Lecture Course: Advanced Systems Theory

Chapter 9-Lecture 9: The well-posedness of the output regulation problem

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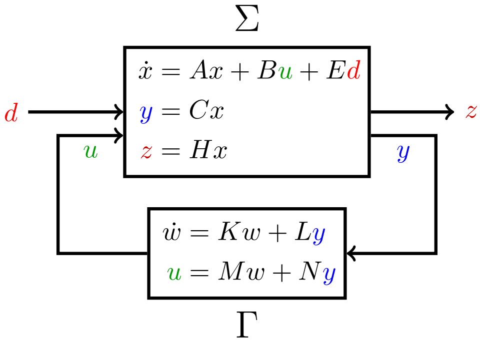
disturbance decoupling with dynamic feedback



- Closed loop system:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}}_{A_e} \begin{bmatrix} x \\ w \end{bmatrix} + \underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix}}_{E_e} d \qquad z = \underbrace{\begin{bmatrix} H & 0 \end{bmatrix}}_{H_e} \begin{bmatrix} x \\ w \end{bmatrix}$$

disturbance decoupling with dynamic feedback

Theorem 6.6+Corollary6.7+Theorem 6.4

DDPM is solvable for $\Sigma = (H, C, A, B, E)$ iff $\exists$ a (C, A, B)-pair $(\mathcal{S}, \mathcal{V})$ s.t.

$$\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H,$$

or, equivalently, $\mathcal{S}^*(\operatorname{im} E) \subseteq \mathcal{V}^*(\ker H)$. If such $(\mathcal{S}, \mathcal{V})$ exists, choose

- $N : \mathcal{Y} \to \mathcal{U}$ s.t. $(A + BNC)\mathcal{S} \subseteq \mathcal{V}$,
- $F : \mathcal{X} \to \mathcal{U}$ s.t. $(A + BF)\mathcal{V} \subseteq \mathcal{V}$,
- $G : \mathcal{Y} \to \mathcal{U}$ s.t. $(A + GC)\mathcal{S} \subseteq \mathcal{S}$,

then Γ is given by

$$\begin{cases} \dot{w} = (A + BF + GC - BNC)w + (BN - G)y \\ u = (F - NC)w + Ny. \end{cases}$$

Theorem 6.6

DDPM is solvable for $\Sigma = (H, C, A, B, E)$ iff $\exists$ a (C, A, B)-pair $(\mathcal{S}, \mathcal{V})$ s.t.

$$\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H.$$

Consider a system $\Sigma = (A, B, E, C, H)$, where

$$A = \begin{bmatrix} a & b & c \\ * & * & * \\ * & * & * \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \; E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Question 1:

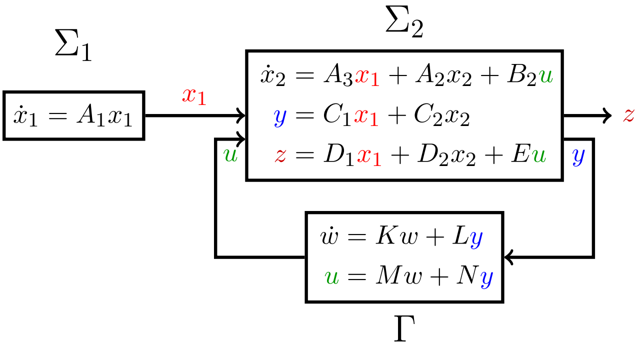
For which choices of a, b, c, the DDPM is solvable for Σ ?
(i) $a = 0$, $b \neq 0$, $c \neq 0$. (ii) $a \neq 0$, $b = 0$, $c \neq 0$. (iii) $a \neq 0$, $b \neq 0$, $c = 0$.

Question 2:

Let $a \neq 0$, $b \neq 0$, $c = 0$. How can we choose $F : (A + BF)\mathcal{V} \subseteq \mathcal{V}$, $G : (A + GC)\mathcal{S} \subseteq \mathcal{S}$ and $N : (A + BNC)\mathcal{S} \subseteq \mathcal{V}$?
(i) $F = \begin{bmatrix} -a & 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} -a & -b & 0 \end{bmatrix}^\top$, $N = -a$. (ii) $F = \begin{bmatrix} 0 & -b & 0 \end{bmatrix}$, $G = \begin{bmatrix} 0 & -b & 0 \end{bmatrix}^\top$, $N = -b$.
(iii) $F = \begin{bmatrix} -a & -b & 0 \end{bmatrix}$, $G = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$, $N = -b$. (iV) $F = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 0 & -a & 0 \end{bmatrix}^\top$, $N = -a$.

9.1 The regulator problem

Output regulation



Goal: Find $\Gamma = (K, L, M, N)$: $\Leftrightarrow \lim_{t\to\infty} z(t) = 0, \ \forall x_1(0)$

9.1 The regulator problem

- The open-loop system is $\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ z = Dx + Eu \end{cases}$ with $A = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, $D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$
- The closed loop system

$$\begin{cases} \dot{x}_1 = A_1 x_1 \\ \dot{x}_{2,e} = A_{2,e} x_{2,e} + A_{3,e} x_1 \\ z = D_{1,e} x_1 + D_{2,e} x_{2,e} \end{cases}$$

$$x_{2,e} = \begin{bmatrix} x_2 \\ w \end{bmatrix}, A_{2,e} = \begin{bmatrix} A_2 + B_2 N C_2 & B_2 M \\ L C_2 & K \end{bmatrix}, A_{3,e} = \begin{bmatrix} A_3 + B_2 N C_1 \\ L C_1 \end{bmatrix}$$

$$D_{2,e} = \begin{bmatrix} D_2 + E N C_2 & E M \end{bmatrix} \quad D_{1,e} = D_1 + E N C_1.$$

- Regulator Problem: Find $\Gamma = (K, L, M, N)$: the closed-loop system satisfies $z(t) \to 0$ as $t \to \infty$ and the closed loop is endostable, i.e. for $x_1(0) = 0$, all variables converge to zero .

Questions

Question 3

Let $A_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} \tilde{\lambda}_1 & 0 \\ 0 & \tilde{\lambda}_2 \end{bmatrix}$. Then $\exists$ a unique T such that $TA_1 - A_2T = A_3$ iff

(i) $(\lambda_1 - \lambda_2)(\tilde{\lambda}_1 - \tilde{\lambda}_2) \neq 0$. (ii) $(\lambda_1 - \tilde{\lambda}_1)(\lambda_2 - \tilde{\lambda}_2) \neq 0$.

(iii) $(\lambda_1 - \lambda_2)(\tilde{\lambda}_1 - \tilde{\lambda}_2)(\lambda_1 - \tilde{\lambda}_1)(\lambda_2 - \tilde{\lambda}_2) \neq 0$.

(iv) $(\lambda_1 - \tilde{\lambda}_2)(\tilde{\lambda}_1 - \lambda_2)(\lambda_1 - \tilde{\lambda}_1)(\lambda_2 - \tilde{\lambda}_2) \neq 0$.

Question 4

In Question 1, let $A_3 = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, if $\exists$ a unique solution T, then $T =$

(i) $\begin{bmatrix} \frac{a}{\lambda_1 - \tilde{\lambda}_2} & \frac{b}{\tilde{\lambda}_1 - \lambda_2} \\ \frac{c}{\lambda_1 - \tilde{\lambda}_1} & \frac{d}{\lambda_2 - \tilde{\lambda}_2} \end{bmatrix}$. (ii) $\begin{bmatrix} \frac{a}{\lambda_1 - \tilde{\lambda}_1} & \frac{b}{\lambda_1 - \tilde{\lambda}_2} \\ \frac{c}{\lambda_2 - \tilde{\lambda}_1} & \frac{d}{\lambda_2 - \tilde{\lambda}_2} \end{bmatrix}$. (iii) $\begin{bmatrix} \frac{a}{\lambda_1 - \tilde{\lambda}_1} & \frac{b}{\lambda_2 - \tilde{\lambda}_1} \\ \frac{c}{\lambda_1 - \tilde{\lambda}_2} & \frac{d}{\lambda_2 - \tilde{\lambda}_2} \end{bmatrix}$.

9.1 The regulator problem

Open loop:

Lemma (9.1)

Consider Σ with A_2 being Hurwitz and $u = 0$. Then $z(t) \to 0$ as $t \to \infty$ if $\exists T : \mathcal{X}_1 \to \mathcal{X}_2$

$$\begin{cases} TA_1 - A_2T = A_3 \\ D_2T + D_1 = 0. \end{cases} \tag{1}$$

If A_1 is antistable (i.e., $\sigma(A_1) \cap \mathbb{C}_{Re<0} = \varnothing$), then the solvability of (1) is also necessary.

Question 5:

If (1) is uniquely solvable, the map $T \mapsto \begin{bmatrix} TA_1 - A_2T \\ D_2T \end{bmatrix}$ is

(i) Injective; (ii) Surjective (iii) Bijective.

9.1 The regulator problem

Closed-loop:

Corollary (9.1a)

The regulator problem for Σ can be solved with controller $\Gamma = (K, L, M, N)$, if $A_{2,e}$ is Hurwitz and $\exists\; T_e : \mathcal{X}_1 \to \mathcal{X}_2 \times \mathcal{W}$ s.t.

$$\begin{cases} T_e A_1 - A_{2,e} T_e = A_{3,e} \\ D_{2,e} T_e + D_{1,e} = 0 \end{cases} \tag{2}$$

Lemma (9.1b)

$\exists \Gamma = (K, L, M, N)$: *equation (2) is solvable iff* $\exists (T, V)$:

$$\begin{cases} T A_1 - A_2 T - B_2 V = A_3 \\ D_1 + D_2 T + E V = 0 \end{cases} \tag{3}$$

Proof sufficiency of Lemma 9.1 b .

If. Let (T, V) solve (3), choose $K = A + GC + BF,\ L = -G,\ M = F,\ N = 0$, i,e,

$$\Gamma : \begin{cases} \dot{w} = (A + GC + BF)w - Gy \\ u = Fw, \end{cases}$$

where $F = [-F_2T + V \quad F_2]$, F_2 be any and $T_e = \begin{bmatrix} T \\ U \end{bmatrix}$ $U = \begin{bmatrix} I \\ T \end{bmatrix}$, then

$$T_eA_1 - A_{2,e}T_e = \begin{bmatrix} T \\ U \end{bmatrix} A_1 - \begin{bmatrix} A_2 & B_2F \\ -GC_2 & A + GC + BF \end{bmatrix} \begin{bmatrix} T \\ U \end{bmatrix} = \begin{bmatrix} TA_1 - A_2T - B_2[-F_2T + V \ \ F_2] \begin{bmatrix} I \\ T \end{bmatrix} \\ UA_1 + GC_2T - (A + GC + BF)U \end{bmatrix}$$

$$= \begin{bmatrix} A_3 \\ A_1 + G_1C_2T - A_1 - G_1C_1 - G_1C_2T \\ TA_1 + G_2C_2T - A_3 - A_2T - G_2C_1 - G_2C_2T - B_2V \end{bmatrix} = \begin{bmatrix} A_3 \\ -G_1C_1 \\ -G_2C_1 \end{bmatrix} = \begin{bmatrix} A_3 \\ -GC_1 \end{bmatrix} = A_{3,e},$$

$$D_{2,e}T_e + D_{1,e} = D_1 + \begin{bmatrix} D_2 & EF \end{bmatrix} \begin{bmatrix} T \\ U \end{bmatrix} = D_1 + D_2T + EV = 0$$

□

›

$$\Sigma : \begin{cases} \dot{x}_1 = A_1 x_1 \\ \dot{x}_2 = A_2 x_2 + A_3 x_1 + B_2 u \\ y = C_1 x_1 + C_2 x_2 \\ z = D_1 x_1 + D_2 x_2 + Eu \end{cases} \quad A = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

Theorem (9.2)

Assume: (i) (A_2, B_2) *stabilizable; (ii)* (C, A) *detectable; (iii)* $\exists (T, V)$:

$$\begin{cases} TA_1 - A_2 T - B_2 V = A_3 \\ D_1 + D_2 T + EV = 0. \end{cases}$$

Then $\exists \Gamma = (K, L, M, N)$ *solves the output regulator problem, i.e.* $\lim_{t\to\infty} z(t) = 0$ *and* $\lim_{t\to\infty} x_2(t) = 0$ *for* $x_1(0) = 0$ *in closed-loop. The converse is true if* A_1 *is anti-stable.*

If (i), (ii), (iii) hold then Γ *is given by* $\Gamma : \begin{cases} \dot{w} = (A + GC + BF)w - Gy \\ u = Fw \end{cases}$

with G *such that* $A + GC$ *is Hurwitz.* $F = [F_1, F_2]$ *where* F_2 *is such that* $A_2 + B_2 F_2$ *is Hurwitz and* $F_1 := -F_2 T + V$

Proof of Thm 9.2.

Due to Corollary 9.1a and Lemma 9.1b, it is sufficient to show that $A_{2,e}$ is Hurwitz.

$$A_{2,e} = \begin{bmatrix} A_2 & B_2F \\ -GC_2 & A+GC+BF \end{bmatrix} \text{ is Hurwitz .}$$

$$\Leftrightarrow \text{ all solutions of } \begin{bmatrix} \dot{x}_2 \\ \dot{w}_1 \end{bmatrix} = A_{2,e} \begin{bmatrix} x_2 \\ w \end{bmatrix} \text{ converge to zero .}$$

$$\Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_{3,e} & A_{2,e} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ w \end{bmatrix} \text{ converge to zero for } x_1(0) = 0.$$

$$\Leftrightarrow \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \cdot F \\ A_3 & A_2 & B_2 \cdot F \\ -GC_1 & -GC_2 & A+GC+BF \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \text{ with } x_1 = 0 \text{ converge to zero .}$$

Define $r : w - x$, we have

$$\dot{x} = (A+BF)x + BFr$$
$$\dot{r} = \dot{w} - \dot{x} = -Ax - BFw - GCx + (A+GC+BF)w = (A+GC)r$$

It follows that $\lim_{t\to\infty} r(t) = 0$, and thus $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} w(t) = 0$. □

9.2-9.4 Well-posedness of the regulator problem.

- Q: Are conditions (i),(ii),(iii) of Theorem 9.2 robust w.r.t. to small perturbation in matrix coefficient?
- (A, B) is stabilizable $\Rightarrow \exists$ small enough A' and B' such that $(A + A', B + B')$ is stabilizable.
- $A + BF$ is Hurwitz then $A + BF + (A' + B'F)$ is also Herwitz for small enough $(A' + B'F)$.
- (C, A) is detectable $\Rightarrow \exists$ small enough C' and A' such that $(C + C', A + A')$ is detectable.
- Rewrite

$$\begin{cases} TA_1 - A_2T - B_2V = A_3 \\ D_1 + D_2T + EV = 0. \end{cases} \Rightarrow \begin{bmatrix} -A_2 & -B_2 \\ D_2 & E \end{bmatrix} \begin{bmatrix} T \\ V \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ V \end{bmatrix} A_1 = \begin{bmatrix} A_2 \\ -D_1 \end{bmatrix}$$

- Clearly, a linear equation $Ax = b$ is well posed if A is surjective.

Corollary

The regulator problem is well-posed $\Leftrightarrow$ *the linear map*

$$(T,V) \mapsto \begin{bmatrix} TA_1 - A_2T - B_2V \\ D_2T + EV \end{bmatrix} \text{ is surjective}$$

(Note that decetability and stabilizability are well-posed properties)
$\Leftrightarrow$ *Linear matrix equation*

$$\begin{bmatrix} -A_2 & -B_2 \\ D_2 & E \end{bmatrix} \begin{bmatrix} T \\ V \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ V \end{bmatrix} A_1 = C$$

is universally solvable (i.e. for any C)

- More general question: when is $\sum_{i=1}^k L_i X R_i = C$ solvable for all C?
- Special case: $LXI + IXR = C$ (Sylvester equation).
- For square L, R, the matrix X is solvable for all $C \Leftrightarrow \sigma(L) \cap \sigma(R) = \varnothing$.
- Unfortunately, there seem to be no such simple test for general L_i and R_i.
- However, for $R_i = q_i(R)$, where R is square and $q_1(s)$,...,$q_k(s)$ are given polynomials, there is a nice test.

Theorem (9.6)

Let L_1, L_2,...,$L_k \in \mathbb{R}^{n\times m}$, $R \in \mathbb{R}^{q\times q}$ and $q_1(s)$, $q_2(s)$,...,$q_k(s)$ be given. Then

$$\sum_{i=1}^k L_i X q_i(R) = C$$

is universally solvable (i.e. for all C there is a matrix X solve the equation above)

$$\Leftrightarrow \operatorname{rank} \sum_{i=1}^k L_i q_i(\lambda) = n \quad \forall \lambda \in \sigma(R)$$

Theorem (9.6)

Let $L_1, L_2,\dots,L_k \in \mathbb{R}^{n\times m}$, $R \in \mathbb{R}^{q\times q}$ *and* $q_1(s), q_2(s),\dots,q_k(s)$ *be given. Then* $\sum_{i=1}^k L_i X q_i(R) = C$ *is universally solvable iff* $\operatorname{rank} \sum_{i=1}^k L_i q_i(\lambda) = n$, $\forall \lambda \in \sigma(R)$.

Proof of Thm 9.6.

"*Only if.*" Assume $\operatorname{rank} \sum_{i=1}^k L_i q_i(\lambda) < n$ for some $\lambda \in \sigma(R)$.

$\Rightarrow \exists v : Rv = \lambda v$ and $\exists\ w : w^\top \sum_{i=1}^k L_i q_i(\lambda) = 0$.

$\Rightarrow w^\top \sum_{i=1}^k L_i X q_i(R) v = w^\top \sum_{i=1}^k L_i X q_i(\lambda) v = w^\top (\sum_{i=1}^k L_i q_i(\lambda)) X v = 0.$

Hence for any C with $w^\top C v \neq 0$ (e.g. $C = wv^\top$) the system is not solvable.

"*If.*" By the right invertibility of $(\sum_{i=1}^k L_i q_i(s))$, choose $M(s)$ such that

$(\sum_{i=1}^k L_i q_i(s)) \cdot M(s) = m(s) \cdot I$ for some scalar polynomial $m(s)$

$\Rightarrow m(R)$ is invertible(because $m(\lambda) \neq 0$, $\forall \lambda \in \sigma(R)$)

and let $X(s) := M(s) \cdot C \cdot m^{-1}(R)$

$\Rightarrow \sum_{i=1}^k L_i X(s) q_i(s) = \sum_{i=1}^k L_i q_i(s) X(s) = m(s) C \cdot m^{-1}(R)$

$\Rightarrow X := X_r(R)$ (right substitution) now solve (Recall Theorem 7.6: if $Q(s)$ commutes with A, then $\forall P(s) : P_r(A) Q_r(A) = (PQ)_r(A)$)

$\sum_{i=1}^k L_i X_r(R) q_i(R) \overset{Thm 7.6}{=} C \cdot m^{-1}(R) m(R) = C$ ∎

- Theorem 7.6: if $Q(s)$ commutes with A, then $\forall P(s) : P_r(A)Q_r(A) = (PQ)_r(A)$.
- Remark: Thm 7.6 is quite powerful, because it allows to plug in matrices also in products of matrix polynomial, e.g., with Thm 7.6, the proof of Cayley-Hamilton Theorem becomes trivial.
- $$(sI - A)^{-1} = \frac{B(s)}{\det(sI - A)} \Leftrightarrow \underbrace{\det(sI - A)}_{P(A)} I = B(s)(sI - A)$$

  Plugging in $s = A$: $P(A) = B(A)(AI - A) = 0$.

Theorem (9.6)

Let $L_1, L_2, \ldots, L_k \in \mathbb{R}^{n\times m}$, $R \in \mathbb{R}^{q\times q}$ and $q_1(s), q_2(s), \ldots, q_k(s)$ be given. Then $\sum_{i=1}^k L_i X q_i(R) = C$ is universally solvable iff $\operatorname{rank} \sum_{i=1}^k L_i q_i(\lambda) = n$, $\forall \lambda \in \sigma(R)$.

For the regulator problem we have

$$\begin{bmatrix} -A_2 & -B_2 \\ D_2 & E \end{bmatrix} \begin{bmatrix} T \\ V \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ V \end{bmatrix} A_1 = C$$

$$L_1 = \begin{bmatrix} -A_2 & -B_2 \\ D_2 & E \end{bmatrix}, \quad q_1(s) = 1, \quad R = A_1.$$

$$L_2 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad q_2(s) = s.$$

Corollary: The regulator problem is well-posed

$$\Leftrightarrow \operatorname{rank} \begin{bmatrix} \lambda I - A_2 & -B_2 \\ D_2 & E \end{bmatrix} = n_2 + r \quad \forall z \in \sigma(A_1).$$

(full row rank)

Summary

- The regulator problem: Lemma 9.1 (open-loop: T), Corollary 9.1 a, Lemma 9.1 b (closed-loop: T, V).
- The main result of the regulator problem Theorem 9.2: (assumptions(i),(ii),(iii), how to choose (K, L, M, N))
- Well-posedness for assumptions(i),(ii),(iii).
- Well-posedness for general matrix equation, Theorem 9.6. (Theorem 7.6 in the proof)