Advanced Systems Theory

Lecture 10: Multiagent systems: Motivation, graph theoretical background

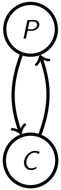
Prof. Dr. Stephan Trenn & Dr. Yahao Chen
Jan C. Willems Center for Systems and Control, University of Groningen

Groningen, Fall 2020-21

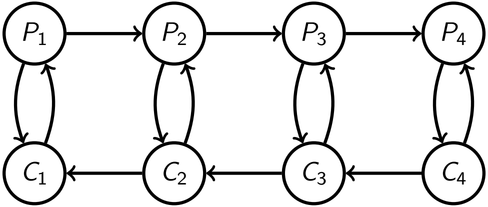
Multi-agent systems: Preliminaries

What are multi-agent systems?

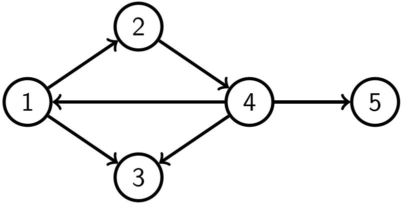
Feedback loop:



Local feedback of connected plants:
(with controller communication)



General interconnection between agents:



Motivation for multi-agent systems

Main motivations

- Complexity: Centralized control not feasible
- Privacy: Centralized control not desired
- Spatially distributed: Centralized control not physically possible
- Safety: No single point of failure

Key questions

$$\text{Local properties} \quad \overset{?}{\Longrightarrow} \quad \text{Global properties}$$

$$\text{Local controller design} \quad \Longrightarrow \quad \text{Global desired behavior}$$

Brain storming (use chat for answers)

What examples of multi-agent systems come to your mind?

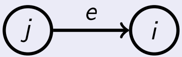
Stephan Trenn, Jan C. Willems Center for Systems and Control, University of Groningen

Interconnection = graph

Definition (Graph)

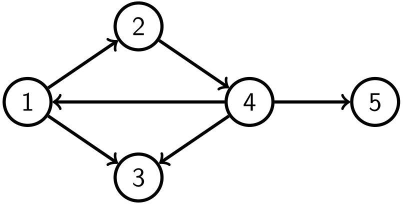
A graph is a tuple $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where

- $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of nodes — agents
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of (directed) edges — influence
- $e = (j, i) \in \mathcal{E}$ is an edge from node j to node i



j and i are adjacent

Example:



$$\mathcal{V} = \{1, 2, 3, 4, 5\}$$
$$\mathcal{E} = \{(1,2),\ (1,3),\ (2,4),\ (4,1),\ (4,3),\ (4,5)\}$$

Question 1

Have you seen graphs in any other lecture before?

Graph related definitions

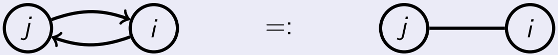
Standard assumptions

In the following we will always assume for $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:

- $\mathcal{V}$ is finite
- $\mathcal{E}$ contains each edge at most once
- no self loops, i.e. $\forall i \in \mathcal{V}: \ (i,i) \notin \mathcal{E}$

Definition (Undirected graphs)

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called undirected $:\Leftrightarrow \ \left[(j,i) \in \mathcal{E} \iff (i,j) \in \mathcal{E} \right]$



Definition (Neighbors)

- $\mathcal{N}_i := \{ \ j \in \mathcal{V} \mid (j,i) \in \mathcal{E} \ \}$ set of neighbors (predecessors), $|\mathcal{N}_i|$ is (in-)degree of node i
- $\mathcal{S}_j := \{ \ i \in \mathcal{V} \mid (j,i) \in \mathcal{E} \ \}$ set of successors, $|\mathcal{S}_j|$ is out-degree of node j

Matrix representations of graphs

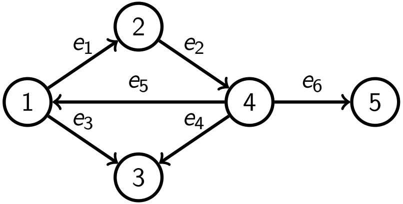
Definition (Adjacency and incidence matrix)

Consider graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E} = \{e_1, e_2, \ldots, e_M\}$

- adjacency matrix: $\mathbf{A} = [a_{ij}] \in \{0,1\}^{N \times N}$ with $a_{ij} = 1 \iff (j,i) \in \mathcal{E}$.
- incidence matrix*: $\mathbf{N} = [n_{ik}] \in \{-1,0,1\}^{N \times M}$ with $n_{ik} = \begin{cases} 1, & \text{if } e_k = (*, i), \\ -1, & \text{if } e_k = (i, *), \\ 0, & \text{otherwise} \end{cases}$

* Convention for undirected graphs: Only one of the two edges (i,j) and (j,i) is considered

Consider the following graph:



Question 2

Which is the adjecency matrix of the graph?

$$(i) \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad (ii) \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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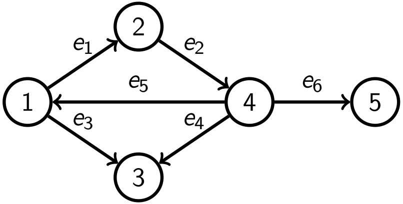
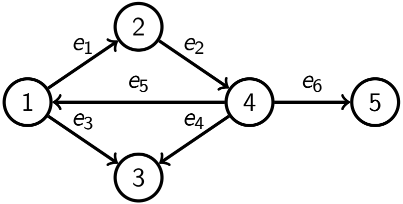


Question 3

Which is the incidence matrix of the graph?

$$(i) \begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad (ii) \begin{bmatrix} -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Some basic properties of adjecency and incidence matrices



$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 4

Which property is wrong in general?

(i) $\mathcal{G}$ is undirected $\iff$ $\mathbf{A} = \mathbf{A}^\top$

(ii) $\operatorname{rank} \mathbf{N} \leq N - 1$

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(iv) the in-degree $|\mathcal{N}_i|$ is the i-th row-sum of $\mathbf{A}$

(v) the difference between out- and in-degree of node i is the i-th row sum $\mathbf{N}$

The Laplacian matrix of a graph

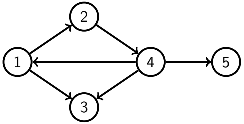
Definition

$\mathbf{L} := \mathbf{D} - \mathbf{A}$ is called Laplacian matrix of $\mathcal{G}$, where

$$\mathbf{D} = \begin{bmatrix} |\mathcal{N}_1| & & & \\ & |\mathcal{N}_2| & & \\ & & \ddots & \\ & & & |\mathcal{N}_N| \end{bmatrix}$$

is the (in-)degree matrix of $\mathcal{G}$.

Laplacian matrix of



is ... $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

Question 5

Which property is NOT true in general?

(i) $\mathbf{L} = \mathbf{L}^\top \iff \mathcal{G}$ is undirected

(ii) 0 is an eigenvalue of $\mathbf{L}$ with eigenvector $(1, 1, \ldots, 1)$

(iii) $\operatorname{rank} \mathbf{L} \leq N - 1$

(iv) $(1, 1, \ldots, 1)$ is in the left-kernel of $\mathbf{L}$

Some further properties of the Laplacian matrix

Theorem (Spectral properties of Laplacian)

All eigenvalues λ *of* $\mathbf{L}$ *satisfy* $\mathrm{Re}(\lambda) \geq 0$.

Proof: ...

It is common to order the eigenvalues of $\mathbf{L}$ as

$$0 = \lambda_1 \leq \mathrm{Re}(\lambda_2) \leq \mathrm{Re}(\lambda_3) \leq \ldots \leq \mathrm{Re}(\lambda_n)$$

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \text{ (undirected case)}$$

Theorem

If $\mathcal{G}$ *is undirected then*

$$\mathbf{L} = \mathbf{N}\mathbf{N}^\top$$

in particular, the product does not depend on the choice of "directions" in $\mathbf{N}$

Proof: ...

Corollary

For an undirected graph

$$\operatorname{rank} \mathbf{L} = \operatorname{rank} \mathbf{N}$$

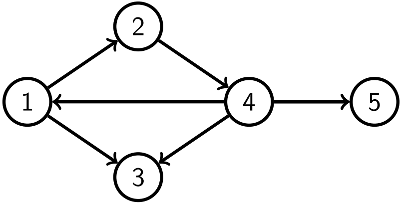
Paths and connectivity

Definition

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

- A path is a sequence of nodes $(i_0, i_1, \ldots, i_K)$ such that $(i_{k-1}, i_k) \in \mathcal{E}$ for all $k = 1, 2, \ldots, K$
- A cycle is a path $(i_0, i_1, \ldots, i_K)$ with $i_0 = i_K$
- $\mathcal{G}$ is called tree $:\Leftrightarrow \exists$ root $r \in \mathcal{V}$: $\forall i \in \mathcal{V} \setminus \{r\}$ $\exists$ unique path from r to i
- $\mathcal{G}$ is said to have a spanning tree $:\Leftrightarrow \exists \mathcal{E}_T \subseteq \mathcal{E}$ such that $\mathcal{G}_T = (\mathcal{V}, \mathcal{E}_T)$ is a tree
- $\mathcal{G}$ is strongly connected $:\Leftrightarrow \forall i, j \in \mathcal{V}$ there is a path connecting i with j

Consider again the graph:



Question 6

Is the graph strongly connected?

Question 7

Does the graph has a spanning tree?

Connectivity properties

Theorem

In general:

$$\mathcal{G} \text{ is strongly connected} \implies \mathcal{G} \text{ has spanning tree}$$

For undirected graph:

$$\mathcal{G} \text{ is strongly connected} \iff \mathcal{G} \text{ has spanning tree} \quad :\Leftrightarrow \mathcal{G} \text{ is connected}$$

Theorem (Connectivity and Laplacian matrix)

$\mathcal{G}$ *has a spanning tree* $\iff \mathrm{Re}(\lambda_2(\mathbf{L})) > 0$

Proof: ...

Summary

Multi-agent systems

- Many motivations and applications
- Interconnection $\leftrightarrow$ graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Basic definitions/properties of graphs

- Adjecancy, incidence, Laplacian matrix
- Connectivity (strongly connected, spanning tree)
- $\mathcal{G}$ has spanning tree $\iff$ $\operatorname{Re} \lambda_2(\mathbf{L}) > 0$

Attention: No lecture next Monday (5 October 2020, 15:00-17:00)