Advanced Systems Theory

Lecture 11: Multiagent systems: Consensus

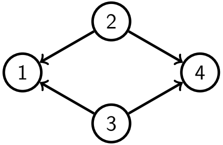
Prof. Dr. Stephan Trenn & Dr. Yahao Chen
Jan C. Willems Center for Systems and Control, University of Groningen

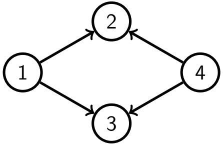
Groningen, Fall 2020-21

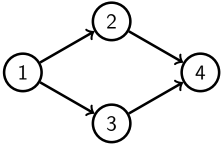
Recap graph theory

Question 1

Which of the following graphs corresponds to the Laplacian $\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$?

(i): 

(ii): 

(iii): 

Question 2

Which of the following statements is correct for the above graph?

(i) $\lambda_2 = 0$ because the graph contains an (undirected) cycle

(ii) $\lambda_2 = 0$ because $\operatorname{rank}(\mathbf{L}) = 2$

(iii) $\lambda_2 > 0$ because it consists of two overlapping spanning trees

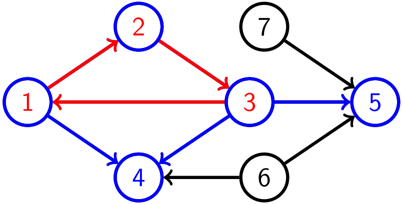
(iv) $\lambda_2 > 0$ because it is conncected

Proof of "no spanning tree $\implies \lambda_2 = 0$" revisited

Question 3

Assume $\mathcal{G}$ has a spanning tree. Is it true that then there is a root of the spanning tree with (in-)degree zero?

Correction of proof:



Step 1: Choose a maximal tree $(\mathcal{V}_T, \mathcal{E}_T)$

Step 2: Choose a maximal root set $\mathcal{V}_R \subseteq \mathcal{V}_T$ and relabel nodes

$\leadsto$ Block structure of Laplacian?
$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_R & 0 & 0 \\ * & \mathbf{L}_{T \setminus R} & * \\ 0 & 0 & \mathbf{L}_{N \setminus T} \end{bmatrix}$$

$\mathbf{L}\mathbf{1} = 0 \implies \mathbf{L}_R \mathbf{1} = 0$ and $\mathbf{L}_{N\setminus T}\mathbf{1} = 0$

$\implies$ rank $\mathbf{L}_R \leq n_R - 1$ and rank $\mathbf{L}_{N\setminus T} = N - n_T - 1 \implies$ rank $\mathbf{L} \leq N - 2$

The consensus problem

for simple integrator dynamics

The consensus problem: setup

Given

- N agents with simple scalar integrator dynamics:

$$\dot{x}_i = u_i, \quad x_i(0) = x_i^0 \in \mathbb{R}$$

- Communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ describing which information the agents can access from their neighbors

Objective

Design coupling laws $(x_i, x_{\mathcal{N}_i}) \mapsto u_i(x_i, x_{\mathcal{N}_i})$ such that

$$\forall i \in \mathcal{V}: \quad x_i(t) \to \overline{x} \text{ as } t \to \infty$$

In particular, $x_i(t) - x_j(t) \to 0$ as $t \to \infty$.

Attention: $\overline{x}$ not known by agents a priori

The consensus protocol for $\dot{x}_i = u_i$

Feasible but undesired choice of feedback

$u_i = -x_i$ leads to $\dot{x}_i = -x_i \quad \rightsquigarrow \quad x_i(t) \to 0$ exponentially

Additional requirement

If for some $t \geq 0$ $x_i(t) = x_j(t)$ for all $j \in \mathcal{N}_i$ (including $\mathcal{N}_i = \emptyset$) then $u_i(t) = 0$.

Simplest case: Agent i has one neighbor j

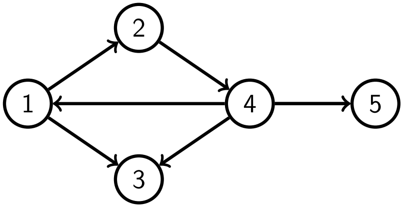
$$\left.\begin{array}{lcl} x_i(t) < x_j(t) & \stackrel{!}{\to} & \dot{x}_i(t) > 0 \leftrightarrow u_i(t) > 0 \\ x_i(t) > x_j(t) & \stackrel{!}{\to} & \dot{x}_i(t) < 0 \leftrightarrow u_i(t) < 0 \end{array}\right\} \to u_i(t) = x_j(t) - x_i(t)$$

More then one neighbor $\quad \rightsquigarrow \quad$ Diffusive coupling

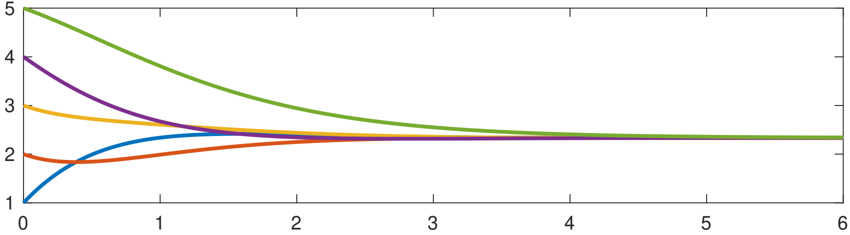
$$u_i(t) = -\sum_{j \in \mathcal{N}_i} \big(x_i(t) - x_j(t)\big)$$

Example

Consider the graph

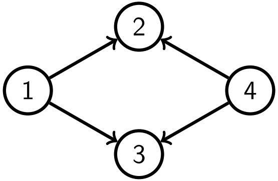


$$\left.\begin{aligned}
u_1 &= -(x_1 - x_4) = -[1, 0, 0, \text{-}1, 0] \cdot \mathbf{x} \\
u_2 &= -(x_2 - x_1) = -[\text{-}1, 1, 0, 0, 0] \cdot \mathbf{x} \\
u_3 &= -\big((x_3 - x_1) + (x_3 - x_4)\big) = -[\text{-}1, 0, 2, \text{-}1, 0] \cdot \mathbf{x} \\
u_4 &= -(x_4 - x_2) = -[0, \text{-}1, 0, 1, 0] \cdot \mathbf{x} \\
u_5 &= -(x_5 - x_4) = -[0, 0, 0, \text{-}1, 1] \cdot \mathbf{x}
\end{aligned}\right\} \rightarrow \mathbf{u} = -\begin{bmatrix} 1 & 0 & 0 & \text{-}1 & 0 \\ \text{-}1 & 1 & 0 & 0 & 0 \\ \text{-}1 & 0 & 2 & \text{-}1 & 0 \\ 0 & \text{-}1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \text{-}1 & 1 \end{bmatrix} \mathbf{x} = -\mathbf{L}\mathbf{x}$$

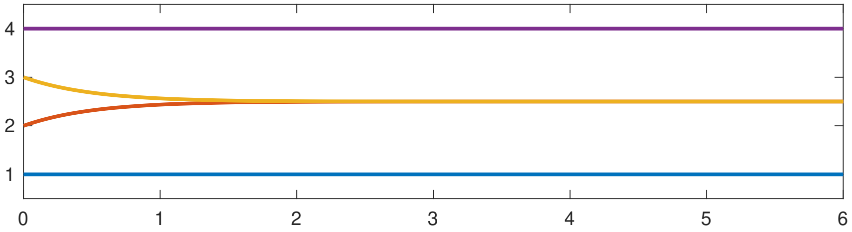


Another example

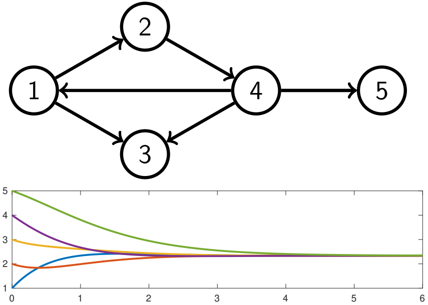
Consider now the graph

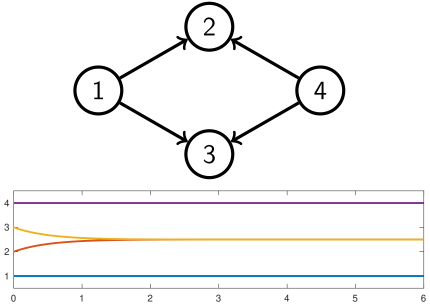


$$
\begin{aligned}
u_1 &= 0\\
u_2 &= -\big((x_2 - x_1) + (x_2 - x_4)\big)\\
u_3 &= -\big((x_3 - x_1) + (x_3 - x_4)\big)\\
u_4 &= 0
\end{aligned}
$$



Consensus: When and what?





Key questions

1. Under which conditions does consensus occur?
2. What is the consensus value?

Brain storming (use chat for answers)

Which property of the graph seems crucial for consensus?
On what does the consensus value probably depend?
Can you say something about the location of the consensus value?

Stephan Trenn, Jan C. Willems Center for Systems and Control, University of Groningen

Consensus: Necessary and sufficient condition

Connectivity

- Clearly, if $\mathcal{G}$ has two (or more) completely disconnected components $\rightsquigarrow$ no consensus
- However, strong connectivity doesn't seem to be necessary (cf. first example)
- Spanning tree necessary? ... Yes!

Desired consensus value is $\mathbf{x}(\infty) = \overline{x} \cdot \mathbf{1}$, where $\mathbf{1} = (1, 1, \ldots, 1)^\top \in \mathbb{R}^n$

Consensus feasible under diffusive coupling?

- Is consensus value equilibrium of closed loop $\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x}$? Yes! $\mathbf{L}\mathbf{1} = 0$
- Consensus only possible for all $\mathbf{x}(0)$ if $\ker \mathbf{L} = \operatorname{span}\{\mathbf{1}\}$

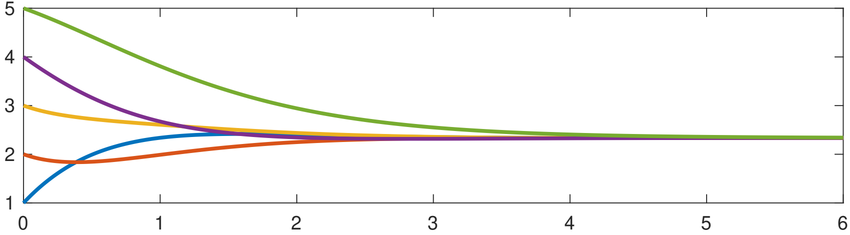
$$\ker \mathbf{L} = \operatorname{span}\{\mathbf{1}\} \iff \dim \ker \mathbf{L} = 1 \iff \lambda_2(\mathbf{L}) \neq 0 \iff \mathcal{G} \text{ has spanning tree}$$

Theorem

Diffusive coupling leads to consensus $\iff$ $\mathcal{G}$ *has a spanning tree*

Proof ...

The consensus value for $\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x}$



Lemma (Proof: Exercise)

For $t_0 \geq 0$ *let* $x_{\min}(t_0) := \min_{j\in\mathcal{V}} x_j(t_0)$ *and* $x_{\max}(t_0) := \max_{j\in\mathcal{V}} x_j(t_0)$, *then*

$$\forall t \geq t_0 \ \forall i \in \mathcal{V}: \quad x_i(t) \in \big[x_{\min}(t_0), x_{\max}(t_0)\big]$$

Key idea to find consensus value: Try to find invariant

Is there a linear combination $\mathbf{w}^\top \mathbf{x}(t)$ which remains constant (i.e. which is an invariant)?

Brain storming (use chat for answers)

Which vectors $\mathbf{w} \in \mathbb{R}^N$ lead to an invariant $\mathbf{w}^\top \mathbf{x}(t)$?

Calculation of consensus value

Lemma

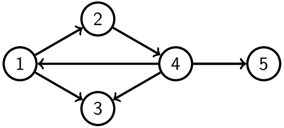
$$\mathbf{w}^\top \mathbf{x}(t) \text{ is invariant for } \dot{\mathbf{x}} = -\mathbf{L}\mathbf{x} \quad \iff \quad \mathbf{w} \in \ker \mathbf{L}^\top$$

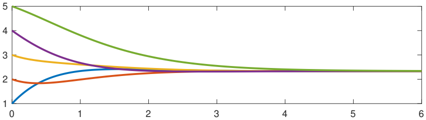
Furthermore, if $\operatorname{rank} \mathbf{L} = N-1$ *then there exists unique* $\widehat{\mathbf{w}} \in \ker \mathbf{L}^\top$ *with* $\widehat{\mathbf{w}}^\top \mathbf{1} = 1$, *in fact*

$$\widehat{\mathbf{w}} = \frac{\mathbf{w}}{\mathbf{w}^\top \mathbf{1}} \quad \text{and} \quad \widehat{\mathbf{w}} \geq 0$$

Corollary

$\mathcal{G}$ *has spanning tree* $\quad \Longrightarrow \quad \mathbf{x}(t) \to \overline{x} \cdot \mathbf{1}$ *with* $\overline{x} = \widehat{\mathbf{w}}^\top \mathbf{x}(0)$ *Proof ...*





$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \text{ with } \ker \mathbf{L}^\top = \operatorname{im} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\leadsto \widehat{\mathbf{w}} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ 1/3 \\ 0 \end{bmatrix} \text{ and } \overline{x} = \frac{x_1(0)+x_2(0)+x_4(0)}{3} = \frac{7}{3}$$

Stephan Trenn, Jan C. Willems Center for Systems and Control, University of Groningen

Interpretation of consensus value

Lemma (without proof)

The i-th component of $\widehat{\mathbf{w}}$ is non-zero $\iff$ *all nodes can be reached from node i*

Corollary

$\overline{x}$ only depends on initial values of agents which are roots of spanning trees

Corollary: Undirected case

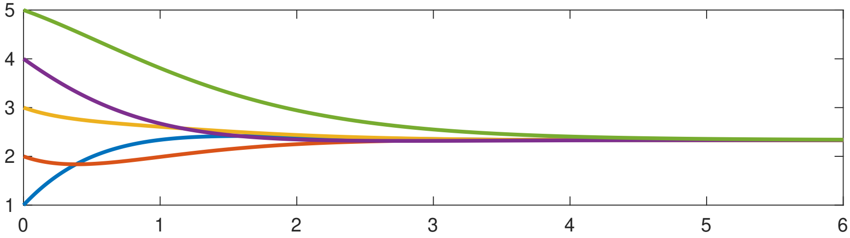
Let $\mathcal{G}$ be undirected and connected, then $\overline{x}$ is the average of initial values, i.e.

$$\overline{x} = \frac{1}{N} \sum_{i=1}^{N} x_i(0)$$

In particular, it is independent of the underlying graph structure!

Stephan Trenn, Jan C. Willems Center for Systems and Control, University of Groningen

Convergence rate



Question

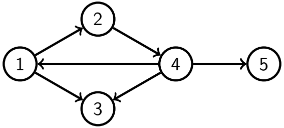
How fast do the agents converge towards the consensus value?

Theorem

$$|x_i(t) - \overline{x}| \le c_{\mathbf{x}(0)} e^{-\operatorname{Re}(\lambda_2) t}$$

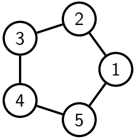
Reminder: For any Hurwitz matrix M we have $\|e^{Mt}\| \le c e^{-\lambda_{\min} t}$ for some $c > 0$ and $\lambda_{\min} = \in \{\ |\operatorname{Re}(\lambda)|\ \mid \lambda \text{ is eigenvalue of } M\ \}$
Proof: ...

Convergence rate for specific graphs

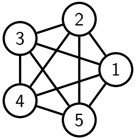


$\rightsquigarrow \quad \lambda_2 = 1$

$\rightsquigarrow \quad \lambda_2 \approx 0.382$



$\rightsquigarrow \quad \lambda_2 \approx 0.382$



$\rightsquigarrow \quad \lambda_2 = 3$

Summary

The consensus problem: The integrator dynamics case

- Agent dynamics: $\dot{x}_i = u_i$
- Goal: $x_i(t) \to \overline{x}$ for all $i \in \mathcal{V}$
- Diffusive coupling: $u_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j)$
- Resulting overall dynamics: $\mathbf{x} = -\mathbf{L}\mathbf{x}$

Consensus results for simple multi-agent system

- Consensus $\iff$ $\mathcal{G}$ has spanning tree (undirected case: is connected)
- $\overline{x} = \widehat{\mathbf{w}}^\top \mathbf{x}(0)$ where $\widehat{\mathbf{w}}^\top L = 0$ and $\mathbf{w}^\top \mathbf{1} = 1$ (undirected case: average of initial values)
- Convergence rate proportional to $e^{-\operatorname{Re}(\lambda_2)t}$

Stephan Trenn, Jan C. Willems Center for Systems and Control, University of Groningen