Lecture Course: Advanced Systems Theory

Chapter 4-Lecture5: Disturbance decoupling problem

Stephan Trenn, Yahao Chen

Jan C. Willems Center for Systems and Control

University of Groningen, Netherlands

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Recapitulation-4.1 controlled invariant subspaces

- A-invariant subspace and invariant subspace of $\dot{x} = Ax$.
- Example: a subspace $\mathcal{V} \subseteq \operatorname{im} A$ is not necessarily A-invariant.
- Definition 4.1: controlled invariant subspace ((A, B)-invariant subspace)

Theorem (4.2)

The following is equivalent:
(i) $\mathcal{V}$ is controlled invariant; (ii) $A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$; (iii) $\exists\, F : (A + BF)\mathcal{V} \subseteq \mathcal{V}$.

- Calculate friend feedback F, i.e., F such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.
- $\mathcal{V}^*(\mathcal{K})$ is the largest controlled invariant subspace contained in $\mathcal{K}$.

Recapitulation-4.2 Disturbance decoupling

- Consider a control system

$$\Sigma_{d,z}: \begin{cases} \dot{x} = Ax + Ed \\ z = Hx, \end{cases}$$

- The output $z(t) = He^{At}x_0 + \int_0^t T(t-\tau)d(\tau)\mathrm{d}\tau$, where $T(t-\tau) = He^{A(t-\tau)}E$.

Definition

$\Sigma_{d,z}$ is called disturbance decoupled if $T(t) = He^{At}E = 0, \forall t \geq 0$, or if $G(s) = H(sI-A)^{-1}E = 0$.

Theorem (4.6)

$\Sigma_{d,z}$ is disturbance decoupled iff $\exists$ an A-invariant subspace $\mathcal{V}$ s.t.

$$\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H.$$

Recapitulation-Questions

Question 1

Let $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^2$ such that (A, B) is controllable, which one of the following subspaces is not (A, B)-invariant?

(i) $\operatorname{im} B$
(ii) $\operatorname{im} AB$
(iii) $\operatorname{im} A$
(iv) $\operatorname{im}(sI - A)$, $s \in \sigma(A)$.

Question 2

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathcal{V} = \operatorname{im} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, which F is a friend feedback of $(A, B, \mathcal{V})$?

(i) $F = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$
(ii) $F = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$.
(iii) $F = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$.

Question 3

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $E = \begin{bmatrix} e \\ f \end{bmatrix}$, $H = \begin{bmatrix} g & h \end{bmatrix}$, $\mathcal{V} = \operatorname{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which of the following satisfies that $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$ and $\mathcal{V}$ is A-inv.?

(i) $c = 0,\ f = 0,\ g = 0.$
(ii) $b = 0,\ e = 0,\ h = 0.$
(iii) $c = 0,\ e = 0,\ h = 0.$
(iv) $b = 0,\ f = 0,\ g = 0.$

Question 4

Except for item (ii), which system in Question 3 is also disturbance decoupled?

(i). (iii). (iv).

Disturbance decoupling problem (DDP)

Disturbance decoupling problem (DDP)

Given a system

$$\Sigma_{u,d,z}: \begin{cases} \dot{x} = Ax + Bu + Ed \\ z = Hx, \end{cases}$$

find $F : \mathcal{X} \to \mathcal{U}$ s.t. the state feedback $u = Fx$ renders the closed loop systems:

$$\Sigma_{d,z}: \begin{cases} \dot{x} = (A + BF)x + Ed \\ z = Hx, \end{cases}$$

is disturbance decoupled.

Remark: Recall from Thm 4.6 that the above system $\Sigma_{d,z}$ is disturbance decoupled iff $\exists$ an $(A + BF)$-invariant subspace $\mathcal{V}_F$ s.t. $\operatorname{im} E \subseteq \mathcal{V}_F \subseteq \ker H$.

Disturbance decoupling problem (DDP)

Disturbance decoupling problem (DDP)

Given a system $\Sigma_{u,d,z} = (A, B, E, H)$ find $F : \mathcal{X} \to \mathcal{U}$ s.t. the state feedback $u = Fx$ renders the closed loop systems $\Sigma_{d,z} = (A + BF, E, H)$ disturbance decoupled.

Q: What doe the system block diagram for DDP looks like ?

Theorem (4.8)

Given a system $\Sigma_{u,d,z} = (A, B, E, H)$*, The DDP is solvable for* $\Sigma_{u,d,z}$ *iff there exists an* (A, B)*-invariant subspace with* $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$*.*

Disturbance decoupling problem (DDP)

Theorem (4.8)

The DDP is solvable iff there exists an (A, B)-invariant subspace with $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$.

Proof.

Only if: DDP is solvable for $\Sigma_{u,d,z}$.
$\Rightarrow \exists F : \exists$ an $(A+BF)$-invariant subspace $\mathcal{V}_F$ s.t. $\operatorname{im} E \subseteq \mathcal{V}_F \subseteq \ker H$.
$\overset{Thm.4.2}{\Rightarrow} \exists$ an (A, B)-invariant subspace $\mathcal{V}$ with $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$.
If: Suppose that $\exists$ an (A, B)-invariant subspace $\mathcal{V}$ with $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$ and denote $\dim \mathcal{V} = k$.
For $\Sigma_{u,d,z} = (A, B, E, H)$, choose new coordinates

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} x = Px,$$

where $P_2 : \mathcal{X} \to \mathbb{R}^{n-k}$ s.t. $\operatorname{rk} P_2 = n - k$ and $\operatorname{im} P_2^T = \mathcal{V}^\perp$, and where $P_1 : \mathcal{X} \to \mathbb{R}^k$ is any s.t. P is invertible. □

Disturbance decoupling problem (DDP)

Proof of Thm 4.8 continue.

Then

$$\Sigma_{u,d,z}: \begin{cases} \dot{x} = Ax + Bu + Ed \\ z = Hx, \end{cases} \overset{\tilde{x}=Px}{\Rightarrow} \tilde{\Sigma}_{u,d,z}: \begin{cases} \dot{\tilde{x}} = PAP^{-1}\tilde{x} + PBu + PEd \\ z = HP^{-1}\tilde{x}. \end{cases}$$

$\mathcal{V}$ is an (A,B)-invariant subspace $\Rightarrow \exists\, F : (A+BF)\mathcal{V} \subseteq \mathcal{V} \Rightarrow$
$\exists\, F : P(A+BF)P^{-1}P\mathcal{V} \subseteq P\mathcal{V} \Rightarrow \exists\, \tilde{F} = FP^{-1} : (\tilde{A}+\tilde{B}\tilde{F})\tilde{\mathcal{V}} \subseteq \tilde{\mathcal{V}}$.
Thus the closed loop system of $\tilde{\Sigma}_{u,d,z}$ takes the form

$$\begin{cases} \dot{\tilde{x}} = PAP^{-1}\tilde{x} + PBu + PEd \\ z = HP^{-1}\tilde{x}. \end{cases} \overset{u=\tilde{F}x}{\Rightarrow} \begin{cases} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & \tilde{A}_4 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} E_1 \\ 0 \end{bmatrix} d \\ z = \begin{bmatrix} 0 & \tilde{H}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \end{cases}$$

Now it is clear that z is independent of d, i.e., $G_{dz}(s) = \tilde{H}(sI - \tilde{A})^{-1}\tilde{E} = 0$. □

Recall that $\mathcal{V}^*(\ker H)$ is the largest controlled invariant subspace contained in $\ker H$, i.e., (i) $\mathcal{V}^*$ is controlled invariant; (ii) $\mathcal{V}^* \subseteq \ker H$; (iii) any other $\mathcal{V} : A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B \Rightarrow \mathcal{V} \subseteq \mathcal{V}^*$.

Corollary 4.9

The DDP is solvable iff $\operatorname{im} E \subseteq \mathcal{V}^*(\ker H)$.

Proof.

Only if. DDP is solvable $\overset{Thm 4.8}{\Rightarrow} \exists\ (A, B)$-inv. $\mathcal{V} : \operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$ and
$\mathcal{V} \subseteq \mathcal{V}^*(\ker H) \subseteq \ker H$ (by Thm 4.5)
$\Rightarrow \operatorname{im} E \subseteq \mathcal{V}^*(\ker H)$.

If. Take $\mathcal{V} = \mathcal{V}^*(\ker H)$, we have $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H \overset{Thm 4.8}{\Rightarrow}$ DDP is solvable . □

4.3 The invariant subspace algorithm (ISA)

- Consider $\Sigma : \dot{x} = Ax + Bu$, given a subspace $\mathcal{K} \subseteq \mathcal{X}$, find $\mathcal{V}^*(\mathcal{K})$.
- Recall definition of inverse image/preimage: $A^{-1}\mathcal{V} := \{x \mid Ax \in \mathcal{V}\}$.

Question 5a

Which one of the following is correct?
(i) $AA^{-1}\mathcal{V} = \mathcal{V}$. (ii) $AA^{-1}\mathcal{V} \subseteq \mathcal{V}$. (iii) $AA^{-1}\mathcal{V} \supseteq \mathcal{V}$.

Question 5b

Which one of the following is correct?
(i) $A^{-1}A\mathcal{V} = \mathcal{V}$. (ii) $A^{-1}A\mathcal{V} \subseteq \mathcal{V}$. (iii) $A^{-1}A\mathcal{V} \supseteq \mathcal{V}$.

Motivation of constructing $\mathcal{V}^*(\mathcal{K})$

$$\mathcal{V}^*(\mathcal{K}) = \{x_0 \mid \exists u \in \mathcal{U} : x_u(x_0, t) \in \mathcal{K}, \forall t \geq 0\} \Rightarrow x(t) \in \mathcal{K} = \mathcal{V}_0$$
$$\Rightarrow \dot{x}(t) \in \mathcal{K} = \mathcal{V}_0 \Rightarrow Ax(t) \in \mathcal{V}_0 + \operatorname{im} B \Rightarrow x(t) \in A^{-1}(\mathcal{V}_0 + \operatorname{im} B)$$
$$\overset{x(t)\in\mathcal{V}_0}{\Rightarrow} x(t) \in \mathcal{V}_1 = A^{-1}(\mathcal{V}_0 + \operatorname{im} B) \Rightarrow \dot{x}(t) \in \mathcal{V}_1 \Rightarrow x(t) \in \mathcal{V}_2 \Rightarrow \cdots$$

4.3 The invariant subspace algorithm (ISA)

Algorithm

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $\mathcal{K} \subseteq \mathbb{R}^n$, define

$$\begin{cases} \mathcal{V}_0 & := \mathcal{K}, \\ \mathcal{V}_{k+1} & := \mathcal{K} \cap A^{-1}(\mathcal{V}_k + \operatorname{im} B), \quad k = 1, 2, \cdots \end{cases}$$

Theorem (4.10)

Let $\mathcal{K} \subseteq X$ *and* $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \ldots$ *as defined in the above algorithm. Then*

(i) $\mathcal{V}_0 \supseteq \mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \cdots$, *(non-increasing)*

(ii) $\exists\, k \leq \dim \mathcal{K} : \mathcal{V}_k = \mathcal{V}_{k+1}$, *(stable index)*

(iii) $\mathcal{V}_k = \mathcal{V}_{k+1} \Rightarrow \mathcal{V}_k = \mathcal{V}_l,\ \forall l \geq k$, *(stable)*

(iv) $\mathcal{V}_k = \mathcal{V}_{k+1} \Rightarrow \mathcal{V}^*(\mathcal{K}) = \mathcal{V}_k$. *(limit)*

4.3 The invariant subspace algorithm (ISA)

Theorem (4.10)

Let $\mathcal{K} \subseteq X$ *and* $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \dots$ *as defined in the above algorithm. Then*
(i) $\mathcal{V}_0 \supseteq \mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \cdots,$ *(ii)* $\exists\, k \leq \dim \mathcal{K} : \mathcal{V}_k = \mathcal{V}_{k+1},$
(iii) $\mathcal{V}_k = \mathcal{V}_{k+1} \Rightarrow \mathcal{V}_k = \mathcal{V}_l,\ \forall l \geq k,$ *(iv)* $\mathcal{V}_k = \mathcal{V}_{k+1} \Rightarrow \mathcal{V}^*(\mathcal{K}) = \mathcal{V}_k.$

Proof.

(i): $\mathcal{V}_0 \supseteq \mathcal{V}_1$ is clear by definition. Induction: suppose that $\mathcal{V}_k \supseteq \mathcal{V}_{k+1}$, then

$$\mathcal{V}_{k+2} = \mathcal{K} \cap A^{-1}(\mathcal{V}_{k+1} + \operatorname{im} B) \overset{\mathcal{V}_{k+1} \subseteq \mathcal{V}_k}{\subseteq} \mathcal{K} \cap A^{-1}(\mathcal{V}_k + \operatorname{im} B) = \mathcal{V}_{k+1}.$$

(iii): $\mathcal{V}_{k+1} = \mathcal{V}_k \Rightarrow \mathcal{V}_{k+2} = \mathcal{K} \cap A^{-1}(\mathcal{V}_{k+1} + \operatorname{im} B) = \mathcal{K} \cap A^{-1}(\mathcal{V}_k + \operatorname{im} B) = \mathcal{V}_{k+1}.$
(ii): (i)+(iii)$\Rightarrow \mathcal{V}_0 \supset \mathcal{V}_1 \supset \mathcal{V}_2 \supset \cdots \supset \mathcal{V}_k = \mathcal{V}_{k+1} = \mathcal{V}_{k+2} = \cdots$. Thus

$$0 \leq \dim \mathcal{V}_k \leq \dim \mathcal{V}_{k-1} - 1 \leq \dim \mathcal{V}_{k-1} - 1 \leq \dim \mathcal{V}_{k-2} - 2 \leq \cdots \leq \dim \mathcal{V}_0 - k = \dim \mathcal{K} - k.$$

4.3 The invariant subspace algorithm (ISA)

Theorem (4.10)

Let $\mathcal{K} \subseteq X$ *and* $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \ldots$ *as defined in the above algorithm. Then (i) ..., (ii)..., (iii)..., (iv)* $\mathcal{V}_k = \mathcal{V}_{k+1} \Rightarrow \mathcal{V}^*(\mathcal{K}) = \mathcal{V}_k$.

Proof.

(iv) $\mathcal{V}_k \subseteq \mathcal{V}_0 = \mathcal{K}$ and $A\mathcal{V}_k = A\mathcal{V}_{k+1} \subseteq \mathcal{V}_k + \operatorname{im} B \Rightarrow \mathcal{V}_k$ is (A, B)-invariant subspace and $\mathcal{V}_k \subseteq \mathcal{K} \Rightarrow \mathcal{V}_k \subseteq \mathcal{V}^*(\mathcal{K})$.
We claim that $\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{V}_l$, $\forall l \geq 0$. Clearly, $\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{V}_0 = \mathcal{K}$. By induction,

$$\begin{aligned} A\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{V}^*(\mathcal{K}) + \operatorname{im} B \subseteq \mathcal{V}_{l-1} + \operatorname{im} B \Rightarrow \mathcal{V}^*(\mathcal{K}) \subseteq A^{-1}(\mathcal{V}_{l-1} + \operatorname{im} B) \\ \Rightarrow \mathcal{K} \cap \mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{K} \cap A^{-1}(\mathcal{V}_{l-1} + \operatorname{im} B) = \mathcal{V}_l. \end{aligned}$$

Thus by claim, we have $\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{V}_l \subseteq \mathcal{V}_k$. □

4.3 The invariant subspace algorithm (ISA)

Example

Let

$$A = \begin{bmatrix} 3 & 2 & 0 \\ -3 & -1 & 1 \\ 4 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

Calculate $\mathcal{V}^*(\ker C)$, i.e., the largest (A, B)-invariant subspace contained in $\ker C$.

Solution

$\mathcal{V}_0 = \ker C = \operatorname{im} \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$

$\mathcal{V}_1 = \ker C \cap A^{-1}(\mathcal{V}_0 + \operatorname{im} B) = \operatorname{im} \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \cap \left(\left[\begin{smallmatrix} 3 & 2 & 0 \\ -3 & -1 & 1 \\ 4 & 2 & -1 \end{smallmatrix}\right]^{-1} \operatorname{im} \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \right) =$

$\operatorname{im} \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \cap \operatorname{im} \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{smallmatrix}\right] = \operatorname{im} \left[\begin{smallmatrix} 1 \\ -1 \\ 1 \end{smallmatrix}\right]$

$\mathcal{V}_2 = \mathcal{V}_1 \Rightarrow \mathcal{V}^*(\ker C) = \mathcal{V}_2 = \mathcal{V}_1.$