**Advanced Systems Theory**

# Lecture 13: Synchronization for heterogenous agents

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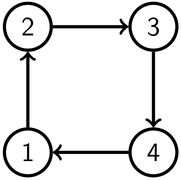
# Recap synchronization

## Question 1

Consider a multi-agent system as shown right. Which of the following agent dynamics are synchronizable by state-coupling?

(i) $\dot{x}_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i$ (ii) $\dot{x}_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_i + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_i$




## Question 2

Which feedback matrix $F$ in the diffusive state-coupling leads to synchronization for above example? (Hint: $\lambda_{2/3} = 1 \pm \mathbf{i}$, $\lambda_3 = 2$)

(i) $F = [1, 0]$ (ii) $F = [0, 1]$ (iii) $F = [2, 0]$ (iv) $F = [0, 2]$ (v) neither

## Question 3

Is the above example synchronizable by output-coupling with output $y_i = [1, 0]x_i$?

## Question 4

Is the above example synchronizable by output-coupling with output $y_i = [0, 1]x_i$?

# Synchronization for heterogenous agents

# Setup

## Given

- Agent dynamics:
$$\dot{x}_i = A_i x_i + b_i u_i$$
$$y_i = c_i x_i$$
- Individual matrices $A_i \in \mathbb{R}^{n_i \times n_i}$, $b_i, c_i^\top \in \mathbb{R}^{n_i}$ for each agent (heterogeneous dynamics)
- Communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

## Objective

Asymptotic output synchronization: $\lim_{t\to\infty} |y_i(t) - y_j(t)| = 0$ for all $i, j \in \mathcal{V}$

## Approach: Diffusive output coupling

$$u_i = -k \sum_{j \in \mathcal{N}_i} (y_i(t) - y_j(t))$$

# Feasible synchronization trajectory

## Convergence $\to$ common output trajectory

Because of linearity and time-invariance it follows from convergence $|y_i(t) - y_j(t)| \to 0$ that

- $\exists y_s : [0,\infty) \to \mathbb{R}$ with $\lim_{t\to\infty} y_i(t) = y_s(t)$ and
- $y_i(0) = y_s(0)\ \forall i \in \mathcal{V} \quad \Longrightarrow \quad y_i(t) = y_s(t)\ \forall t \geq 0\ \forall i \in \mathcal{V}$

## Definition (Possible output trajectories)

$$\begin{aligned}\mathcal{Y}_i &:= \{\ y_i(\cdot) = c_i x_i(\cdot)\ \mid \exists x_{i,0} \in \mathbb{R}^{n_i} : \dot{x}_i = A_i x_i + 0,\ x_i(0) = x_{i,0}\ \} \\ &= \left\{\ t \mapsto c_i e^{A_i t} x_{i,0}\ \middle|\ x_{i,0} \in \mathbb{R}^{n_i}\ \right\}\end{aligned}$$

## Definition (Common dynamics)

Two agents $i$ and $j$ are said to have common dynamics $:\Longleftrightarrow$

$$\mathcal{Y}_i \cap \mathcal{Y}_j \neq \{0\}$$

# Common dynamics: Necessary conditions

$$\{\ t \mapsto c_i e^{A_i t} x_{i,0} \ \big|\ x_{i,0} \in \mathbb{R}^{n_i}\ \} \cap \{\ t \mapsto c_j e^{A_j t} x_{j,0} \ \big|\ x_{j,0} \in \mathbb{R}^{n_j}\ \} \neq \{0\}$$

## Question 5

Which of the following conditions is necessary for agents $i$ and $j$ having common dynamics?

(i) $c_i A_i^k = c_j A_j^k$ for $k = 0, 1, 2 \ldots, n-1$

(ii) $A_i$ and $A_j$ have a common eigenvector

(iii) $n_i = n_j$

(iv) $A_i$ and $A_j$ have a common eigenvalue

(v) neither

## Fact: Linear independence of exponential functions

Let $\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{C}$ pairwise distinct. Then

$$\sum_{j=1}^{p} \alpha_j e^{\lambda_j t} = 0 \ \forall t \in \mathbb{R} \quad \iff \quad 0 = \alpha_1 = \alpha_2 = \ldots = \alpha_p$$

# Common dynamics: Necessary conditions

$$\{\ t \mapsto c_i e^{A_i t} x_{i,0} \ \big|\ x_{i,0} \in \mathbb{R}^{n_i}\ \} \cap \{\ t \mapsto c_j e^{A_j t} x_{j,0} \ \big|\ x_{j,0} \in \mathbb{R}^{n_j}\ \} \neq \{0\}$$

## Question 5

Which of the following conditions is necessary for agents $i$ and $j$ having common dynamics?

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(v) neither

## Fact: Linear independence of exponential functions

Let $\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{C}$ pairwise distinct. Then

$$\sum_{j=1}^{p} \sum_{k=0}^{\kappa_j} \beta_{jk} t^k e^{\lambda_j t} = 0\ \forall t \in \mathbb{R} \quad \iff \quad \beta_{jk} = 0\ \forall j, k$$

# Common dynamics: When and what?

$$\begin{aligned}\dot{x}_i &= A_i x_i + b_i u_i \\ y_i &= c_i x_i\end{aligned} \qquad \mathcal{Y}_i = \left\{\, t \mapsto c_i e^{A_i t} x_{i,0} \;\middle|\; x_{i,0} \in \mathbb{R}^{n_i} \,\right\}$$

Assumption: $(c_i, A_i)$ observable

**Theorem**

*The multi-agent system has common dynamics, i.e.* $\bigcap_{i\in\mathcal{V}} \mathcal{Y}_i \neq \{0\}$
$\iff$ *all* $A_i$ *have at least one common eigenvalue, i.e.* $\bigcap_{i\in\mathcal{V}} \sigma(A_i) \neq \emptyset$

Proof ...

**Remaining question**

How to find $A_s$, $c_s$ such that

$$\bigcap_{i\in\mathcal{V}} \mathcal{Y}_i =: \mathcal{Y}_s = \left\{\, y_s(\cdot) \;\middle|\; \exists x_{s,0} : \begin{array}{l} \dot{x}_s = A_s x_s, \;\; x_s(0) = x_{s,0} \\ y_s = c_s x_s \end{array} \right\}$$


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# The internal model principle

**Lemma**

$\bigcap_{i\in\mathcal{V}} \sigma(A_i) =: \sigma_s \neq \emptyset \iff \exists$ *coordinate transformations* $T_i \in \mathbb{R}^{n_i \times n_i}$ *such that*

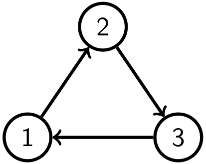
$$T_i^{-1} A_i T_i = \begin{bmatrix} A_s & * \\ 0 & * \end{bmatrix}, \quad c_i T_i = [c_s, *]$$

*and* $\sigma(A_s) = \sigma_s$

**Corollary**

Heterogenous multi-agent systems with diffusive coupling can synchronize nontrivially
$\implies$ all agents share common dynamics (internal model principle)

# Example

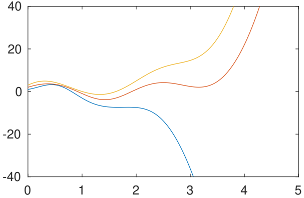


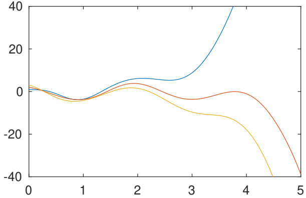

$$(A_i, b_i, c_i) = \left( \begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & 1 \\ 0 & 0 & -\gamma_i \end{bmatrix}, \begin{pmatrix} 0 \\ 0 \\ \gamma_i \end{pmatrix}, \begin{pmatrix} 1 & 10 & 0 \end{pmatrix} \right)$$

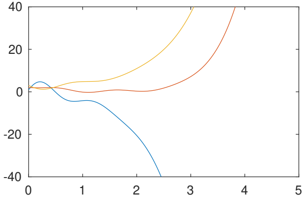
with $\gamma_i := 1/i$ and $\sigma(A_i) = \{\pm 3\mathbf{i}, \gamma_i\}$

$$\dot{x}_i = A_i x_i + b_i u_i \qquad u_i = -k \sum_{j \in \mathcal{N}_i} (y_i - y_j)$$
$$y_i = c_i x_i$$

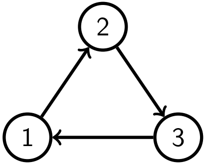
Simulations for $k = 1$



# Example

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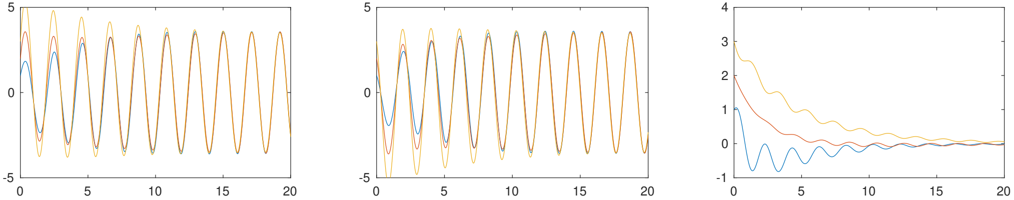
$$\dot{x}_i = A_i x_i + b_i u_i$$
$$y_i = c_i x_i$$

$$u_i = -k \sum_{j \in \mathcal{N}_i} (y_i - y_j)$$

Simulations for $k = 0.05$



# Error dynamics

$$\dot{x}_i = A_i x_i + b_i u_i \qquad u_i = -k \sum_{j \in \mathcal{N}_i} (y_i - y_j) \qquad e_i := y_i - y_1$$
$$y_i = c_i x_i$$

## Assumption: Common internal model

$$\exists T_i \in \mathbb{R}^{n_i \times n_i}: \qquad (T_i^{-1} A_i T_i,\, T_i^{-1} b_i,\, c_i T_i) = \left( \begin{bmatrix} A_s & A_{q_i} \\ 0 & A_{p_i} \end{bmatrix}, \begin{pmatrix} b_{q_i} \\ b_{p_i} \end{pmatrix}, \begin{pmatrix} c_s & c_{p_i} \end{pmatrix} \right)$$

## Lemma (Error dynamics)

*The error* $\mathbf{e} := (e_2, e_2, \ldots, e_N)^\top$ *satisfies* $\mathbf{e} = \begin{bmatrix} -c_{p_1} & c_2 & & \\ \vdots & & \ddots & \\ -c_{p_1} & & & c_N \end{bmatrix} \mathbf{x}_e$ *where* $\dot{\mathbf{x}}_e = \mathbf{A}_e(k)\mathbf{x}_e$ *with*

$$\mathbf{A}_e(k) = \begin{bmatrix} A_{p_1} & 0 & \ldots & 0 \\ -T_2 \begin{bmatrix} A_{q_1} \\ 0 \end{bmatrix} & A_2 & & 0 \\ \vdots & & \ddots & \\ -T_N \begin{bmatrix} A_{q_1} \\ 0 \end{bmatrix} & 0 & & A_N \end{bmatrix} - \begin{bmatrix} b_{p_1} & 0 & \ldots & 0 \\ -T_2 \begin{bmatrix} b_{q_1} \\ 0 \end{bmatrix} & b_2 & & 0 \\ \vdots & & \ddots & \\ -T_N \begin{bmatrix} b_{q_1} \\ 0 \end{bmatrix} & 0 & & b_N \end{bmatrix} k\mathbf{L} \begin{bmatrix} c_{p_1} & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_N \end{bmatrix}$$

# Synchronization characterization

$$\dot{x}_i = A_i x_i + b_i u_i \qquad u_i = -k \sum_{j \in \mathcal{N}_i} (y_i - y_j) \qquad e_i := y_i - y_1$$
$$y_i = c_i x_i$$

**Theorem**

*Synchronization to a nontrivial trajectory occurs* $\iff$

1. *All agents have a common internal model (i.e.* $\bigcap_{i\in\mathcal{V}} \sigma(A_i) \neq \emptyset$*)*
2. $\mathbf{A}_e(k)$ *is Hurwitz*

Example: $\mathbf{A}_e(k)$ for $k = 1$ and $k = 0.05$ are

$$\mathbf{A}_e(1) = \begin{bmatrix} -1 & 1 & 10 & 0 & 1 & 10 & 0 \\ -1 & 0 & 3 & 1 & 0 & 0 & 0 \\ -1 & -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -10 & -\frac{1}{2} & \frac{1}{2} & 5 & 0 \\ -1 & 0 & 0 & 0 & 0 & 3 & 1 \\ -1 & 0 & 0 & 0 & -3 & 0 & 1 \\ 0 & \frac{1}{3} & \frac{10}{3} & 0 & -\frac{2}{3} & -\frac{20}{3} & -\frac{1}{3} \end{bmatrix} \quad \mathbf{A}_e(0.05) = \begin{bmatrix} -1 & \frac{1}{20} & \frac{1}{2} & 0 & \frac{1}{20} & \frac{1}{2} & 0 \\ -1 & 0 & 3 & 1 & 0 & 0 & 0 \\ -1 & -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{20} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{40} & \frac{1}{4} & 0 \\ -1 & 0 & 0 & 0 & 0 & 3 & 1 \\ -1 & 0 & 0 & 0 & -3 & 0 & 1 \\ 0 & \frac{1}{60} & \frac{1}{6} & 0 & -\frac{1}{30} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$