Lecture Course: Advanced Systems Theory

Chapter 4: Controlled invariant subspaces

Stephan Trenn, Yahao Chen

Jan C. Willems Center for Systems and Control

University of Groningen, Netherlands

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A-invariant subspace

Definition

Given a linear map $A : \mathcal{X} \to \mathcal{X}$ and a subspace $\mathcal{V} \subseteq \mathcal{X}$, we say $\mathcal{V}$ is A-invariant if $\forall x \in \mathcal{V}$, $Ax \in \mathcal{V}$.

Question 1

Which one of the following spaces is NOT necessarily A-invariant?
(i). $\mathcal{X}$; (ii). $\ker A$; (iii). $\operatorname{im} A$;
(iv). $\mathcal{V} \subseteq \ker A$; (v). $\mathcal{V} \subseteq \operatorname{im} A$.

- Consider a differential equation

$$\dot{x} = Ax. \tag{1}$$

Definition (2.3)

A subspace $\mathcal{V}$ is called an invariant subspace of (1) $:\Leftrightarrow \forall x_0 \in \mathcal{V},\ \forall t \geq 0 : x(t, x_0) \in \mathcal{V}$.

A-invariant subspace

Theorem (2.4)

A subspace $\mathcal{V}$ is an invariant subspace of (1) iff $\mathcal{V}$ is A-invariant.

Proof.

"*If*": $\mathcal{V}$ is A-inv. $\Rightarrow$ For any $x_0 \in \mathcal{V}$: $Ax_0 \subseteq \mathcal{V}$, $A^2 x_0 \subseteq \mathcal{V}$, $\ldots \overset{\text{induction}}{\Rightarrow} A^k x_0 \subseteq \mathcal{V}$, $\forall k \geq 0$.

$$\Rightarrow e^{At} x_0 = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} x_0 \in \mathcal{V}.$$

"*Only if*": $x(t, x_0) \in \mathcal{V}$, $\forall t \geq 0$ and $\forall x_0 \in \mathcal{V}$

$$\Rightarrow Ax_0 = (e^{At} x_0)'(0) = \lim_{t \to 0} \frac{e^{At} x_0 - x_0}{t} = \lim_{t \to 0} \frac{x(t, x_0) - x_0}{t} \in \mathcal{V}.$$

□

A-invariant subspace

- Given a matrix A and an A-invariant subspace $\mathcal{V} \subseteq \mathcal{X}$, how to construct a basis (coordinates) transformation matrix T (isomorphism) such that

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$

- Why T is called a coordinates transformation matrix for the ODE

$$\dot{x} = Ax.$$

  What T does for the state variables x?
- Can we generalize the notion of A-invariant subspace to the pair (A, B) (or correspondingly, the system $\dot{x} = Ax + Bu$)?

4.1 Controlled invariance

- Controlled and conditioned invariant subspace: Basile and Marro (1969), Wonham and Morse (1970).
- Consider an LTI control system
$$\Sigma : \dot{x} = Ax + Bu,$$
denoted by $\Sigma = (A, B)$;
- $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$, $u \in \boldsymbol{U}$,
- in particular, $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$, $\boldsymbol{U} = L_{1,\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)$.

Definition (4.1)

A subspace $\mathcal{V}$ is controlled invariant (or (A,B)-invariant):$\Leftrightarrow$ $\forall x_0 \in \mathcal{V}$, $\exists\, u \in \boldsymbol{U}$: $x_u(t, x_0) \in \mathcal{V}$, $\forall t \geq 0$.

4.1 Controlled invariance

Theorem (4.2)

Consider a system $\Sigma = (A, B)$ and a subspace $\mathcal{V} \subseteq \mathcal{X}$, the following is equivalent

(i) $\mathcal{V}$ *is controlled invariant,*

(ii) $A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$,

(iii) $\exists$ *linear* $F : \mathcal{X} \to \mathcal{U}$: $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.

Proof.

(Handwriting proof)

(i)$\Rightarrow$ (ii): $\exists\, u \in \boldsymbol{U}$: $x_u(t, x_0) \in \mathcal{V}$, $\forall t \geq 0$, $\forall x_0 \in \mathcal{V} \Rightarrow$

$$Ax_0 + Bu(0) = \lim_{t\to 0} \frac{e^{At}x_0 + \int_0^t Be^{t-\tau}u(\tau)\mathrm{d}\tau - x_0 - Bu(0)}{t - 0} = \lim_{t\to 0} \frac{x_u(t, x_0) - x_0 - Bu(0)}{t - 0}$$

$$\in \mathcal{V} + \operatorname{im} B \Rightarrow Ax_0 \in \mathcal{V} + \operatorname{im} B.$$

□

4.1 Controlled invariance

Theorem (4.2)

Consider a system $\Sigma = (A, B)$ and a subspace $\mathcal{V} \subseteq \mathcal{X}$, the following is equivalent

(i)$\mathcal{V}$ *is controlled invariant,*

(ii)$A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$,

(iii)$\exists$ *linear* $F : \mathcal{X} \to \mathcal{U}$: $(A + BF)\mathcal{V} \subseteq V$.

Proof.

(ii)$\Rightarrow$ (iii): Let $q_1, \ldots, q_k$ $(k \leq n)$ be a basis of $\mathcal{V}$. Then

$$A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} \mathcal{V} \Rightarrow \exists v_i \in \mathcal{V}, u_i \in \mathcal{U} : Aq_i = v_i + Bu_i, \ \forall 1 \leq i \leq k.$$

Define $F : \mathcal{X} \to \mathcal{U}$ s.t. $Fq_i = -u_i, \forall 1 \leq i \leq k$ (why such an F always exists?). $\Rightarrow$ $(A + BF)q_i = v_i \in \mathcal{V}$.

(iii)$\Rightarrow$ (i): $(A + BF)\mathcal{V} \subseteq \mathcal{V} \Rightarrow \mathcal{V}$ is $(A + BF)$-inv. $\Rightarrow x(t, x_0) \in \mathcal{V}$, where $x(t, x_0)$ solves $\dot{x} = (A + BF)x \Rightarrow \exists u$ s.t. $x_u(t, x_0) \in \mathcal{V}$, where $x_u(t, x_0)$ solves $\dot{x} = Ax + Bu$ (take $u = Fx(t)$). □

4.1 Controlled invariance (Questions Page)

Theorem (4.2)

Consider a system $\Sigma = (A, B)$ and a subspace $\mathcal{V} \subseteq \mathcal{X}$, the following is equivalent (i) $\mathcal{V}$ is controlled invariant, (ii) $A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$, (iii) $\exists$ linear $F : \mathcal{X} \to \mathcal{U} : (A + BF)\mathcal{V} \subseteq V$.

Question 2

Which style of presenting the proves of theorems do you like better?
(i) Typing proof.
(ii) Handwriting proof.
(iii) Both.
(iv) I do not care!

Question 3

Which one of the following subspaces may not be (A, B)-inv.?
(i) any A-inv.
(ii) $A \operatorname{im} B$.
(iii) $A^{-1} \operatorname{im} B$.

Question 4

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathcal{V} = \operatorname{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which F satisfies $(A + BF)\mathcal{V} \subset \mathcal{V}$?
(i) $F = [0 \ 0]$.
(ii) $F = [-1 \ 0]$.
(iii) $F = [0 \ -2]$.

Example-finding F (a friend feedback) such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$

Consider a system $\Sigma = (A, B)$ and a subspace $\mathcal{V} \subset \mathcal{X}$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{V} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Solution

Let $q_1 = [0\ 0\ 1]^T$ and $q_2 = [1\ 1\ 0]^T$, then

$$Aq_1 = v_1 + Bu_1 \Leftrightarrow \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot -1, \quad Aq_2 = v_2 + Bu_2 \Leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot 0$$

Then F is any such that

$$F \cdot \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Controlled invariance within a given subspace

- Consider the system $\Sigma : \dot{x} = Ax + Bu$.

Definition (4.1 controlled invariant subspace)

A subspace $\mathcal{V}$ is controlled invariant (or (A,B)-invariant):$\Leftrightarrow$ $\forall x_0 \in \mathcal{V}$, $\exists\, u \in \boldsymbol{U}$: $x_u(t, x_0) \in \mathcal{V}$, $\forall t \geq 0$.

- What is the largest controlled invariance in $\mathcal{X}$?: $\mathcal{X}$ itself.
- For any subspace $\mathcal{K} \subseteq \mathcal{X}$, what is the largest c.i.s. in $\mathcal{K}$?

Definition (4.4 controlled invariant subspace within $\mathcal{K}$)

Given a subspace $\mathcal{K} \subseteq \mathcal{X}$, define

$$\mathcal{V}^*(\mathcal{K}) := \{x_0 \,|\, \exists\, u \in \boldsymbol{U} : x_u(t, x_0) \in \mathcal{K},\ \forall t \geq 0\}$$

Remark: (i) $\mathcal{V}^*(\mathcal{K})$ is a subspace itself. (ii) $\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{K}$.

The largest controlled invariant subspace in $\mathcal{K}$

Definition (4.4)

For a subspace $\mathcal{K} \subseteq \mathcal{X}$, define

$$\mathcal{V}^*(\mathcal{K}) := \{x_0 \,|\, \exists\, u \in \boldsymbol{U} : x_u(t, x_0) \in \mathcal{K},\ \forall t \geq 0\}$$

Theorem ((4.5))

$\mathcal{V}^*(\mathcal{K})$ is the largest (A, B)-invariant subspace contained in $\mathcal{K}$, i.e.,

(i) $\mathcal{V}^*(\mathcal{K})$ *is (A, B)-invariant;*

(ii) $\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{K}$;

(iii) $\mathcal{V} \subseteq \mathcal{K}$ *is (A, B)-invariant* $\Rightarrow \mathcal{V} \subseteq \mathcal{V}^*$.

Proof.

(Handwriting proof), see also page 78 of the book. □

The largest controlled invariant subspace in $\mathcal{K}$

Theorem ((4.5))

$\mathcal{V}^*(\mathcal{K})$ is the largest (A,B)-invariant subspace contained in $\mathcal{K}$, i.e., (i) $\mathcal{V}^*(\mathcal{K})$ is (A,B)-invariant; (ii) $\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{K}$; (iii) $\mathcal{V} \subseteq \mathcal{K}$ is (A,B)-invariant $\Rightarrow \mathcal{V} \subseteq \mathcal{V}^*$.

Proof.

(i) Take any $x_0 \in \mathcal{V}^*(\mathcal{K})$, then by definition, $\exists\, u = u(t) : x_u(t, x_0) \in \mathcal{K},\ \forall t \geq 0$. We will show by contradictions that such $u(t)$ renders $x_u(t, x_0) \in \mathcal{V}^*(\mathcal{K}), \forall t \geq 0$ as well. Suppose that for some $T \geq 0 : x_1 = x_u(T, x_0) \in \mathcal{K}/\mathcal{V}^*(\mathcal{K})$. However, by definition, $x_1 \in \mathcal{V}^*(\mathcal{K})$ since $x_u(t, x_1) \in \mathcal{K},\ \forall t \geq 0$. Thus $x_1 \notin \mathcal{V}^*(\mathcal{K})$. Since T is any, we have that $x_u(t, x_0) \in \mathcal{V}^*(\mathcal{K}), \forall t \geq 0$, which implies that $\mathcal{V}^*(\mathcal{K})$ is (A,B)-invariant;
(ii) Clear;
(iii) Take any $x_0 \in \mathcal{V}$, then $\exists u : x_u(t, x_0) \in \mathcal{V} \subseteq \mathcal{K},\ \forall t \geq 0$, which by definition, implies that $x_0 \in \mathcal{V}^*(\mathcal{K})$.

□

4.2 Disturbance decoupling

- Consider a control system

$$\Sigma_{d,z}: \begin{cases} \dot{x} = Ax + Ed \\ z = Hx, \end{cases}$$

- $d \in \mathbb{D}$ denotes an unknown disturbance, and in particular, $\mathbb{D} = L_{1,\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)$,
- The output $z(t) = He^{At}x_0 + \int_0^t T(t-\tau)d(\tau)\mathrm{d}\tau$, where $T(t-\tau) = He^{A(t-\tau)}E$.
- $\Sigma_{d,z}$ is called disturbance decoupled if $T(t) = He^{At}E = 0, \forall t \geq 0$, i.e., $z(t)$ does not depend on $d(t)$, or if $G(s) = H(sI-A)^{-1}E = 0$.

Question 5

Let $A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$, which system with E, H below is disturbance decoupled?

$$(i): E = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & 0 \end{bmatrix}, \quad (ii) E = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & 0 \end{bmatrix},$$

$$(iii): E = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & H_2 \end{bmatrix}, \quad (iv): E = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & H_2 \end{bmatrix}.$$

Theorem (4.6)

$\Sigma_{d,z}$ *is disturbance decoupled iff* $\exists$ *an A-invariant subspace* $\mathcal{V}$ *s.t.*

$$\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H.$$

Proof.

"*If*": $T(t) = He^{At}E = 0,\ \forall t \geq 0. \Rightarrow T^k(t) = HA^k e^{At}E = 0,\ \forall t \geq 0,\ \forall k \geq 0.$

$\overset{t=0}{\Rightarrow} HA^kE = 0,\ \forall k \geq 0.$

Let $\mathcal{V} = \operatorname{im}[E, AE, \ldots, A^{n-1}E]$, then $\mathcal{V}$ is A-invariant subspace with $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$.

"*Only if*": $\mathcal{V}$ is an A-invariant subspace with $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$.

$\Rightarrow \operatorname{im} A^kE \subseteq \mathcal{V} \subseteq \ker H,\ \forall k \geq 0.$

$\Rightarrow HA^kE = 0,\ \forall k \geq 0.$

$\Rightarrow T(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} HA^kE = 0,\ \forall k \geq 0.$ $\square$

Summary

Recapitulation

- A-invariant subspace
- Controlled invariant (or (A, B)-invariant) subspace
- Controlled invariant subspace within a given subspace $\mathcal{K}$
- The largest controlled invariant subspace in $\mathcal{K}$
- Disturbance decoupled system

Section 2.2, Section 2.6, Section 4.1 and Section 4.2 of the book