Lecture Course: Advanced Systems Theory

Chapter 4 and 5-Lecture6: Controllability subspace and conditioned invariance

Stephan Trenn, Yahao Chen

Jan C. Willems Center for Systems and Control

University of Groningen, Netherlands

September 2020

Recapitulation-4.2 Disturbance decoupling problem (DDP)

Theorem (4.8)

Given a system $\Sigma_{u,d,z} = (A, B, E, H)$, The DDP is solvable for $\Sigma_{u,d,z}$ iff there exists an (A, B)-invariant subspace $\mathcal{V}$ s.t. $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$.

Question 1

Consider a system $\Sigma = (A, B, E, H)$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e \\ f \end{bmatrix}$, $E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$, for which one of the following that the DDP is not solvable ?

(i) $b = 0$, $e = 1$. (ii) $b = 1$, $e = 1$. (iii) $b = 0$, $e = 0$. (iv) $b = 1$, $e = 0$.

Corollary 4.9

The DDP is solvable iff $\operatorname{im} E \subseteq \mathcal{V}^*(\ker H)$.

Recapitulation-4.3 The invariant subspace algorithm (ISA)

Algorithm

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $\mathcal{K} \subseteq \mathbb{R}^n$, define

$$\begin{cases} \mathcal{V}_0 := \mathcal{K}, \\ \mathcal{V}_{k+1} := \mathcal{K} \cap A^{-1}(\mathcal{V}_k + \operatorname{im} B), \quad k = 1, 2, \cdots \end{cases}$$

Theorem (4.10)

Let $\mathcal{K} \subseteq X$ and $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \ldots$ as defined in the above algorithm. Then

(i) $\mathcal{V}_0 \supseteq \mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \cdots$, *(non-increasing)*

(ii) $\exists\, k \leq \dim \mathcal{K} : \mathcal{V}_k = \mathcal{V}_{k+1}$, *(stable index)*

(iii) $\mathcal{V}_k = \mathcal{V}_{k+1} \Rightarrow \mathcal{V}_k = \mathcal{V}_l,\ \forall l \geq k$, *(stable)*

(iv) $\mathcal{V}_k = \mathcal{V}_{k+1} \Rightarrow \mathcal{V}^*(\mathcal{K}) = \mathcal{V}_k$. *(limit)*

Recapitulation-4.3 The invariant subspace algorithm (ISA)

Algorithm

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $\mathcal{K} \subseteq \mathbb{R}^n$, define

$$\begin{cases} \mathcal{V}_0 & := \mathcal{K}, \\ \mathcal{V}_{k+1} & := \mathcal{K} \cap A^{-1}(\mathcal{V}_k + \operatorname{im} B), \quad k = 1, 2, \cdots \end{cases}$$

Question 2

If $\mathcal{K} = \mathbb{R}^n$, then which one is correct?

(i) $\mathcal{V}^*(\mathcal{K}) = 0$.

(ii) $\mathcal{V}^*(\mathcal{K}) = \mathbb{R}^n$.

(iii) $\mathcal{V}^*(\mathcal{K}) = A^{-1} \operatorname{im} B$.

Question 3

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, then $\mathcal{V}^*(\ker C) =?$

(i) $\ker C$.

(ii) $\{0\}$.

(iii) $A^{-1} \ker C$.

Question 4

In Question 3, what is the smallest interger k such that $\mathcal{V}_k = \mathcal{V}_{k+1}$?

(i) $k = 0$.

(ii) $k = 1$.

(iii) $k = 2$.

4.4 Controllability subspace

Definition (4.11)

Consider $\Sigma : \dot{x} = Ax + Bu$. A subspace $\mathcal{R} \subseteq$ is called a controllability subspace of Σ if

$$\forall x_0 \in \mathcal{R}, \exists T > 0, u \in \boldsymbol{U} : x_u(t, x_0) \in \mathcal{R}, \forall 0 \leq t \leq T \text{ and } x_u(T, x_0) = 0.$$

(i)$\mathcal{R}$ is a controllability space $\Rightarrow$ $\mathcal{R}$ is controlled invariant subspace (just choose $u(t) = 0$, $\forall t \geq T$).

(ii)$\langle A|B\rangle = \operatorname{im}[B, AB, \dots, A^{n-1}B]$ is a controllability subspace (actually the largest possible)

Theorem (4.12)

A subspace $\mathcal{R}$ is a controllability subspace iff $\exists\, F : \mathcal{U} \to \mathcal{X}$, $L : \mathcal{U} \to \mathcal{U}$ s.t. $\mathcal{R} = \langle A + BF | \operatorname{im} BL\rangle$.

How F and L define a feedback transformation?

4.4 Controllability subspace

Theorem (4.12)

A subspace $\mathcal{R}$ is a controllability subspace iff $\exists\, F : \mathcal{U} \to \mathcal{X}$, $L : \mathcal{U} \to \mathcal{U}$ *s.t.*
$\mathcal{R} = \langle A + BF | \operatorname{im} BL \rangle$.

Proof.

Only if. Let $F : (A + BF)\mathcal{R} \subseteq \mathcal{R}$ ($\mathcal{R}$ is an (A, B)-invariant subspace) and $\operatorname{im} L = B^{-1}\mathcal{R}$.
$\Rightarrow$ $u(t)$ (which renders $x_u(x_0, t) \in \mathcal{R}$) is of the form $u(t) = Fx(t) + Lw(t)$ for some $w \in \boldsymbol{U}$
(Note that $x(t) = e^{A_F t} x_0 + \int_0^t BLw(\tau)d\tau \in \mathcal{R}$).
Then consider

$$\dot{\bar{x}} = (A + BF)\bar{x} + BLw, \quad \bar{x}(0) = x_0, \quad \bar{x}(T) = 0 \qquad (*)$$

$\mathcal{R}$ is controllability subspace $\Rightarrow$
$\forall x_0 \in \mathcal{R}, \exists T > 0, u \in \boldsymbol{U} : x_u(t, x_0) \in \mathcal{R}, \forall t \geq 0$ and $x_u(T, x_0) = 0$. $\Rightarrow$ $\mathcal{R}$ is the reachable space of (*), i.e., $\mathcal{R} = \langle A + BF | \operatorname{im} BL \rangle$.

□

4.4 Controllability subspace

Theorem (4.12)

A subspace $\mathcal{R}$ is a controllability subspace iff $\exists\, F : \mathcal{U} \to \mathcal{X}$, $L : \mathcal{U} \to \mathcal{U}$ s.t.
$\mathcal{R} = \langle A + BF | \operatorname{im} BL \rangle$.

Proof.

If. $\exists\, F : \mathcal{U} \to \mathcal{X}$, $L : \mathcal{U} \to \mathcal{U}$ s.t. $\mathcal{R} = \langle A + BF | \operatorname{im} BL \rangle$
$\Rightarrow \dot{\bar{x}} = (A + BF)\bar{x} + BLw$ is a linear system with reachable space (or null-controllable space) $\mathcal{R}$.
$\Rightarrow \forall x_0 \in \mathcal{R}$, $\exists\, w : \bar{x}_w(T, x_0) = 0$.
$\Rightarrow \exists\, u = Fx + Lw$ s.t. $\dot{x} = Ax + Bu$ can be controlled to zero while remaining in $\mathcal{R}$. □

Corollary (4.13)

$\mathcal{R} = \langle A + BF | \operatorname{im} B \cap \mathcal{R} \rangle$. *(since* $\operatorname{im} L = B^{-1}\mathcal{R} \Rightarrow \operatorname{im} BL = \operatorname{im} B \cap \mathcal{R}$*.)*

Controllability subspace within a subspace

Definition (4.14)

Let $\mathcal{K}$ be a subspace of $\mathcal{X}$, define

$$\mathcal{R}^*(\mathcal{K}) := \{x_0 \in \mathcal{K} \,|\, \exists\, u \in \boldsymbol{U}, T > 0 : x_u(t, x_0) \in \mathcal{K}, \forall\; 0 \leq t \leq T \text{ and } x_u(T, x_0) = 0.\}$$

Theorem (4.15)

$\mathcal{R}^*(\mathcal{K})$ *is the largest controllability subspace contained in* $\mathcal{K}$, *i.e.,*

(i) $\mathcal{R}^*(\mathcal{K})$ *is a controllability subspace.*

(ii) $\mathcal{R}^*(\mathcal{K}) \subseteq \mathcal{K}$.

(iii) $\mathcal{R} \subseteq \mathcal{K} \Rightarrow \mathcal{R} \subseteq \mathcal{R}^*(\mathcal{K})$

Controllability subspace within a subspace

Terminology

$\mathcal{V}^*(\mathcal{K})$ is the largest controlled invariant subspace contained in $\mathcal{K}$.
$\mathcal{R}^*(\mathcal{K})$ is the largest controllability subspace contained in $\mathcal{K}$.

Lemma 4.16

Let $\mathcal{K}$ be any subspace of $\mathcal{X}$. Then $\operatorname{im} B \cap \mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{R}^*(\mathcal{K})$.

Proof

Let L be a linear map s.t. $\operatorname{im} L = B^{-1}\mathcal{V}^*$. Then

$$\operatorname{im} B \cap \mathcal{V}^* = \operatorname{im} BL \subseteq \langle A + BF \,|\, BL \rangle \subseteq \mathcal{R}^*$$

Remark

$\operatorname{im} B \cap \mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{R}^*(\mathcal{K}) = \mathcal{R}^*(\mathcal{V}^*(\mathcal{K})) \subseteq \mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{K}$.

Controllability subspace within a subspace

Theorem (4.17)

Let $\mathcal{K} \subseteq \mathcal{X}$ be a subspace. Then any $F : \mathcal{X} \to \mathcal{U}$ s.t. $(A + BF)\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{V}^*(\mathcal{K})$ satisfies $(A + BF)\mathcal{R}^*(\mathcal{K}) \subseteq \mathcal{R}^*(\mathcal{K})$ and

$$\mathcal{R}^*(\mathcal{K}) = \langle A + BF \,|\, \operatorname{im} B \cap \mathcal{V}^*(\mathcal{K}) \rangle \,.$$

Proof.

Choose an F s.t. $(A + BF)\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{V}^*(\mathcal{K})$, since $\mathcal{R}^* \subseteq \mathcal{V}^*$, we have $(A + BF)\mathcal{R}^* \subseteq \mathcal{V}^*$.
$\mathcal{R}^*$ is controlled invariance $\Rightarrow (A + BF)\mathcal{R}^* \subseteq \mathcal{R}^* + \operatorname{im} B$
$\Rightarrow (A + BF)\mathcal{R}^* \subseteq \mathcal{V}^* \cap (\mathcal{R}^* + \operatorname{im} B) = \mathcal{R}^* + \mathcal{V}^* \cap \operatorname{im} B \overset{Lemma 4.16}{=} \mathcal{R}^*$.
Now by Thm 4.12, $\mathcal{R}^* = \langle A + BF | \operatorname{im} BL \rangle$ for $L = B^{-1}\mathcal{R}^*$ (note that $\operatorname{im} BL = \mathcal{R}^* \cap \operatorname{im} B$).
We have $\mathcal{R}^* \subseteq \mathcal{V}^*$ and $\mathcal{V}^* \cap \operatorname{im} B \subseteq \mathcal{R}^*$.
$\Rightarrow \mathcal{R}^* \cap \operatorname{im} B \subseteq \mathcal{V}^* \cap \operatorname{im} B \subseteq \mathcal{R}^* \cap \operatorname{im} B$
$\Rightarrow \operatorname{im} BL = \mathcal{V}^* \cap \operatorname{im} B$. □

Conditioned invariant subspaces

Definition (Conditioned invariant subspaces)

Consider the system $\Sigma = (C, A)$

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cx(t)$$

A subspace $\mathcal{S} \subseteq \mathbb{R}^n$ is called conditioned invariant if there exists $G \in \mathbb{R}^{n\times p}$ such that $(A + GC)\mathcal{S} \subseteq \mathcal{S}$

Remark

(i) G defines an output injection transformation: Observer problem.
(ii) $\mathcal{S}$ could be defined in terms of the existence of a certain observer ! (Definition 5.2: Does always exist an observer for $x/\mathcal{S}$?)

Theorem (5.5 Conditioned invariant subspaces)

A subspace $\mathcal{S} \subseteq \mathbb{R}^n$ is conditioned invariant iff $A(\mathcal{S} \cap \ker C) \subseteq \mathcal{S}$.

Conditioned invariant subspaces

Theorem (5.5 Conditioned invariant subspaces)

A subspace $\mathcal{S} \subseteq R^n$ *is conditioned invariant iff* $A(\mathcal{S} \cap \ker C) \subseteq \mathcal{S}$

Proof.

"*If*" Let $q_1, \ldots, q_k$ be a basis of $\mathcal{S}$ and $q_1, \ldots, q_l$ $(l \leq k)$ be a basis of $\mathcal{S} \cap \ker C$.
Choose $G : \mathcal{Y} \to \mathcal{X}$ s.t. $GCq_i = -Aq_i$ for $i = l+1, \ldots, k$.
$\Rightarrow$ $(A + GC)q_i = Aq_i \in \mathcal{S}$ for $i = 1, \ldots, l$ and $(A + GC)q_i = 0$ for $i = l+1, \ldots, k$.
Hence $(A + GC)\mathcal{S} \subseteq \mathcal{S}$.
"*Only if*" $(A + GC)\mathcal{S} \subseteq \mathcal{S} \Rightarrow (A + GC)(\mathcal{S} \cap \ker C) \subseteq \mathcal{S} \Rightarrow A(\mathcal{S} \cap \ker C) \subseteq \mathcal{S}$ ☐

Question 5

Given $A : \mathcal{X} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$, which of the following subspaces is not necessarily conditioned invariant? (i) $\mathcal{S} = \operatorname{im} A$ (ii) $\mathcal{S} = \ker A$ (iii) $\mathcal{S} = \ker C$ (iv) $\mathcal{S} = (\ker C)^\perp$

Conditioned invariant subspaces

Terminology ((A,B)-and (C,A)-invariant)

$$(A,B)\text{-invariant} \Leftrightarrow \text{controlled invariant for } \Sigma = (A,B)$$
$$(C,A)\text{-invariant} \Leftrightarrow \text{conditioned invariant for } \Sigma = (C,A)$$

Theorem (5.6 Duality between controlled and conditioned invariant)

A subspace $\mathcal{C}$ is (C,A)-invariant iff $\mathcal{S}^\perp$ is (A^T, C^T)-invariant.

Proof

This follows from the fact that $A\mathcal{V} \subseteq \mathcal{V} \Leftrightarrow A^{-T}\mathcal{V}^\perp \supseteq \mathcal{V}^\perp \Leftrightarrow A^T\mathcal{V}^\perp \subseteq \mathcal{V}^\perp$ and $\mathcal{S}^\perp$ is an (A^T, C^T)-invariant subspace $\overset{Thm 4.2}{\Leftrightarrow} \exists F : (A^T + C^T F)S^\perp \subseteq S^\perp$ $(G := F^T)$.

Conditioned invariant subspaces

Theorem (5.7 The smallest conditioned invariant subspace containing a given subspace)

Consider the system $\Sigma = (C, A)$. Let $\mathcal{E} \subseteq \mathbb{R}^n$ be a subspace. Then

$$\mathcal{S}^*(\mathcal{E}, C, A) = (\mathcal{V}^*(A^T, C^T, \mathcal{E}^\perp))^\perp$$

is the smallest conditioned invariant subspace containing $\mathcal{E}$, i.e.,

(i) $\mathcal{S}^*(\mathcal{E})$ *is conditioned invariant,*

(ii) $\mathcal{E} \subseteq \mathcal{S}^*(\mathcal{E})$,

(iii) $\mathcal{S}$ *is a conditioned invariant subspace with* $\mathcal{E} \subseteq \mathcal{S} \Rightarrow \mathcal{S}^*(\mathcal{E}) \subseteq \mathcal{S}$.

Terminology

$\mathcal{V}^*(A, B, \mathcal{K})$: the largest (A, B) – invariant subspace contained in $\mathcal{K}$

$\mathcal{S}^*(\mathcal{E}, C, A)$: the smallest (C, A) – invariant subspace containing $\mathcal{E}$

Theorem (5.7 The smallest conditioned invariant subspace containing a given subspace)

Consider $\Sigma = (C, A)$ and $\mathcal{E} \subseteq \mathcal{X}$. Then

$$\mathcal{S}^*(\mathcal{E}, C, A) = (\mathcal{V}^*(A^T, C^T, \mathcal{E}^\perp))^\perp$$

is the smallest conditioned invariant subspace containing $\mathcal{E}$.

Proof.

(i) $\mathcal{V}^*(A^T, C^T, \mathcal{E}^\perp)$ is an (A^T, C^T)-invar. $\overset{Thm.5.6}{\Rightarrow}$ $\mathcal{S}^*(\mathcal{E}, C, A)$ is a (C, A)-invar.
(ii) $\mathcal{V}^* \subseteq \mathcal{E}^\perp \Rightarrow \mathcal{E} = (\mathcal{E}^\perp)^\perp \subseteq (\mathcal{V}^*)^\perp = \mathcal{S}^*$.
(iii) Assume $\mathcal{S}$ is a (C, A)-inv. with $\mathcal{E} \subseteq \mathcal{S} \Rightarrow \mathcal{S}^\perp$ is a (A^T, C^T)-inv. with $\mathcal{S}^\perp \subseteq \mathcal{E}^\perp$.
Hence

$$\mathcal{S}^\perp \subseteq \mathcal{V}^* \Rightarrow \mathcal{S}^* = (\mathcal{V}^*)^\perp \subseteq ((\mathcal{S})^\perp)^\perp = \mathcal{S}.$$

□

Summary

- Controllability subspace: definition (Def. 4.11), characterization (Thm 4.12).
- Controllability subspace within $\mathcal{K}$: definition (Def.4.14, Thm 4.16), relations of $\mathcal{V}^*(\mathcal{K})$ and $\mathcal{R}^*(\mathcal{K})$ (Thm 4.17).
- Conditioned invariance definition, characterization (Thm 5.5)
- Duality between controlled and conditioned invariant (Thm 5.6)
- The smallest conditioned invariant subspace containing $\mathcal{E}$. (Thm 5.7)