Advanced Systems Theory

Lecture 12: Multiagent systems: Synchronization

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Recap consensus

Question 1

Consider simple multi-agent system $\dot{x}_i = u_i$ whose interaction graph $\mathcal{G}$ has a spanning tree. Is it true that diffusive coupling with additional gain $k > 0$

$$u_i = -k \sum_{j \in \mathcal{N}_i} (x_i - x_j)$$

leads to consensus with the same consensus value for any k?

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How does the choice of the gain k effect the convergence rate?

(i) The larger k is the faster the agents converge

(ii) The smaller k is the faster the agents converge

(iii) There is no simple connection between the size of k and the convergence rate

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Question 3

Does consensus to the same value still occurs when individual gains k_i are used, i.e. $u_i = -k_i \sum_{j \in \mathcal{N}_i} (x_i - x_j)$?

Synchronization by state-coupling

for arbitrary homogeneous linear agent dynamics

The synchronization by state-feedback problem: setup

Given

- Agent dynamics: $\dot{x}_i = Ax_i + Bu_i$

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Objective

Asymptotic synchronization: $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all $i, j \in \mathcal{V}$

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Approach: Diffusive coupling with common feedback matrix $F \in \mathbb{R}^{m \times n}$

$$u_i = -F \sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t))$$

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Question 4

Which is the overall feedback expression?

(i) $\mathbf{u} = -F\mathbf{L}\mathbf{x}$ (ii) $\mathbf{u} = -\mathbf{L}F\mathbf{x}$ (iii) neither

Overall coupled dynamics

Problem

Neither $\underbrace{\mathbf{u}}_{N\cdot m} = -\underbrace{F}_{m\times n}\underbrace{\mathbf{L}}_{N\times N}\underbrace{\mathbf{x}}_{N\cdot n}$ nor $\underbrace{\mathbf{u}}_{N\cdot m} = -\underbrace{\mathbf{L}}_{N\times N}\underbrace{F}_{m\times n}\underbrace{\mathbf{x}}_{N\cdot n}$ are well-defined!

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$$\mathbf{u} = \ldots$$

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Some properties of the Kronecker product

For $P = [p_{ij}] \in \mathbb{R}^{m_P \times n_P}$, $Q \in \mathbb{R}^{m_Q \times n_Q}$

$$P \otimes Q := \begin{bmatrix} p_{11}\cdot Q & p_{12}\cdot Q & \cdots & p_{1,n_P}\cdot Q \\ p_{21}\cdot Q & p_{22}\cdot Q & \cdots & p_{2,n_P}\cdot Q \\ \vdots & \vdots & & \vdots \\ p_{m_P,1}\cdot Q & p_{m_P,2}\cdot Q & \cdots & p_{m_P,n_P}\cdot Q \end{bmatrix} \in \mathbb{R}^{m_P\cdot m_Q \times n_P\cdot n_Q}$$

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- $(P\otimes Q)^\top = P^\top \otimes Q^\top$

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- $(P \otimes Q) \cdot (M \otimes N) = (P \cdot M) \otimes (Q \cdot N)$
- $(P \otimes Q)^\top = P^\top \otimes Q^\top$
- $(P \otimes Q) + (P \otimes M) = P \otimes (Q + M)$

Coupled dynamics in terms of Kronecker product

$$\begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} = I \otimes A \qquad \begin{bmatrix} B & & & \\ & B & & \\ & & \ddots & \\ & & & B \end{bmatrix} = I \otimes B$$

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$\rightsquigarrow$ Overall dynamics:

$$\begin{aligned} \dot{\mathbf{x}} &= (I \otimes A)\,\mathbf{x} + (I \otimes B)\,\mathbf{u} \\ \mathbf{u} &= -(\mathbf{L} \otimes F)\,\mathbf{x} \end{aligned}$$

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$$\dot{\mathbf{x}} = (I \otimes A)\,\mathbf{x} - (I \otimes B)(\mathbf{L} \otimes F)\,\mathbf{x} = (I \otimes A - \mathbf{L} \otimes BF)\,\mathbf{x}$$

Coupled dynamics in compact form

The multi-agent system $\dot{x}_i = Ax_i + Bu_i$ with diffusive feedback $u_i = -F\sum_{j \in \mathcal{N}_i}(x_i - x_j)$ has the compact form

$$\dot{\mathbf{x}} = (I \otimes A - \mathbf{L} \otimes BF)\,\mathbf{x}$$

Synchronization condition

$$\dot{x}_i = A\,x_i + B\,u_i \qquad\qquad \dot{\mathbf{x}} = (I \otimes A)\,\mathbf{x} + (I \otimes B)\,\mathbf{u}$$

$$u_i = -F \sum_{j \in \mathcal{N}_i} (x_i - x_j) \qquad\qquad \mathbf{u} = -(\mathbf{L} \otimes F)\,\mathbf{x}$$

Theorem

Assume $\mathcal{G}$ contains a spanning tree, then synchronization occurs if, and only if

$$A - \lambda_i BF \text{ is Hurwitz for each nonzero eigenvalue } \lambda_2, \lambda_3, \ldots, \lambda_N \text{ of } \mathbf{L}$$

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Furthermore, the limit trajectory x_{lim} with $|x_i(t) - x_{\text{lim}}(t)| \to 0$ for all $i \in \mathcal{V}$ is then given by

$$\dot{x}_{\text{lim}} = A\,x_{\text{lim}}, \quad x_{\text{lim}}(0) = (\widehat{w} \otimes I)\,\mathbf{x}(0)$$

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Proof: ...

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Proof: ...

Consensus result as special case

$A = 0,\ B = 1,\ F = 1$ (or $F = k > 0$) $\rightsquigarrow$ Consensus result

Existence of synchronizing coupling feedback F

Theorem

Assume $0 < \mathrm{Re}(\lambda_2) \leq \mathrm{Re}(\lambda_3) \leq \ldots \leq \mathrm{Re}(\lambda_N)$.
$\exists F$ *such that* $A - \lambda_i BF$ *Hurwitz for* $i = 2, 3, \ldots, N \iff (A, B)$ *is stabilizable*

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For the proof we need the following two facts (see Thms. 3.28 and 10.13 in the CTLS-book):

(F1) $H \in \mathbb{C}^{n \times n}$ is Hurwitz if there exists $P = P^\top > 0$ (sym. pos. def.) with

$$H^* P + PH =< 0 \qquad \text{(Lyapunov equation)}$$

(F2) If (A, B) is stabilizable, then for any $Q = Q^\top > 0$ exists $P = P^\top > 0$ with

$$A^\top P + PA - PBB^\top P = -Q \qquad \text{(Riccati equation)}$$

Proof: ...

Synchronizing coupling feedback F

Synchronization is achieved for any $F = B^\top P$ where $P = P^\top > 0$ is such that

$$A^\top P + PA - 2\,\mathrm{Re}(\lambda_2) PBB^\top P < 0$$

Synchronization by output coupling

for arbitrary homogeneous linear agent dynamics

The synchronization by output coupling: setup

Given

- Agent dynamics: $\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i$

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Objective

Asymptotic synchronization: $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all $i, j \in \mathcal{V}$

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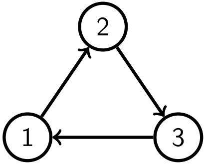
$$u_i(t) = -F \sum_{j\in\mathcal{N}_i} (y_i(t) - y_j(t)) = -FC \sum_{j\in\mathcal{N}_i} (x_i(t) - x_j(t))$$

Corollary

Synchronization via static output coupling occurs $\iff$ $\exists F \in \mathbb{R}^{m\times p}$ *such that*

$$A - \lambda_i BFC \text{ is Hurwitz} \quad \forall i \in \{2, 3, \dots, N\}$$

Example



$$\mathbf{L} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \text{ with eigenvalues } \lambda_1 = 0,\ \lambda_{2/3} = \frac{3 \pm \sqrt{3}\,\mathbf{i}}{2}$$

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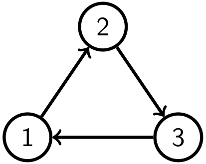

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Agent dynamics ($i = 1, 2, 3$): $\dot{x}_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i, \qquad y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i$

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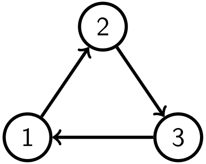

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Static output coupling: $u_i = -F \sum_{j \in \mathcal{N}_i} (y_i - y_j), \qquad F \in \mathbb{R}$

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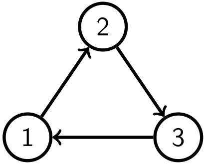

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Static output coupling: $u_i = -F \sum_{j \in \mathcal{N}_i} (y_i - y_j),$ $F \in \mathbb{R}$

Question 5

Synchronization occurs, i.e. $A - \lambda_2 BFC$ and $A - \lambda_3 BFC$ are Hurwitz for

(i) all $F \in \mathbb{R}$ (ii) all $F > 0$ (iii) no $F \in \mathbb{R}$

Synchronization by dynamic output coupling

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- Same coefficient matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ for all agents
- Communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Objective

Asymptotic synchronization: $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all $i, j \in \mathcal{V}$

Approach

- Each agent runs observer

$$\dot{w}_i = Pw_i + Q\sum_{j\in\mathcal{N}_i}(u_i - u_j) + R\sum_{j\in\mathcal{N}_i}(y_i - y_j)$$

with P, Q, R chosen such that $\left|w_i(t) - \sum_{j\in\mathcal{N}_i}(x_i - x_j)\right| \to 0$ as $t \to \infty$

Synchronization by dynamic output coupling

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  with P, Q, R chosen such that $\left|w_i(t) - \sum_{j\in\mathcal{N}_i}(x_i - x_j)\right| \to 0$ as $t \to \infty$

- Use diffusive state feedback with w_i instead of $\sum_{j\in\mathcal{N}_i}(x_i - x_j)$, i.e.

$$u_i = -F\, w_i$$

Distributed observer approach

For some $G \in \mathbb{R}^{n \times p}$ consider the follower observer candidate:

$$\dot{w}_i = Aw_i + B \sum_{j \in \mathcal{N}_i} (u_i - u_j) + G \left(\sum_{j \in \mathcal{N}_i} (y_i - y_j) - Cw_i \right)$$

i.e. $P = A - GC$, $Q = B$ and $R = G$.

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Error dynamics

$e_i := w_i - \sum_{j\in\mathcal{N}_i}(x_i - x_j)$ satisfies

$$\dot{e}_i = \dots$$

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Corollary

$\exists G \in \mathbb{R}^{n \times p}$ *such that* $e_i(t) \underset{t \to \infty}{\to} 0 \quad \Longleftrightarrow \quad (C, A)$ *is detectable*

Synchronization result for dynamic output coupling

Agent dynamics:

$$\dot{x}_i = Ax_i + Bu_i \qquad \dot{w}_i = Aw_i + B\sum_{j\in\mathcal{N}_i}(u_i - u_j) + G\left(\sum_{j\in\mathcal{N}_i}(y_i - y_j) - Cw_i\right)$$

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Full coupled dynamics:

$$\begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{w}}\end{pmatrix} = \begin{bmatrix} I\otimes A & -I\otimes BF \\ L\otimes GC & I\otimes(A-GC) - L\otimes BF\end{bmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix}$$

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Theorem

There exists F and G such that synchronization occurs
$\iff$ *(A,B) is stabilizable and (C,A) is detectable.*

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Theorem

There exists F and G such that synchronization occurs
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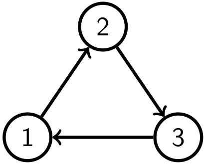
Key observation for proof: Separation principle also holds in multi-agent context!

$$\begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{e}}\end{pmatrix} = \begin{bmatrix} I\otimes A - L\otimes BF & -I\otimes BF \\ 0 & I\otimes(A-GC)\end{bmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{e}\end{pmatrix}$$

Example revisited



$$\mathbf{L} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$ with eigenvalues $\lambda_1 = 0$, $\lambda_{2/3} = \frac{3 \pm \sqrt{3}\,\mathbf{i}}{2}$

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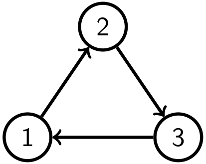
Example revisited



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Agent dynamics ($i = 1, 2, 3$): $\dot{x}_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i, \qquad y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i$

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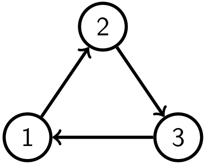

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Question 6

Is this multi-agent system synchronizable by dynamic output coupling?

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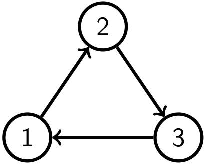

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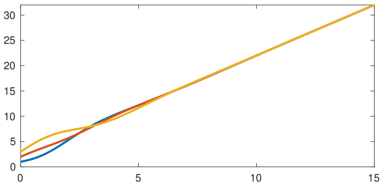
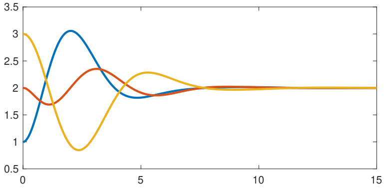
Question 6

Is this multi-agent system synchronizable by dynamic output coupling?

Simulations with $F = [1, 1]$ and $G = [1, 1]^\top$ (1st components of x_i left, 2nd components right):





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Summary: Synchronization for $\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i$

Synchronization by state-coupling $u_i = -F \sum_{j \in \mathcal{N}_i} (x_i - x_j)$

- Coupled system: $\dot{\mathbf{x}} = (I \otimes A + \mathbf{L} \otimes BF)\mathbf{x}$
- Synchronization $\iff$ $A - \lambda_i BF$ Hurwitz for eigenvalues $\lambda_2, \ldots, \lambda_N$ of $\mathbf{L}$
- Synchronizable $\iff$ (A, B) stabilizable
- Limit trajectory $\dot{x}_{\text{lim}} = A\, x_{\text{lim}}$ with $x_{\text{lim}}(0) = (\widehat{w} \otimes I)\mathbf{x}(0)$

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Synchronization by static output-coupling $u_i = -F \sum_{j \in \mathcal{N}_i}(y_i - y_j)$

- Coupled system: $\dot{\mathbf{x}} = (I \otimes A + \mathbf{L} \otimes BFC)\mathbf{x}$
- Synchronization by static output coupling $\iff$ $A - \lambda_i BFC$ Hurwitz for $\lambda_2, \ldots, \lambda_N$

Summary: Synchronization for $\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i$

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Synchronization by static output-coupling $u_i = -F\sum_{j\in\mathcal{N}_i}(y_i - y_j)$

- Coupled system: $\dot{\mathbf{x}} = (I \otimes A + \mathbf{L} \otimes BFC)\mathbf{x}$
- Synchronization by static output coupling $\iff$ $A - \lambda_i BFC$ Hurwitz for $\lambda_2, \ldots, \lambda_N$

Synchronization by dynamic output-coupling $u_i = -Fw_i,\ w_i \approx \sum_{j\in\mathcal{N}_i}(x_i - x_j)$

- Closed loop error behavior: $\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{pmatrix} = \begin{bmatrix} I\otimes A - L\otimes BF & -I\otimes BF \\ 0 & I\otimes(A-GC) \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{e} \end{pmatrix}$
- Synchronization $\iff$ $A - \lambda_i BF$ and $(A - GC)$ Hurwitz for $\lambda_2, \ldots, \lambda_N$
- Synchronizable $\iff$ (A, B) stabilizable and (A, C) detectable