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Lecture Course: Advanced Systems Theory

Chapter 6 and 9-Lecture 8: DDP by dynamical feedback and the output regulation problem

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6.1 (C, A, B)-pairs

Definition (6.1)

A pair of subspace $(\mathcal{S}, \mathcal{V})$ of $\mathcal{X}$ is called (C, A, B)-pair if
(i) $\mathcal{S} \subseteq \mathcal{V}$; (ii) $\mathcal{S}$ is a (C, A)-invariant subspace; (iii) $\mathcal{V}$ is an (A, B)-invariant subspace.

Theorem (6.2)

Consider a subspace $\mathcal{V}_e \subseteq \mathcal{X} \times \mathcal{W}$ and let

$$p(\mathcal{V}_e) := \{x \in \mathcal{X} \mid \exists w \in \mathcal{W} : \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{V}_e\} \quad \text{(projection)}$$
$$i(\mathcal{V}_e) := \{x \in \mathcal{X} \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{V}_e\}. \quad \text{(intersection)}$$

If $\mathcal{V}_e$ is $A_e = \begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}$-inv. then $(i(\mathcal{V}_e), p(\mathcal{V}_e))$ is a (C, A, B)-pair.

Lemma (6.3)

If $(\mathcal{S}, \mathcal{V})$ is a (C, A, B)-pair, then $\exists$ linear $N : \mathcal{Y} \to \mathcal{U}$ s.t. $(A + BNC)\mathcal{S} \subseteq \mathcal{V}$.

Questions

Consider $\Sigma = (A, B, C)$, where $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, i.e.,

$$\Sigma : \begin{cases} \dot{x}_1 = x_1 + x_2 + u_1 \\ \dot{x}_2 = -x_1 + x_2 \\ \dot{x}_3 = u_2 \end{cases} \qquad y = x_1.$$

Question 1

Which $(\mathcal{S}, \mathcal{V})$ is a (C, A, B)-pair? (i) $\mathcal{S} = \mathcal{X}_1$, $\mathcal{V} = \mathcal{X}_1$ (ii) $\mathcal{S} = \mathcal{X}_3$, $\mathcal{V} = \mathcal{X}_2 \times \mathcal{X}_3$ (iii) $\mathcal{S} = \mathcal{X}_2$, $\mathcal{V} = \mathcal{X}_2 \times \mathcal{X}_3$ (iv) $\mathcal{S} = \mathcal{X}_2$, $\mathcal{V} = \mathcal{X}_2$.

Question 2

Let $\mathcal{S} = \mathcal{X}_3$, $\mathcal{V} = \mathcal{X}_2 \times \mathcal{X}_3$, then which N does not satisfy that $(A + BNC)\mathcal{S} \subseteq \mathcal{V}$?
(i) $N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (ii) $N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (iii) $N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (iv) none of the above.

Question 3

Let $\mathcal{S} = \mathcal{X}_3 = i(\mathcal{V}_e)$, $\mathcal{V} = \mathcal{X}_2 \times \mathcal{X}_3 = p(\mathcal{V}_e)$, then $\mathcal{V}_e \subseteq \mathcal{X} \times \mathbb{R}$ could be?
(i) $\operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ (ii) $\operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ (iii) $\operatorname{im} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (iv) $\operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

6.1 (C, A, B)-pairs

Theorem 6.4 (using (C, A, B) pairs to construct Γ)

Let $(\mathcal{S}, \mathcal{V})$ be a (C, A, B)-pair. Then there exists controller Γ and an A_e-invariant subspace $\mathcal{V}_e \subseteq \mathcal{X} \times \mathcal{W}$ s.t. $\mathcal{S} = i(\mathcal{V}_e)$ and $\mathcal{V} = p(\mathcal{V}_e)$.

In fact, choose

- $N : \mathcal{Y} \to \mathcal{U}$ s.t. $(A + BNC)\mathcal{S} \subseteq \mathcal{V}$,
- $F : \mathcal{X} \to \mathcal{U}$ s.t. $(A + BF)\mathcal{V} \subseteq \mathcal{V}$,
- $G : \mathcal{Y} \to \mathcal{U}$ s.t. $(A + GC)\mathcal{S} \subseteq \mathcal{S}$.

Then Γ is given by

$$\begin{cases} \dot{w} = (A + BF + GC - BNC)w + (BN - G)y \\ u = (F - NC)w + Ny, \end{cases}$$

where $\mathcal{W} = \mathcal{X}$ and $\mathcal{V}_e = \{\left[\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} x_2 \\ x_2 \end{smallmatrix}\right] \mid x_1 \in \mathcal{S}, x_2 \in \mathcal{V}\}$

disturbance decoupling with dynamic feedback

Problem (DDP with dynamic measurement feedback (DDPM))

Given the system $\Sigma = (H, C, A, B, E)$

$$\begin{aligned}\dot{x} &= Ax + Bu + Ed\\ y &= Cx\\ z &= Hx\end{aligned}$$

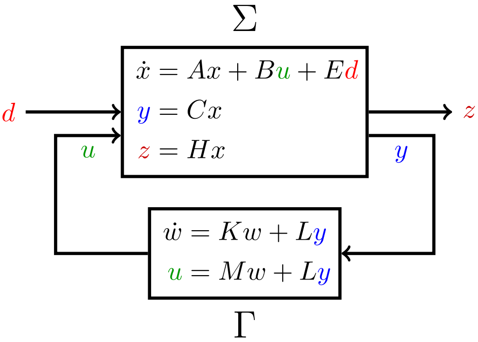
find K, L, M, N such that the dynamic controller $\Gamma(M, K, L, N)$

$$\begin{aligned}\dot{w} &= Kw + Ly\\ u &= Mw + Ny\end{aligned}$$

renders the closed loop system disturbance decoupled:

$$\begin{bmatrix}\dot{x}\\ \dot{w}\end{bmatrix} = \underbrace{\begin{bmatrix}A + BNC & BM\\ LC & K\end{bmatrix}}_{A_e}\begin{bmatrix}x\\ w\end{bmatrix} + \underbrace{\begin{bmatrix}E\\ 0\end{bmatrix}}_{E_e} d \qquad z = \underbrace{\begin{bmatrix}H & 0\end{bmatrix}}_{H_e}\begin{bmatrix}x\\ w\end{bmatrix}$$

disturbance decoupling with dynamic feedback



- Closed loop system:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}}_{A_e} \begin{bmatrix} x \\ w \end{bmatrix} + \underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix}}_{E_e} d \qquad z = \underbrace{\begin{bmatrix} H & 0 \end{bmatrix}}_{H_e} \begin{bmatrix} x \\ w \end{bmatrix}$$

disturbance decoupling with dynamic feedback

Definition 6.5 DDPM

Find $\Gamma = (K, L, M, N)$ s.t.

$$T_{\Gamma(t)} := H_e e^{A_e t} E_e = 0, \ \forall t \geq 0$$

or, equivalently, $G_\Gamma(s) = H_e(sI - A_e)^{-1} E_e = 0$.

Corollary of the result of (DDP): Thm.4.8

DDPM is solvable for $\Sigma = (H, C, A, B, E)$ iff there exists an A_e invariant subspace $\mathcal{V}_e$ such that $\operatorname{im} E_e \subseteq \mathcal{V}_e \subseteq \ker H_e$

Theorem 6.6+Corollary6.7

DDPM is solvable for $\Sigma = (H, C, A, B, E)$ iff $\exists$ a (C, A, B)-pair s.t.

$$\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H,$$

or, equivalently, $\mathcal{S}^*(\operatorname{im} E) \subseteq \mathcal{V}^*(\ker H)$.

disturbance decoupling with dynamic feedback

Theorem 6.6+Corollary6.7

DDPM is solvable for $\Sigma = (H, C, A, B, E)$ iff $\exists$ a (C, A, B)-pair s.t.

$$\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H,$$

or, equivalently, $\mathcal{S}^*(\operatorname{im} E) \subseteq \mathcal{V}^*(\ker H)$.

Proof.

"*Only if*": Assume the closed loop system

$$\Sigma_e : \quad \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = A_e \begin{bmatrix} x \\ w \end{bmatrix} + E_e d, \qquad y_e = H_e \begin{bmatrix} x \\ w \end{bmatrix},$$

is disturbance decoupled $\Rightarrow \exists$ A_e-inv. $\mathcal{V}_e$ s.t. $\operatorname{im} E_e \subseteq \mathcal{V}_e \subseteq \ker H_e$,

Let $\mathcal{S} := i(\mathcal{V}_e)$, $\mathcal{V} := p(\mathcal{V}_e) \overset{Thm.6.2}{\Rightarrow} (\mathcal{S}, \mathcal{V})$ is a (C, A, B)-pair.

Let $x \in \operatorname{im} E \Rightarrow \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \operatorname{im} E_e \subseteq \mathcal{V}_e \Rightarrow x \in i(\mathcal{V}_e) = \mathcal{S} \Rightarrow \operatorname{im} E \subseteq \mathcal{S}$.

Let $x \in \mathcal{V} = p(\mathcal{V}_e) \Rightarrow \exists w \in \mathcal{W} : \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{V}_e \subseteq \ker H_e \Rightarrow Hx = H_e \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = 0 \Rightarrow x \in \ker H$. □

disturbance decoupling with dynamic feedback

Theorem 6.6+Corollary6.7

DDPM is solvable for $\Sigma = (H, C, A, B, E)$ iff $\exists$ a (C, A, B)-pair s.t.

$$\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H,$$

or, equivalently, $\mathcal{S}^*(\operatorname{im} E) \subseteq \mathcal{V}^*(\ker H)$.

Proof.

"*If*": $\exists$ a (C, A, B)-pair s.t. $\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H$, $\overset{Thm6.4}{\Rightarrow}$ $\exists \Gamma = (K, L, M, N)$ and A_e-inv. $\mathcal{V}_e$ with $\mathcal{S} = i(\mathcal{V}_e)$ and $\mathcal{V} = p(\mathcal{V}_e)$.

We claim that $\operatorname{im} E_e \subseteq \mathcal{V}_e \subseteq \ker H_e$.

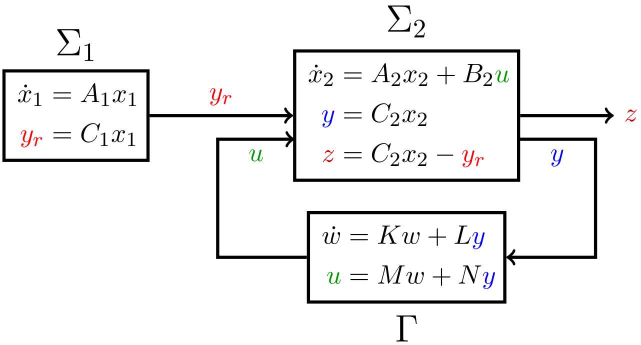
Let $\begin{bmatrix} x \\ w \end{bmatrix} \in \operatorname{im} E_e \Rightarrow w = 0$ and $x \in \operatorname{im} E \subseteq \mathcal{S} = i(\mathcal{V}_e) \Rightarrow \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{V}_e$.

Let $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{V}_e \Rightarrow x \in \mathcal{V} \subseteq \ker H \Rightarrow H_e \begin{bmatrix} x \\ w \end{bmatrix} = Hx = 0 \Rightarrow \begin{bmatrix} x \\ w \end{bmatrix} \in \ker H_e$.

Thus the claim is true and by Thm 4.6, Σ_e is disturbance decoupled. $\square$

9.1 The regulator problem

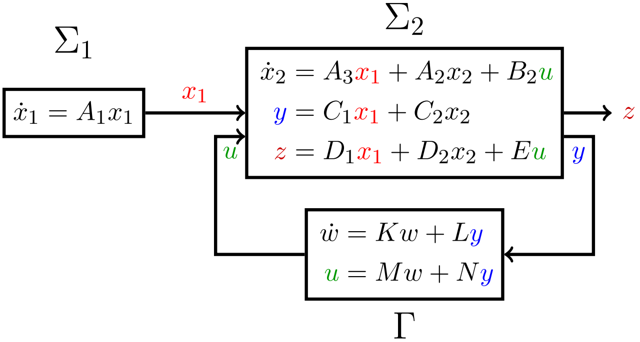
Tracking Problem:



Goal: Find $\Gamma = (K, L, M, N)$: $\lim_{t\to\infty} y(t) - \lim_{t\to\infty} y_r(t) = 0$ $(\Leftrightarrow \lim_{t\to\infty} z(t) = 0)$

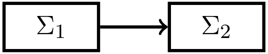
9.1 The regulator problem

Output regulation



Goal: Find $\Gamma = (K, L, M, N)$: $\Leftrightarrow \lim_{t\to\infty} z(t) = 0,\ \forall x_1(0)$

9.1 The regulator problem

- Consider the cascade system: 

, where

$$\Sigma_1 : \dot{x}_1 = A_1 x_1, \quad \Sigma_2 : \begin{cases} \dot{x}_2 = A_3 x_1 + A_2 x_2 + B_2 u \\ y = C_1 x_1 + C_2 x_2 \\ z = D_1 x_1 + D_2 x_2 + E u \end{cases}$$

- The overall system is $\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ z = Dx + Eu \end{cases}$ with $A = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$,

  $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, $D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$

Definition (Regulator Problem)

Find $\Gamma = (K, L, M, N)$ such that closed loop system satisfies

(i) $z(t) \to 0$ as $t \to \infty$

(ii) closed loop is endostable, i.e. for $x_1(0) = 0$, all variables converge to zero (Σ_2 is internally stable).

9.1 The regulator problem

Lemma (9.1)

Consider Σ with A_2 being Hurwitz and $u = 0$. Then $z(t) \to 0$ as $t \to \infty$ if $\exists T : \mathcal{X}_1 \to \mathcal{X}_2$

$$\begin{cases} TA_1 - A_2T = A_3 \\ D_2T + D_1 = 0. \end{cases} \tag{1}$$

If A_1 is antistable (i.e., $\sigma(A_1) \cap \mathbb{C}_{Re<0} = \varnothing$), then the solvability of (1) is also necessary.

Proof.

Necessity: assume A_1 is antistable $\Rightarrow \sigma(A_1) \cap \sigma(A_2) = \varnothing$.
Hence Sylvester's Theorem ensures the existence of (an unique) T such that
$TA_1 - A_2T = A_3$.
Let $v = x_2 - Tx_1$, then

$$z = D_1x_1 + D_2x_2 \Rightarrow z = D_2v + (D_1 + D_2T)x_1.$$

□

Proof of Lemma 9.1

Proof of Lemma 9.1 continue.

Observe that

$$\dot{v} = A_2 v + \overbrace{(A_2 T - T A_1 + A_3) x_1}^{=0} \Rightarrow \lim_{t\to\infty} v(t) = 0$$

Thus

$$\lim_{t\to\infty} z(t) = 0 \Rightarrow \lim_{t\to\infty} (D_1 + D_2 T) x_1(t) = z(t) - D_2 v(t) = 0 \Rightarrow D_1 + D_2 T = 0.$$

"$\Rightarrow$" because $\lim_{t\to\infty} x_1(t) \neq 0$ by A_1 is antistable.

Sufficiency: Assume (1) holds. Let $v = x_2 - T x_1$, then

$$z = D_2 v + (D_1 + D_2 T) x_1 = D_2 v + 0.$$

Thus $\dot{v} = A_2 v + \underbrace{(A_2 T - T A_2 + A_3)}_{=0} x_1 \Rightarrow \lim_{t\to\infty} v(t) = 0 \Rightarrow \lim_{t\to\infty} z(t) = 0.$ □

9.1 The regulator problem

- Next goal: Find $\Gamma = (K, L, M, N)$ such that conditions of Lemma 9.1 satisfied for closed loop:

$$\Sigma_{ce}: \begin{cases} \dot{x}_1 = A_1 x_1 \\ \dot{x}_2 = (A_2 + B_2 N C_2) x_2 + (A_3 + B_2 N C) x_1 + B_2 M w \\ \dot{w} = K w + L C_1 x_1 + L C_2 x_2 \\ z = (D_1 + E N C_1) x_1 + (D_2 + E N C_2) x_2 + E M w \end{cases}$$

- or equivalently

$$\begin{cases} \dot{x}_1 = A_1 x_1 \\ \dot{x}_{2,e} = A_{2,e} x_{2,e} + A_{3,e} x_1 \\ z = D_{1,e} x_1 + D_{2,e} x_{2,e} \end{cases}$$

- with

$$x_{2,e} = \begin{bmatrix} x_2 \\ w \end{bmatrix}, A_{2,e} = \begin{bmatrix} A_2 + B_2 N C_2 & B_2 M \\ L C_2 & K \end{bmatrix}, A_{3,e} = \begin{bmatrix} A_3 + B_2 N C_1 \\ L C_1 \end{bmatrix}$$

$$D_{2,e} = \begin{bmatrix} D_2 + E N C_2 & E M \end{bmatrix} \quad D_{1,e} = D_1 + E N C_1.$$

9.1 The regulator problem

Corollary (9.1a)

The regulator problem for Σ can be solved with controller $\Gamma = (K, L, M, N)$, if $A_{2,e}$ is Hurwitz and $\exists\, T_e : \mathcal{X}_1 \to \mathcal{X}_2 \times \mathcal{W}$ s.t.

$$\begin{cases} T_e A_1 - A_{2,e} T_e = A_{3,e} \\ D_{2,e} T_e + D_{1,e} = 0 \end{cases} \qquad (2)$$

Lemma (9.1b)

$\exists \Gamma = (W, K, M, N)$: *equation (2) is solvable iff* $\exists (T, V)$:

$$\begin{cases} T A_1 - A_2 T - B_2 V = A_3 \\ D_1 + D_2 T + E V = 0 \end{cases}$$

Lemma (9.1b)

$\exists \Gamma = (K, L, M, N)$: *equation (2) is solvable iff* $\exists(T, V)$:

$$\begin{cases} TA_1 - A_2T - B_2V = A_3 \\ D_1 + D_2T + EV = 0 \end{cases} \tag{3}$$

Proof.

Only if. Let $T_e = \left[\begin{smallmatrix} T \\ U \end{smallmatrix}\right]$ be a solution of (2). Then $T_eA_1 - A_{2,e}T_e = A_{3,e} \Rightarrow$

$$\begin{aligned} &TA_1 - (A_2 + B_2NC_2)T - B_2MU = A_3 + B_2NC_1 \\ &\Leftrightarrow TA_1 - A_2T - B_2 \underbrace{(NC_2T + MU + NC_1)}_{=:V} = A_3 \end{aligned}$$

$$\begin{aligned} 0 = D_{2,e}T_e + D_{1,e} &= (D_2 + ENC_2)T + EMU + D_1 + ENC_1 \\ &= D_1 + D_2T + E \underbrace{(NC_2T + MU + NC_1)}_{=V} \end{aligned}$$

□

Proof of Lemma 9.1 b continue.

If. Let (T, V) solve (3), choose $K = A + GC + BF$, $L = -G$, $M = F$, $N = 0$, i,e,

$$\Gamma : \begin{cases} \dot{w} = (A + GC + BF)w - Gy \\ u = Fw, \end{cases}$$

where $F = [-F_2 T + V \quad F_2]$, F_2 be any and $T_e = \begin{bmatrix} T \\ U \end{bmatrix}$ $U = \begin{bmatrix} I \\ T \end{bmatrix}$, then

$$T_e A_1 - A_{2,e} T_e = \begin{bmatrix} T \\ U \end{bmatrix} A_1 - \begin{bmatrix} A_2 & B_2 F \\ -GC_2 & A + GC + BF \end{bmatrix} \begin{bmatrix} T \\ U \end{bmatrix} = \begin{bmatrix} TA_1 - A_2 T - B_2 [-F_2 T + V \;\; F_2] \begin{bmatrix} I \\ T \end{bmatrix} \\ UA_1 + GC_2 T - (A + GC + BF)U \end{bmatrix}$$

$$= \begin{bmatrix} A_3 \\ A_1 + G_1 C_2 T - A_1 - G_1 C_1 - G_1 C_2 T \\ TA_1 + G_2 C_2 T - A_3 - A_2 T - G_2 C_1 - G_2 C_2 T - B_2 V \end{bmatrix} = \begin{bmatrix} A_3 \\ -G_1 C_1 \\ -G_2 C_1 \end{bmatrix} = \begin{bmatrix} A_3 \\ -GC_1 \end{bmatrix} = A_{3,e},$$

$$D_{2,e} T_e + D_{1,e} = D_1 + \begin{bmatrix} D_2 & EF \end{bmatrix} \begin{bmatrix} T \\ U \end{bmatrix} = D_1 + D_2 T + EV = 0$$

□