e., regular), then $\exists$ invertible Q, P s.t.

$$(QEP^{-1}, QAP^{-1}) = \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right),$$

- Index of linear DAEs

$$\nu_{lin} := \begin{cases} 0, & \text{if } n_1 = n, \\ \min \{ k \in \mathbb{N} \mid N^k = 0 \}, & \text{if } n_1 < n. \end{cases}$$

- Various notions of index for nonlinear DAEs: differentiation index, geometric index, perturbation index, strangeness index, uniform differentiation index, etc.

- Some survey or survey-like papers on index of DAEs: Griepentrog et al. (1992), Campbell (1995), Campbell and Gear (1995), Mehrmann (2015).

Plan

- A *solution* of Ξ is a $\mathcal{C}^1$-curve $x : I \to X$ with an open interval I such that for all $t \in I$, $x(t)$ solves (3).
- x_a: an admissible point ($\exists\ x(t)$ and t_0 s.t. $x(t_0) = x_a$), S_a: the admissible set ($S_a := \{x_0 \,|\, x_0 = x_a\}$)

Definition 1 (Locally invariant submanifold)

For a DAE Ξ defined on X, fix x_a, a smooth connected submanifold M of X is called locally *invariant* if $\exists$ a neighborhood U of x_a s.t $\forall x_0 \in M \cap U$, there exists a solution $x : I \to X$ of Ξ such that $x(t_0) = x_0$ and $x(t) \in M$, $\forall t \in I$.

- An invariant submanifold M^* is called locally *maximal*, if locally any other invariant submanifold $M \subseteq M^*$.
- What is the relation of M^* and S_a?
- How to calculate/construct M^*?

Algorithm 1

For a DAE $\Xi_{l,n} = (E, F)$, set $M_0 = X$. Step $k > 0$: assume that

(A) *for some open neighborhood $U_{k-1} \subseteq X$ of x_p that the intersection $M_{k-1} \cap U_{k-1}$ is a smooth submanifold.*

Set

$$M_k := \left\{ x \in M^c_{k-1} \,|\, F(x) \in E(x) T_x M^c_{k-1} \right\}. \tag{5}$$

where M^c_{k-1}: the connected component of $M_{k-1} \cap U_{k-1}$ s.t. $x_p \in M^c_{k-1}$

Definition 2 (Geometric index)

The geometric index of $\nu_g \in \mathbb{N}$ of a DAE Ξ is defined by

$$\nu_g := \min \left\{ k \geq 0 \,|\, (M_k = M_{k+1}) \wedge (M_k \neq \emptyset) \right\}.$$

Proposition 1

If ν_g exists, then $M^* = M_{\nu_g}$ is a locally maximal invariant submanifold.

Realization of the maximal invariant submanifold algorithm

- Step 1: Consider $\Xi_{l,n} = (E, F)$, fix $x_p \in X$, assume (to produce zero-level set)

(A1) $\exists U_1 \subseteq X$ of x_p s.t. $\operatorname{rank} E(x) = const. = l_1$, $\forall x \in U_1$.

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- Find $Q : U_1 \to Gl(l, \mathbb{R})$ s.t. E^1 below if of *full row rank* l_1:

$$QE(x) = \begin{bmatrix} E^1(x) \\ 0 \end{bmatrix}, \quad QF(x) = \begin{bmatrix} F^1(x) \\ F^2(x) \end{bmatrix},$$

Following (5), define

$$M_1 = \{x \in U_1 \mid F(x) \in \operatorname{Im} E(x)\} = \left\{x \in U_1 \mid F^2(x) = 0\right\}.$$

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(A2) $x_p \in M_1$ and $\operatorname{rank} \mathrm{D}F^2(x) = const. = n - n_1$ for $x \in M_1 \cap U_1$.

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- Then choose new coordinates $z = (\bar{z}_1, z_1) = \psi(x)$ on U_1 such that

$$M_1^c = \{(\bar{z}_1, z_1) \mid z_1 = 0\},$$

where $\bar{z}$ are any complementary coordinates s.t. ψ is a local diffeomorphism.

- Using $Q(x)$ and $z = (\bar{z}_1, z_1) = \psi(x)$, we get

$$Q(x)E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1}\left(\frac{\partial\psi(x)}{\partial x}\right)\dot{x} = Q(x)F(x) \Leftrightarrow \begin{bmatrix} \bar{E}^1(z) & \tilde{E}^1(z) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\bar{z}}_1 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} F^1(z) \\ F^2(z) \end{bmatrix}.$$

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- Set $z_1 = 0$ to get the reduced DAE:
$$\Xi|_{M_1^c} : \bar{E}^1(\bar{z}_1, 0)\dot{\bar{z}}_1 = F^1(\bar{z}_1, 0),$$
where $\bar{E}^1 : M_1^c \to \mathbb{R}^{l_1 \times n_1}$. Notice that $\Xi|_{M_1^c}$ is again a DAE of form (3).

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- Repeat the above procedure until Step k^*, where k^* is the smallest k s.t. $M_k^c = M_{k+1}$ (note that $k^* = \nu_g$ and $M^* = M_{k^*}^c$).

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$\bar{E}^{k^*} : M^* \to \mathbb{R}^{l_{k^*} \times n_{k^*}}$ of the reduced DAE $\Xi|_{M^*} : \bar{E}^{k^*}(\bar{z}_{k^*})\dot{\bar{z}}_{k^*} = F^{k^*}(\bar{z}_{k^*})$ is of full row rank and thus $\Xi|_{M^*}$ can be seen as an ODE with free variables.

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$$\bar{E}^{k^*}(\bar{z}_{k^*})\dot{\bar{z}}_{k^*} = F^{k^*}(\bar{z}_{k^*}) \Rightarrow \begin{bmatrix} \bar{E}_1(\bar{z}_{k^*}) & \bar{E}_2(\bar{z}_{k^*}) \end{bmatrix}\begin{bmatrix} \dot{\bar{z}}_{k^*}^1 \\ \dot{\bar{z}}_{k^*}^2 \end{bmatrix} = F^{k^*}(\bar{z}_{k^*})$$

$$\Rightarrow \begin{cases} \dot{\bar{z}}_{k^*}^1 = (\bar{E}_1)^{-1}F^{k^*}(\bar{z}_{k^*}) - (\bar{E}_1)^{-1}E_2(\bar{z}_{k^*})u \\ \dot{\bar{z}}_{k^*}^2 = u \end{cases}$$

Theorem 1

Consider a DAE $\Xi_{l,n} = (E, F)$ and fix a point $x_p \in X$. Assume the constant assumptions (A1) and (A2) of the algorithm are satisfied. Then locally (on $U^* = U_{k^*}$)

(i) $M^* = M^c_{k^*} = S_a$; *(existence)*

(ii) *there exists a diffeomorphism which maps any solution of Ξ to that of*

$$\bar{E}^{k^*}(\bar{z}_{k^*})\dot{\bar{z}}_{k^*} = F^{k^*}(\bar{z}_{k^*}), z_{k^*} = 0, \ldots, z_1 = 0;$$

where $M^* = \{z \,|\, z_{k^*} = 0, \ldots, z_1 = 0\}$, $\bar{E}^{k^*} : M^* \to \mathbb{R}^{l_{k^*} \times n_{k^*}}$ is of full row rank.

(iii) *for any $x_0 \in M^*$, there passes only one solution if and only if*

$$\dim M^* = \dim E(x) T_x M^*$$

for all $x \in M^*$, i.e., $n_{k^*} = l_{k^*}$. (uniqueness)

Remarks on the geometric index

- Relation to the existence of solutions: if $x_0 \in U_{\nu_g}/M_{\nu_g}$ around x_p, then there does not exist a solution $x(t)$ such that $x(0) = x_0$.

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- The geometric index *does not* concern the uniqueness of the solutions. As an example, consider two DAEs

$$\Xi_{2,2} : \begin{cases} \dot{x}_1 = f(x_1, x_2) \\ 0 = x_2, \end{cases} \quad \text{and} \quad \tilde{\Xi}_{2,3} : \begin{cases} \dot{x}_1 = f(x_1, x_2) \\ 0 = x_3, \end{cases}$$

where $f : X_1 \times X_2 \to \mathbb{R}$ is smooth. For both Ξ and $\tilde{\Xi}$, $\nu_g = 1$.

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where $f : X_1 \times X_2 \to \mathbb{R}$ is smooth. For both Ξ and $\tilde{\Xi}$, $\nu_g = 1$.

- Apply the above algorithm to a linear DAE $E\dot{x} = Ax$ and denote $\mathscr{V} = M$, we get (the Wong sequence(1973))

$$\mathscr{V}_0 = \mathbb{R}^n, \ \mathscr{V}_k = \left\{ x \in \mathscr{V}_{k-1} \,|\, Ax \in E\mathscr{V}_{k-1} \right\} = A^{-1}E\mathscr{V}_{k-1}.$$

Note that $\nu_{lin} = k^*$. As a result, the geometric index ν_g of nonlinear DAEs is a nonlinear generalization of the index ν_{lin} of linear DAEs.

Plan

Classical definition of the differentiation index

- The notion of differentiation index is originally proposed for (see Campbell(1995)):

$$\Xi^{gen}: G(t, x, x') = 0.$$

- Define the differential array by

$$G_k(t, x, x', w) = \begin{bmatrix} G \\ D_t G + D_x G x' + D_{x'} G x'' \\ \vdots \\ \frac{\mathrm{d}^k}{\mathrm{d}t^k} G \end{bmatrix} (t, x, x', w) = 0, \tag{6}$$

where $w = \left[x^{(2)}, \ldots, x^{(k+1)}\right]$.

- The differentiation index ν_d is the least integer k such that equation (6) uniquely determines x' as a function of (x, t), i.e., $x' = v(x, t)$.

Drawbacks of the classical definition of the differentiation index

Example 1

$$\Xi_{2,2} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = F(x_1, x_2) \end{cases} \quad \text{and} \quad \tilde{\Xi}_{3,2} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = F(x_1, x_2) \\ 0 = x_1 \end{cases},$$

where $F(0,0) \neq 0$. Initial point: $(x_{10}, x_{20}) = (0,0)$. Both $\Xi_{2,2}$ and $\tilde{\Xi}_{3,2}$ have $\nu_d = 0$.

- Such a definition does not identify M^* and thus does not allows for a conclusion about existence of solutions (Ξ has a solution and $\tilde{\Xi}$ has not).

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- Mix up the difference between an ODE and an "over-determined" DAE. E.g., in order to derive solutions: $\Xi_{2,2}$-no differentiation needed and $\tilde{\Xi}_{3,2}$-need two times of differentiation.

Our definition of the differentiation index (inspired by (Griepentrog1991))

- Consider a nonlinear DAE $\Xi_{l,n} = (E, F)$, let $H(x,\zeta_1) = E(x)\zeta_1 - F(x)$, denote $(\frac{\mathrm{d}^k}{\mathrm{d}t^k} H) = H^{(k)}$ and define

$$H_k(x,\bar{\zeta}_{k+1}) = \begin{bmatrix} H^{(0)}(x,\zeta_1) \\ \vdots \\ H^{(k)}(x,\bar{\zeta}_{k+1}) \end{bmatrix} = 0, \tag{7}$$

  where $\bar{\zeta}_{k+1} = (\zeta_1, \ldots, \zeta_{k+1})$.

- Set $N_0 = X$, $\mathcal{Z}_1^0 = \mathbb{R}^n$ and for $k > 1$, define

$$\begin{aligned} N_k &:= \left\{ x \in X \mid H_{k-1}(x,\bar{\zeta}_k) = 0 \right\}, \\ \mathcal{Z}_1^k &:= \left\{ \zeta_1 \in \mathbb{R}^n \mid H_{k-1}(x,\bar{\zeta}_k) = 0, x \in N_k \right\}, \end{aligned} \tag{8}$$

  and assume that for each $k > 0$, N_k is a smooth connected submanifold.

Definition 3 (Differentiation index)

The differentiation index ν_d of Ξ is defined by

$$\nu_d := \begin{cases} 0, & \text{if } (l = n) \wedge (E : X \to Gl(l,\mathbb{R})), \\ \min \left\{ k > 0 \;\middle|\; \begin{array}{l} N_k \neq \emptyset \wedge (\mathcal{Z}_1^k = \mathcal{Z}_1^k(x) \text{ is a singleton}) \\ \wedge (\mathcal{Z}_1^k(x) \in T_x N_k, \forall x \in N_k) \end{array} \right\}, & \text{otherwise.} \end{cases}$$

Theorem 2

For a DAE $\Xi_{l,n} = (E, F)$, fix a point x_p and assume locally that

(A1)' *N_k is a smooth embedded submanifold and $x_p \in N_k$;*

(A2)' $\dim E(x)T_x N_{k-1}^c = const.$.

Then we have locally around x_p that for each $k \geq 0$,

$$N_k = M_k.$$

Thus there exists a smallest k, denoted by $k^* \leq n$ such that $N_{k^*+1} = N_{k^*}^c$ and the geometric index $\nu_g = k^*$.
Moreover, ν_d of Ξ exists and satisfies $\nu_d = \nu_g$ if and only if

$$\dim N_{k^*}^c = \dim E(x)T_x N_{k^*}^c.$$

Consider a mathematical model of a pendulum with a mass attached to its end (Rabier(1994)):

$$\Xi : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ x_4 \\ -x_5 x_3 - g \\ x_1^2 + x_3^2 - l^2 \end{bmatrix}. \tag{9}$$

We consider the point $x_0 = (x_{10}, x_{20}, x_{30}, x_{40}, x_{50})$, where $x_{10} = 0$, $x_{20} = 0$, $x_{30} = -l$, $x_{40} = 0$, $x_{50} = g/l$.

- By applying the algorithm, we get

$$\begin{aligned} M_1 &= \left\{ x \in X \,|\, x_1^2 + x_3^2 - l^2 = 0 \right\}. \\ M_2 &= \left\{ x \in M_1 \,|\, x_3 x_4 + x_1 x_2 = 0 \right\}. \\ M^* &= M_3 = \left\{ x \in M_2 \,|\, x_4^2 + x_2^2 - x_5 l^2 - g x_3 = 0 \right\} \end{aligned}$$

  Notice that the assumptions of (A1) and (A2) are satisfied and the solution passing through x_0 exists and is unique (since $\dim M^* = \operatorname{rank} E_{k^*} = 2$). Indeed, since

$$\Xi|_{M^*} : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \end{bmatrix},$$

  where $x_5 = \frac{1}{l^2}(x_1^2 x_2^2 (l - x_1^2)^{-1} + x_2^2 + g(l^2 - x_1^2)^{1/2})$.

- By calculating the differential array, it is seen that $N_k = M_k$ for $k = 0, 1, 2, 3$. Moreover, Since $\dim M^* = \dim(E(x) T_x M^*) = 2$ we have $\nu_g = \nu_d = 3$.

Plan

Conclusions

- Maximal invariant submanifold algorithm (how to solve a DAE using geometric method).
- Existence and uniqueness of solutions.
- The two indices related to the existence and uniqueness of solutions in a different manner.
- The two indices coincide with each other when some constant rankness and smoothness assumptions are satisfied

Thank you for listening !!!