university of groningen
faculty of science and engineering
bernoulli institute for mathematics, computer science and artificial intelligence

Lecture Course: Advanced Systems Theory

Chapter 6-Lecture 7: (C, A, B)-pairs and DDP by dynamical feedback

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Recapitulation-4.4 Controllability subspace

Definition (4.11)

Consider $\Sigma : \dot{x} = Ax + Bu$. A subspace $\mathcal{R} \subseteq$ is called a controllability subspace of Σ if

$$\forall x_0 \in \mathcal{R}, \exists T > 0, u \in \boldsymbol{U} : x_u(t, x_0) \in \mathcal{R}, \forall\ 0 \leq t \leq T \quad \text{and} \quad x_u(T, x_0) = 0.$$

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Theorem (4.12)

A subspace $\mathcal{R}$ is a controllability subspace iff $\exists\, F : \mathcal{U} \to \mathcal{X}$, $L : \mathcal{U} \to \mathcal{U}$ s.t.
$\mathcal{R} = \langle A + BF | \operatorname{im} BL \rangle$.

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Question 1

Which one of the following is not necessarily a controllability subspace of $\Sigma = (A, B)$?
(i) $\langle A|B \rangle$. (ii) $\{0\}$. (iii) $\mathcal{X}$. (iv) $\langle A| \operatorname{im} B \cap \mathcal{W} \rangle$, where $\mathcal{W}$ is the reachable space of Σ.

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Recapitulation-Controllability subspace within a subspace

Theorem (4.15)

$\mathcal{R}^*(\mathcal{K})$ *is the largest controllability subspace contained in* $\mathcal{K}$, *i.e.,*

(i) $\mathcal{R}^*(\mathcal{K})$ *is a controllability subspace.*

(ii) $\mathcal{R}^*(\mathcal{K}) \subseteq \mathcal{K}$.

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Theorem (4.17)

Let $\mathcal{K} \subseteq \mathcal{X}$ *be a subspace. Then any* $F : \mathcal{X} \to \mathcal{U}$ *s.t.* $(A + BF)\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{V}^*(\mathcal{K})$ *satisfies* $(A + BF)\mathcal{R}^*(\mathcal{K}) \subseteq \mathcal{R}^*(\mathcal{K})$ *and* $\mathcal{R}^*(\mathcal{K}) = \langle A + BF \mid \operatorname{im} B \cap \mathcal{V}^*(\mathcal{K}) \rangle$.

Questions

Consider $\Sigma = (A, B, C)$, where $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, i.e.,

$$\Sigma : \begin{cases} \dot{x}_1 = x_1 + x_2 + u_1 \\ \dot{x}_2 = -x_1 + x_2 \\ \dot{x}_3 = u_2 \end{cases}, \quad y = x_1,$$

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Which is $\mathcal{V}^*(\ker C)$? (i) $\mathcal{X}_1 \times \mathcal{X}_2$, (ii) $\mathcal{X}_2 \times \mathcal{X}_3$, (iii) $\mathcal{X}_2$, (iv) $\mathcal{X}_3$.

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For $\Sigma = (A, B)$ and $\mathcal{K} \subseteq K$, then

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Conditioned invariant subspaces

Definition (Conditioned invariant subspaces)

Consider the system $\Sigma = (C, A)$

$$\begin{aligned}\dot{x}(t) &= Ax(t)\\ y(t) &= Cx(t)\end{aligned}$$

A subspace $\mathcal{S} \subseteq \mathbb{R}^n$ is called conditioned invariant if there exists $G \in \mathbb{R}^{n\times p}$ such that $(A + GC)\mathcal{S} \subseteq \mathcal{S}$

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A subspace $\mathcal{S} \subseteq \mathbb{R}^n$ is conditioned invariant iff $A(\mathcal{S} \cap \ker C) \subseteq \mathcal{S}$.

Theorem (5.6 Duality between controlled and conditioned invariant)

A subspace $\mathcal{S}$ is (C, A)-invariant iff $\mathcal{S}^\perp$ is (A^T, C^T)-invariant.

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Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}^T$, $C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$ and $\mathcal{S} = \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, is $\mathcal{S}$ a conditioned invariance of (C, A) ?

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Which G satisfies that $(A + GC)\mathcal{S} \subseteq \mathcal{S}$?

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Theorem (5.7 The smallest conditioned invariant subspace containing a given subspace)

Consider the system $\Sigma = (C,A)$. Let $\mathcal{E} \subseteq \mathbb{R}^n$ be a subspace. Then

$$\mathcal{S}^*(\mathcal{E},C,A) = (\mathcal{V}^*(A^T,C^T,\mathcal{E}^\perp))^\perp$$

Invariant subspace algorithm for $\mathcal{S}^*(\mathcal{E}, C, A)$

ISA Algorithm (Invariant subspace algorithm for controlled invariance)

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $\mathcal{K} \subseteq \mathbb{R}^n$, define

$$\begin{cases} \mathcal{V}_0 := \mathcal{K}, \\ \mathcal{V}_{k+1} := \mathcal{K} \cap A^{-1}(\mathcal{V}_k + \operatorname{im} B), \quad k = 0, 1, 2, \cdots \end{cases}$$

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Given $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, and $\mathcal{E} \subseteq \mathbb{R}^n$, define

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Theorem (4.10 Invariant subspace algorithm for controlled invariance)

Let $\mathcal{K} \subseteq X$ and $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \ldots$ as defined in the Algorithm (ISA). Then

(i) $\mathcal{V}_0 \supseteq \mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \cdots$, *(non-increasing)*

(ii) $\mathcal{V}_k = \mathcal{V}_{k+1} \Rightarrow \mathcal{V}_k = \mathcal{V}_l,\ \forall l \geq k$, *(stable)*

(iii) $\exists\, k \leq \dim \mathcal{K} : \mathcal{V}_k = \mathcal{V}_{k+1}$, *(stable index)*

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Let $\mathcal{S}_\ell$ be defined by the Algorithm (CISA). Then, we have:

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Let $\mathcal{S}_\ell$ be defined by the Algorithm (CISA). Then, we have:

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Invariant subspace algorithm for $\mathcal{S}^*(\mathcal{E}, C, A)$

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6.1 (C, A, B)-pairs

- Consider a system $\Sigma = (C, A, B)$ with $x(t) \in \mathcal{X}$:

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- The closed loop system is $\Sigma_e = (C_e, A_e)$ with $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{X} \times \mathcal{W}$:

$$\Sigma_e : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\ y_e = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \end{cases}$$

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Theorem (6.2)

Consider a subspace $\mathcal{V}_e \subseteq \mathcal{X} \times \mathcal{W}$ and let

$$p(\mathcal{V}_e) := \{x \in \mathcal{X} \mid \exists w \in \mathcal{W} : \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{V}_e\} \quad \textit{(projection)}$$
$$i(\mathcal{V}_e) := \{x \in \mathcal{X} \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{V}_e\}. \quad \textit{(intersection)}$$

If $\mathcal{V}_e$ is $A_e = \begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}$-inv. then $(i(\mathcal{V}_e), p(\mathcal{V}_e))$ is a (C, A, B)-pair.

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For $x \in p(\mathcal{V}_e)$, choose $w \in \mathcal{W}$ s.t. $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{V}_e$. Then $A_e \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} Ax + B(NCx + Mw) \\ LCx + Kw \end{bmatrix} \in \mathcal{V}_e$

$$\Rightarrow Ax + B(NCx + Mw) \in p(\mathcal{V}_e)$$

6.1 (C, A, B)-pairs

Theorem (6.2)

Consider a subspace $\mathcal{V}_e \subseteq \mathcal{X} \times \mathcal{W}$ and let

$$p(\mathcal{V}_e) := \{x \in \mathcal{X} \mid \exists w \in \mathcal{W} : \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{V}_e\}, \qquad i(\mathcal{V}_e) := \{x \in \mathcal{X} \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{V}_e\}.$$

If $\mathcal{V}_e$ is A_e-inv. then $(i(\mathcal{V}_e), p(\mathcal{V}_e))$ is a (C, A, B)-pair.

Proof.

Clearly, $i(\mathcal{V}_e) \subseteq p(\mathcal{V}_e)$. Let $x \in i(\mathcal{V}_e) \cap \ker C$. Then

$$A_e \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Ax + BNCx \\ LCx \end{bmatrix} = \begin{bmatrix} Ax \\ 0 \end{bmatrix} \in \mathcal{V}_e \Rightarrow Ax \in i(\mathcal{V}_e) \Rightarrow i(\mathcal{V}_e) \text{ is } (C, A)\text{-inv}$$

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$$\Rightarrow Ax + B(NCx + Mw) \in p(\mathcal{V}_e) \Rightarrow Ax \in p(\mathcal{V}_e) + \operatorname{im} B \Rightarrow p(\mathcal{V}_e) \text{ is } (A, B)\text{-inv.}$$

6.1 (C, A, B)-pairs

Lemma (6.3)

If $(\mathcal{S}, \mathcal{V})$ *is a* (C, A, B)*-pair, then* $\exists$ *linear* $N : \mathcal{Y} \to \mathcal{U}$ *s.t.* $(A + BNC)\mathcal{S} \subseteq \mathcal{V}$.

Proof.

□

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Proof.

Let $q_1, q_2, \ldots, q_k$ a basis of $\mathcal{S}$ s.t. $q_1, \ldots, q_l$ is a basis of $\mathcal{S} \cap \ker C$ $(l \leq k)$. Then

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$\mathcal{S} \subseteq \mathcal{V}$ and $\mathcal{V}$ is (A, B)-inv. $\Rightarrow$ $Aq_i = v_i + Bu_i$, for $v_i \in \mathcal{V}$ and $u_i \in \mathcal{U}$, for $i = 1, \ldots, k$.

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We claim that $Cq_{l+1}, Cq_{l+2}, \ldots, Cq_k$ are linear independent.

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$$\alpha_{l+1}Cq_{l+1} + \alpha_{l+2}Cq_{l+2} + \cdots + \alpha_k Cq_k = 0.$$

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Thus $\tilde{q} := \alpha_{l+1}q_{l+1} + \alpha_{l+2}q_{l+2} + \cdots + \alpha_k q_k \in \mathcal{S} \cap \ker C$.

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$$\mathcal{S} \cap \ker C \cap \operatorname{span}\{q_{l+1}, \ldots, q_k\} = \{0\} \Rightarrow \tilde{q} = 0 \overset{q_{l+1},\ldots,q_k \text{ independent}}{\Rightarrow} \alpha_{l+1}, \ldots, \alpha_k = 0,$$

which proves that the claim is true. $\square$

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If $(\mathcal{S}, \mathcal{V})$ is a (C, A, B)-pair, then $\exists$ linear $N : \mathcal{Y} \to \mathcal{U}$ s.t. $(A + BNC)\mathcal{S} \subseteq \mathcal{V}$.

Proof.

Let $q_1, q_2, \dots, q_k$ a basis of $\mathcal{S}$ s.t. $q_1, \dots, q_l$ is a basis of $\mathcal{S} \cap \ker C$ $(l \leq k)$. Then $\mathcal{S} \subseteq \mathcal{V}$ and $\mathcal{V}$ is (A, B)-inv. $\Rightarrow$ $Aq_i = v_i + Bu_i$, for $v_i \in \mathcal{V}$ and $u_i \in \mathcal{U}$, for $i = 1, \dots, k$. We claim that $Cq_{l+1}, Cq_{l+2}, \dots, Cq_k$ are linear independent. Suppose $\exists$ non-zero $\alpha_{l+1}, \dots, \alpha_k$ s.t.

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which proves that the claim is true.

Therefore, $\exists N : \mathcal{Y} \to \mathcal{U}$ with $NCq_i = -u_i$, $i = l+1, \dots, k$ $\Rightarrow$ $(A + BNC)q_i = v_i \in \mathcal{V}$, $i = l+1, \dots, k \Rightarrow (A + BNC)\mathcal{S} \subseteq \mathcal{V}$. □

6.1 (C, A, B)-pairs

Theorem 6.4 (using (C, A, B) pairs to construct Γ)

Let $(\mathcal{S}, \mathcal{V})$ be a (C, A, B)-pair. Then there exists controller Γ and an A_e-invariant subspace $\mathcal{V}_e \subseteq \mathcal{X} \times \mathcal{W}$ s.t. $\mathcal{S} = i(\mathcal{V}_e)$ and $\mathcal{V} = p(\mathcal{V}_e)$.
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- $N : \mathcal{Y} \to \mathcal{U}$ s.t. $(A + BNC)\mathcal{S} \subseteq \mathcal{V}$,
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- $N : \mathcal{Y} \to \mathcal{U}$ s.t. $(A + BNC)\mathcal{S} \subseteq \mathcal{V}$,
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Then Γ is given by

$$\begin{cases} \dot{w} = (A + BF + GC - BNC)w + (BN - G)y \\ u = (F - NC)w + Ny, \end{cases}$$

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Then Γ is given by

$$\begin{cases} \dot{w} = (A + BF + GC - BNC)w + (BN - G)y \\ u = (F - NC)w + Ny, \end{cases}$$

where $\mathcal{W} = \mathcal{X}$ and $\mathcal{V}_e = \{\left[\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} x_2 \\ x_2 \end{smallmatrix}\right] \mid x_1 \in \mathcal{S}, x_2 \in \mathcal{V}\}$

Summary

- Invariant subspace algorithm for $\mathcal{S}^*(\mathcal{E}, C, A)$, characterization (Thm 5.8). Notice its duality with $\mathcal{V}^*(\mathcal{K}, A, B)$.

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Summary

- Invariant subspace algorithm for $\mathcal{S}^*(\mathcal{E}, C, A)$, characterization (Thm 5.8). Notice its duality with $\mathcal{V}^*(\mathcal{K}, A, B)$.
- Dynamic feedback controller Γ
- (C, A, B)-pairs: definition (Def 6.1), constructing (C, A, B)-pairs from A_e-inv.:$\mathcal{V}_e$ (Thm 6.2)
- Using (C, A, B)-pairs to construct dynamic feedback controller Γ. (Thm 6.4)

disturbance decoupling with dynamic feedback

Problem (DDP with dynamic measurement feedback (DDPM))

Given the system $\Sigma = (H, C, A, B, E)$

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed \\ y &= Cx \\ z &= Hx \end{aligned}$$

find K, L, M, N such that the dynamic controller $\Gamma(M, K, L, N)$

$$\begin{aligned} \dot{w} &= Kw + Ly \\ u &= Mw + Ny \end{aligned}$$

renders the closed loop system disturbance decoupled:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}}_{A_e} \begin{bmatrix} x \\ w \end{bmatrix} + \underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix}}_{E_e} d \qquad z = \underbrace{\begin{bmatrix} H & 0 \end{bmatrix}}_{H_e} \begin{bmatrix} x \\ w \end{bmatrix}$$

disturbance decoupling with dynamic feedback

- Closed loop system:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}}_{A_e} \begin{bmatrix} x \\ w \end{bmatrix} + \underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix}}_{E_e} d \qquad z = \underbrace{\begin{bmatrix} H & 0 \end{bmatrix}}_{H_e} \begin{bmatrix} x \\ w \end{bmatrix}$$

Definition 6.5 DDPM

Find $\Gamma = (K, L, M, N)$ s.t.

$$T_{\Gamma(t)} := H_e e^{A_e t} E_e = 0, \;\; \forall t \geq 0$$

or, equivalently, $G_\Gamma(s) = H_e(sI - A_e)^{-1} E_e = 0$.

Corollary of the result of (DDP): Thm.4.8

DDPM is solvable for $\Sigma = (H, C, A, B, E)$ iff there exists an A_e invariant subspace $\mathcal{V}_e$ such that $\operatorname{im} E_e \subseteq \mathcal{V}_e \subseteq \ker H_e$

disturbance decoupling with dynamic feedback

Theorem 6.6+Corollary6.7

DDPM is solvable for $\Sigma = (H, C, A, B, E)$ iff $\exists$ a (C, A, B)-pair s.t.

$$\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H,$$

or, equivalently, $\mathcal{S}^*(\operatorname{im} E) \subseteq \mathcal{V}^*(\ker H)$.

Proof.

"*If*": Assume the closed loop system

$$\Sigma_e : \quad \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = A_e \begin{bmatrix} x \\ w \end{bmatrix} + E_e d, \qquad y_e = H_e \begin{bmatrix} x \\ w \end{bmatrix}.$$

is disturbance decoupled $\Rightarrow \exists$ A_e-inv. $\mathcal{V}_e$ s.t. $\operatorname{im} E_e \subseteq \mathcal{V}_e \subseteq \ker H_e$,

Let $\mathcal{S} := i(\mathcal{V}_e)$, $\mathcal{V} := p(\mathcal{V}_e)$ $\overset{Thm.6.2}{\Rightarrow}$ $(\mathcal{S}, \mathcal{V})$ is a (C, A, B)-pair.

Let $x \in \operatorname{im} E \Rightarrow \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \operatorname{im} E_e \subseteq \mathcal{V}_e \Rightarrow x \in i(\mathcal{V}_e) = \mathcal{S} \Rightarrow \operatorname{im} E \subseteq \mathcal{S}$.

Let $x \in \mathcal{V} = p(\mathcal{V}_e) \Rightarrow \exists w \in \mathcal{W} : \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{V}_e \subseteq \ker H_e \Rightarrow Hx = H_e \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = 0 \Rightarrow x \in \ker H$. □

disturbance decoupling with dynamic feedback

Theorem 6.6+Corollary6.7

DDPM is solvable for $\Sigma = (H, C, A, B, E)$ iff $\exists$ a (C, A, B)-pair s.t.

$$\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H,$$

or, equivalently, $\mathcal{S}^*(\operatorname{im} E) \subseteq \mathcal{V}^*(\ker H)$.

Proof.

"*Only if*": exists a (C, A, B)-pair s.t. $\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H$, $\overset{Thm 6.4}{\Rightarrow} \exists \Gamma = (K, L, M, N)$ and A_e-inv. $\mathcal{V}_e$ with $\mathcal{S} = i(\mathcal{V}_e)$ and $\mathcal{V} = p(\mathcal{V}_e)$. We claim that $\operatorname{im} E_e \subseteq \mathcal{V}_e \subseteq \ker H_e$.
Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \operatorname{im} E_e \Rightarrow w = 0$ and $x \in \operatorname{im} E \subseteq \mathcal{S} = i(\mathcal{V}_e) \Rightarrow \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{V}_e$.
Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{V}_e \Rightarrow x \in \mathcal{V} \in \ker H \Rightarrow H_e \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = Hx = 0 \Rightarrow \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \ker H_e$.
Thus the claim is true and by Thm 4.6, Σ_e is disturbance decoupled. $\square$