Solutions, stability and stabilization of state-dependent linear switched differential-algebraic equations

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Abstract

Recent advancements in the analysis and stabilization of linear switched differential-algebraic equations (SwDAEs) have yielded fruitful results, particularly for time-dependent switching signals. However, the state-dependent case, which finds significant applications in switched ordinary differential equations (SwODEs) [13], has received considerably less attention in the literature regarding SwDAEs. This paper aims to address this gap by introducing a novel jump rule that resolves the contradiction between the consistency projector [14, 15] and the state-dependent switching rule. Building upon this new jump rule, we investigate solutions and various sliding modes that encompass both jump and flow dynamics for state-dependent SwDAEs. Additionally, we extend well-established stability and stabilization results from state-dependent SwODEs to SwDAEs. These extensions include stability criteria applicable to arbitrary state-dependent switching signals, stable convex combinations, and the "min-max-switching rule" for stabilization via state-dependent signal. Both numerical and physical examples are presented to showcase the application of these state-dependent stabilization rules in SwDAEs.

Key words: Switched systems; differential-algebraic equations; state-dependent switching rule; jump and flow solutions; arbitrary switching; convex combinations; stability and stabilization

1 Introduction

We consider a linear switched differential-algebraic equation (SwDAE)

$$\Delta_\sigma : \quad E_\sigma \dot{x} = H_\sigma x, \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the vector of generalized states, and $\sigma : \mathbb{R}^n \to \mathcal{N}$ represents a state-dependent switching signal with a locally finite number of jumps. Here, $\mathcal{N} := \{1, \ldots, N\}$, with $N \in \mathbb{N}$ being the number of DAE modes. For each $p \in \mathcal{N}$, the maps $E_p : \mathbb{R}^n \to \mathbb{R}^n$ and $H_p : \mathbb{R}^n \to \mathbb{R}^n$ are linear maps. SwDAEs have been proved to be powerful tools for modeling various physical systems, including electrical circuits with switching devices [25, 19], power grids [9], and structure changing mechanics/robotics [8].

The analysis of linear SwDAEs with time-dependent switching signals has been a subject of considerable interest among researchers for several decades. Numerous topics closely related to the subject matter of this paper have been explored. For instance, stability analysis under arbitrary (time-dependent) switching has been extensively investigated [14, 31, 26, 22], as well as stability analysis under slow switching rules such as average dwell time method [14, 34]. Stabilization techniques utilizing fast switching and averaging methods have also been explored [17, 18]. Some extensions of results for discrete-time time-dependent SwDAEs can be found in e.g., [32, 1]. It is worth noting that most of these results have their counterparts for switched ordinary differential equations (ODEs), which can be found in the comprehensive book by Liberzon [13] and the references therein.

Despite the frequent occurrence of state-dependent SwDAEs in physical systems, a general theory on their solutions and stability is rare to find. Typically, the focus has been on studying specific systems rather than establishing a broad theoretical framework. For example, in [19], the passivity of a state-dependent SwDAE-modelled circuit was discussed, providing insights into a specific application. In [21] and [2], numerical methods and Modelica tools were utilized, respectively, to simulate physical examples that involved state-dependent

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SwDAEs. These studies shed light on the behavior of particular instances of state-dependent SwDAEs but do not provide a comprehensive theory applicable to a wide range of systems.

Defining state jumps that connect the continuous solutions of different modes is a primary challenge in solving state-dependent SwDAEs. In the case of time-dependent SwDAEs, the jumps of linear SwDAEs were typically defined using a set-value map known as the consistency projector (refer to [24, 15]). However, state-dependent switching laws divide the (generalized) state space into distinct active regions. Applying the consistency projector directly to state-dependent SwDAEs may lead to a consistent point that *contradicts* the active region. Additionally, *sliding modes* can occur on the switching surface of state-dependent switched ODEs [13]. Similar phenomena can be expected in the case of SwDAEs. Given that both state jumps and continuous solutions can traverse the switching surface, characterizing the sliding behavior that combines jumps and continuous solutions becomes a challenging task. Therefore, the primary objective of this paper is to propose a suitable definition of state jumps and to address the sliding mode behavior for state-dependent linear SwDAEs.

Another significant challenge lies in the stabilization of linear SwDAEs. It is well-known [13] that for switched ODEs of the form $\dot{x} = A_\sigma x$, stabilization can be achieved through two approaches: employing fast switching to approximate the solution of $\dot{x} = A_\sigma x$, or using a hysteresis state-dependent switching strategy [29]. The extension of the fast switching approach to SwDAEs can be found in [31, 17, 18, 27]. In these works, assumptions such as commutativity of flow matrices [16] are often required to approximate the solution of a SwDAE with that of a switched ODE perturbed by small terms [17]. Weaker assumptions, such as the **(PA)** assumption on the consistency projectors, are made in [18] to derive an averaged ODE model with jumps for SwDAEs. The second objective of this paper is to explore the application of convex combinations to SwDAEs in order to establish hysteresis state-dependent switching rules that facilitate their stabilization.

It is worth noting that state-dependent DAEs have close connections to complementarity systems [4, 11]. Complementarity systems are widely used to model various systems, including circuits with diodes and transistors, mechanics with unilateral constraints, and optimization problems [23]. Therefore, the results presented in this paper regarding the solutions of linear state-dependent DAEs may also contribute to the understanding and analysis of linear complementarity systems.

The structure of the paper is as follows: Section 2 provides essential concepts and notions related to linear DAEs. In Section 3, we introduce novel definitions of state jumps and jump-flow solutions for SwDAEs, and we discuss different sliding modes for both the jumps and the flow solutions. Section 4 presents stability and stabilization results, including stability analysis under arbitrary state-dependent switching rules and the hysteresis stabilization law for SwDAEs with and without Hurwitz convex combinations. Finally, in Section 5, we summarize the conclusions drawn from this work and discuss potential future research directions.

2 Preliminaries

The following notation is used throughout the paper. $\mathbb{N}$ and $\mathbb{R}$ are the natural numbers and real numbers, respectively. For a matrix $M \in \mathbb{R}^{n\times m}$, the kernel (null space) of M is denoted by $\ker M$, the image of M is denoted by $\operatorname{im} M$, the transpose of M is $M^\top$. The image of a set $S \subseteq \mathbb{R}^n$ under M is $MS := \{Mx \in \mathbb{R}^n \mid x \in S\}$ and the pre-image of S under M is $M^{-1}S := \{x \in \mathbb{R}^n \mid Mx \in S\}$. The identity matrix of size $n \times n$ is denoted by I_n.

The non-switching case of (1) is a DAE $E\dot{x} = Hx$, denoted by $\Delta = (E, H)$. A $\mathcal{C}^1$-curve $x : [0, \infty) \to \mathbb{R}^n$ is called a $\mathcal{C}^1$*-solution* of Δ if $E\dot{x}(t) = Hx(t)$ for all $t \in [0, \infty)$. A point $x_0 \in \mathbb{R}^n$ is called *consistent* if there exists a $\mathcal{C}^1$-solution $x(\cdot)$ starting from x_0, i.e., $x(0) = x_0$. The set of all consistent points is called *consistency space*, denoted by $\mathfrak{C}$, which coincides with the limit $\mathscr{V}^* = \mathscr{V}_n$ of the sequence of subspace $\mathscr{V}_k$, which, together with $\mathscr{W}_k$, are called the Wong sequences [30]:

$$\begin{cases} \mathscr{V}_0 = \mathbb{R}^n, & \mathscr{V}_{k+1} = H^{-1}E\mathscr{V}_k,\ k \geq 1, \\ \mathscr{W}_0 = \{0\}, & \mathscr{W}_{l+1} = E^{-1}H\mathscr{W}_l,\ l \geq 1. \end{cases} \tag{2}$$

The DAE Δ is called *regular* if $\det(sE - H)$ is not identically zero. The regularity guarantees the existence and uniqueness of $\mathcal{C}^1$-solutions. We assume throughout that all DAE modes in the present paper are regular. Any regular DAE can be always transformed, via two constant invertible matrices Q and P, into the Weierstrass form [28, 3] $\tilde{\Delta} = (QEP^{-1}, QHP^{-1})$:

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{3}$$

where $A_1 \in \mathbb{R}^{n_1\times n_1}$ and $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with nilpotency index ν, i.e. $N^{\nu-1} \neq 0$ and $N^\nu = 0$, where $n_1 + n_2 = n$. The index of Δ is defined to be the nilpotency index ν of N, thus we have $N = 0$ for index-1 DAEs. Note that the Wong sequences the matrices Q, P can be constructed with the help of $\mathscr{V}^*$ and $\mathscr{W}^* := \mathscr{W}_n$ [3], the variables $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are coordinates on $\mathscr{V}^*$ and $\mathscr{W}^*$, respectively, and $\mathscr{V}^* \oplus \mathscr{W}^* = \mathbb{R}^n$.

Discontinuous solutions are considered for differential-algebraic equations (DAEs) when the initial point $x_0^- \notin$

$\mathfrak{C}$ is not consistent. One approach to achieve a consistent initialization is to introduce a jump (an instant change) from x_0^- to x_0^+, where $x_0^+ \in \mathfrak{C}$ is a consistent point that can be uniquely defined using the consistency projector $\Pi_{E,H}$ [24, 14, 25]. For a regular DAE $\Delta = (E, H)$, the consistency projector $\Pi_{E,H}$ is defined as follows:

$$\Pi_{E,H} := P^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} P, \tag{4}$$

where P is from (3). Now we construct two matrices A^{df}, which is called the flow matrix [16, 25], and A^{jp}, which we introduce in the present paper for the jump dynamics, given by, respectively,

$$A^{\mathrm{df}} := P^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} P \text{ and } A^{\mathrm{jp}} := P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -I_{n_2} \end{bmatrix} P. \tag{5}$$

where A_1 and P are the matrices from (3).

Remark 2.1. The ODE $\dot{x} = A^{\mathrm{df}}x$ has the same $\mathcal{C}^1$-solution with Δ for any consistent initial point x_0^+, and any solution $J : [0, \infty) \to \mathbb{R}^n$ of the ODE $\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} = A^{\mathrm{jp}}J(\tau)$ starting from $x_0^- = J(0)$ is a jump solution/trajectory in the sense of Definition 3.1 below (by setting $\Omega_p = \mathbb{R}^n$). It is remarked that the consistency projector $\Pi_{E,H}$ and the matrix A^{jp} are related by

$$\Pi_{E,H} = \Phi_\infty^{A^{\mathrm{jp}}}, \tag{6}$$

where $\Phi_\tau^{A^{\mathrm{jp}}} := e^{A^{\mathrm{jp}}\tau}$ is the flow map of A^{jp}. Moreover,

$$\operatorname{im} A^{\mathrm{jp}} = \mathscr{W}^* \text{ and } \ker A^{\mathrm{jp}} = \mathscr{V}^*.$$

Note that one may replace $-I_{n_2}$ of A^{jp} by any other Hurwitz matrix, then equation (6) still holds. However, such a replacement results in a trajectory $J(\tau)$ which may not be the "shortest" path (i.e., straight line) connecting x_0^- and x_0^+.

3 Solutions of state-dependent SwDAEs

3.1 State-dependent jumps and jump sliding modes

Given a SwDAE Δ_σ with a state-dependent switching signal $\sigma(x) = p$ for $x \in \Omega_p \subseteq \mathbb{R}^n$, consider an inconsistent initial point $x_0^- \notin \mathfrak{C}_p$ and $x_0^- \in \Omega_p$. If we directly apply the consistency projector Π_p to x_0^-, we obtain the consistent point $x_0^+ = \Pi_p x_0^- \in \mathfrak{C}_p$. However, in general, $x_0^+ \notin \Omega_p$, which means that *the resulting consistent point violates the switching rule*. Therefore, it becomes necessary to introduce a new definition of jumps for state-dependent SwDAEs to address this issue.

In our recent paper [5], we proposed a generalization of the notion of jumps for nonlinear DAEs. Instead of considering a jump as a simple instantaneous change, we introduce the concept of a parametrized curve that connects x_0^- with a consistent point x_0^+, while adhering to specific jump rules. By adopting this approach, we establish a definition of jumps for SwDAEs with state-dependent switching as follows.

Definition 3.1 (state-dependent jumps)**.** Consider a SwDAE Δ_σ with a state-dependent switching signal: $\sigma(x) = p$ if $x \in \Omega_p \subseteq \mathbb{R}^n$, for $p \in \mathcal{N}$. Given an initial point $x_0^- \in \mathbb{R}^n$, if an absolutely continuous curve $J : [0, \infty) \to \mathbb{R}^n$ satisfies $J(0) = x_0^-$ and $\forall \tau \in [0, \infty)$:

$$\begin{cases} J(\tau) \cap \bigcup\limits_{p=1}^{N} (\mathfrak{C}_p \cap \Omega_p) = \emptyset, \\ \dfrac{dJ(\tau)}{d\tau} = A_p^{\mathrm{jp}} J(\tau) \text{ if } J(\tau) \in \Omega_p, \end{cases} \tag{7}$$

and $J(\infty) = x_0^+ \in \bigcup\limits_{p=1}^{N} (\mathfrak{C}_p \cap \Omega_p)$, then $J(\tau)$ is called a (convergent) jump solution/trajectory. If $J(\infty) \notin \bigcup\limits_{p=1}^{N} (\mathfrak{C}_p \cap \Omega_p)$, then it is called a (divergent) jump solution/trajectory. The change $x_0^- \to x_0^+$ associated with the jump trajectory $J(\cdot)$ is called a (convergent or divergent) jump of Δ_σ.

Remark 3.2. The motivation of the above jump rule is to keep the jump direction $\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau}$ in the subspace $\mathscr{W}_p^*$ (recall that $\operatorname{im} A_p^{\mathrm{jp}} = \mathscr{W}_p^*$) when $J(\tau)$ is in the active region Ω_p. For each sub-mode Δ_p, the idea of keeping the jump direction in $\mathscr{W}_p^*$ can be easily seen via the Weierstrass form (3). Indeed, any inconsistent initial point (x_{10}, x_{20}) of (3) jumps into $(x_{10}, 0)$, i.e., only x_2-variables are allowed to jump (see e.g. [24] for the distributional arguments about its reason). Recall that x_2 are coordinates on $\mathscr{W}^*$, which means that the jump direction stays in $\mathscr{W}^*$. Moreover, the jump solution $J(\tau)$ in each Ω_p is the "shortest" path along $\mathscr{W}_p^*$ towards $\mathscr{V}_p^*$, see Remark 2.1.

By the above definition, each jump solution of a state-dependent SwDAE can be seen as a solution of a state-dependent switched ODE system

$$\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} = A_\sigma^{\mathrm{jp}} J(\tau), \quad \sigma(J) = p \text{ if } J \in \Omega_p, \tag{8}$$

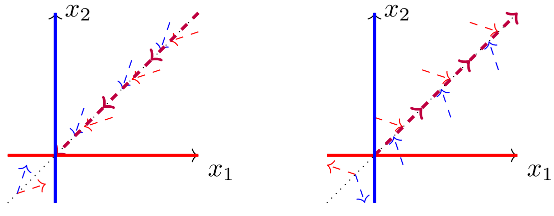
where, for each p, A_p^{jp} is the jump matrix of Δ_p defined in (5), the trajectory starts from a given initial point x_0^-, then it either reaches a consistent point $x_0^+ \in \bigcup\limits_{p=1}^{N} (\mathfrak{C}_p \cap \Omega_p)$ and stops or diverges to infinity as $\tau \to \infty$.

It is known that sliding behaviors can happen on the switching surface of a state-dependent switched ODE [13]. More specifically, let S_{pq} be the switching surface of the p-th and the q-th mode of (8), if both the vectors $f_p(J) = A_p^{\mathrm{jp}} J$ and $f_q(J) = A_q^{\mathrm{jp}} J$ point towards S_{pq}, then there exists a convex combination $F(J) = \alpha f_p(J) + (1-\alpha) f_q(J)$ for $0 \leq \alpha \leq 1$ such that $F(J) \in T_J S_{pq}$, where $f_p(J) = A_p^{\mathrm{jp}} J$, $f_q(J) = A_q^{\mathrm{jp}} J$ and $T_J S_{pq}$ is the tangent space of S_{pq} at $J \in S_{pq}$, then a solution (in the sense of Filippov [10]) of the differential inclusion $\frac{\mathrm{d}J}{\mathrm{d}\tau} \in F(J)$ starting from any point $x_0^- \in S_{pq}$ is a solution of (8) starting from x_0^-. Thus for the SwDAE Δ_σ, a jump solution $J(\tau)$ can slide on the switching surface *before* reaching any consistent point $x_0^+ \in \bigcup_{p=1}^{N} \mathfrak{C}_p$, which we will call a *jump sliding mode* of Δ_σ.

Example 3.3. Consider a SwDAE $\Delta_\sigma = (E_\sigma, H_\sigma)$ defined on $\mathbb{R}^2$ with two modes $\Delta_1 : \left[\begin{smallmatrix} 1 & -\gamma \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ and $\Delta_2 : \left[\begin{smallmatrix} -\gamma & 1 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$. The state-dependent switching signal is

$$\sigma = \begin{cases} 1 & \text{if } \gamma(x_1 - x_2) \geq 0, \\ 2 & \text{if } \gamma(x_1 - x_2) < 0. \end{cases}$$

By a direct calculation, we get $A_1^{\mathrm{jp}} = P_1^{-1} \left[\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right] P_1 = \left[\begin{smallmatrix} -1 & 0 \\ -\frac{1}{\gamma} & 0 \end{smallmatrix}\right]$ and $A_2^{\mathrm{jp}} = P_2^{-1} \left[\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right] P_2 = \left[\begin{smallmatrix} 0 & -\frac{1}{\gamma} \\ 0 & -1 \end{smallmatrix}\right]$, where $P_1 = \left[\begin{smallmatrix} 1 & -\gamma \\ 1 & 0 \end{smallmatrix}\right]$ and $P_2 = \left[\begin{smallmatrix} -\gamma & 1 \\ 0 & 1 \end{smallmatrix}\right]$. It is seen that the switching surface $S_{12} = \left\{ J = (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1 = x_2 \right\}$ and we have $\alpha f_1(J) + (1-\alpha) f_2(J) \in T_J S_{12}$, where $f_1(J) = A_1^{\mathrm{jp}} J$, $f_2(J) = A_2^{\mathrm{jp}} J$, $\alpha = 0.5$ and $T_J S_{12} = \operatorname{im} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Moreover, $f_1(J)$ and $f_2(J)$ point towards S_{pq} on the first quadrant (not on the third quadrant). Thus given any inconsistent point $x_0^- = \left[\begin{smallmatrix} x_{10}^- \\ x_{20}^- \end{smallmatrix}\right] \in S_{12} \cap \{x \in \mathbb{R} \,|\, x_1 \geq 0, x_2 \geq 0\}$, i.e., $x_0^- \neq 0$ and $x_{10}^- = x_{20}^- > 0$, we have jump sliding modes. As seen from Fig 1, the jump sliding mode $J(\tau)$ converges to 0 (implying that $x_0^+ = 0$ is the resulting consistent point) if $\gamma > 1$, and $J(\tau)$ diverges if $\gamma < -1$.



(a) $\gamma > 1$ (b) $\gamma < -1$

Fig. 1. Red and blue dashed arrows: Jump directions of Δ_1 and Δ_2, Red and blue lines: $\mathfrak{C}_1$ and $\mathfrak{C}_2$, purple dashed line with arrows: Jump sliding modes.

Now given an initial point x_0^- and a state-dependent switching signal $\sigma(x) = p$ for $x \in \Omega_p$, we assume without loss of generality that

(A1). $\bigcup_{p=1}^{N} \Omega_p = \mathbb{R}^n$.

(A2). $\sigma(x) = 1$ if $x \in \bigcap_{i=1}^{l} \Omega_i$ for $l \geq 2$.

Assumption (A1) is to ensure that $\sigma(x)$ exists for all $x \in \mathbb{R}^n$ and that $\bigcap_{p=1}^{N} (\mathfrak{C}_p \cap \Omega_p) \neq \emptyset$ (the origin $0 \in \bigcap_{p=1}^{N} \mathfrak{C}_p$ so $0 \in \bigcap_{p=1}^{N} (\mathfrak{C}_p \cap \Omega_p)$ if (A1) holds). Assumption (A2) is to guarantee that $\sigma(x)$ is uniquely defined by x. Then we show that it is possible to calculate the jump $x_0^- \to x_0^+$ of Δ_σ via the following algorithm with the help of the consistency projectors.

Algorithm 1 State dependent jumps algorithm

Input: $x_0^- \in \mathbb{R}^n$
Output: $x_0^+ \in \bigcup_{p=1}^{N} (\mathfrak{C}_p \cap \Omega_p)$

1: **if** $x_0^- \in \bigcup_{p=1}^{N} (\mathfrak{C}_p \cap \Omega_p)$ **then**
2:     **return** $x_0^+ = x_0^-$
3: **end if**
4: Permute the index of the subsystems of Δ_σ such that $x_0^- \in \bigcap_{i=1}^{l} \Omega_i$ and $\sigma(x_0^-) = 1$, for a maximal integer $1 \leq l \leq N$.
5: Set $\hat{x}_0^+ \leftarrow \Pi_{E_1, H_1} x_0^-$.
6: **if** $\forall 0 \leq \alpha \leq 1 : (1-\alpha) x_0^- + \alpha \hat{x}_0^+ \in \Omega_1$ **then**
7:     **return** $x_0^+ = \hat{x}_0^+$
8: **else**
9:     Find the smallest $0 < \alpha^* \leq 1$ such that $(1-\alpha^*) x_0^- + \alpha^* \hat{x}_0^+ \in \partial\Omega_1$, where $\partial\Omega_1$ denotes the boundary of Ω_1.
10:     Set $x_0^- \leftarrow (1-\alpha^*) x_0^- + \alpha^* \hat{x}_0^+$.
11:     Go to Step 1.
12: **end if**

Proposition 3.4. *Given a SwDAE Δ_σ. Assume* (A1)-(A3) *hold, where*

(A3). *There are no jump sliding modes.*

If Algorithm 1 returns to a point $x_0^+ \in \mathbb{R}^n$, then the change $x_0^- \to x_0^+$ is a (convergent) jump of Δ_σ in the sense of Definition 3.1.

The proof of Proposition 3.4 is omitted as Algorithm 1 follows exactly the same jump rule as in Definition 3.1

due to the relation $\Pi_{E,H} = \Phi_{\infty}^{A^{\mathrm{jp}}}$. We give the following remarks for Algorithm 1. For a point $x_0^- \in S \subseteq \partial\Omega_1$, it is possible that sliding jump mode exists on the switching surface S. In that case, $(1-\alpha)x_0^- + \alpha\hat{x}_0^+ \notin \Omega_1$ for all $0 < \alpha \leq 1$, thus α^* in Algorithm 1 does not exist, so the cases of sliding jump modes are *excluded* in Proposition 3.4. Moreover, it is possible that Algorithm 1 does not return to any point, a simple case is that $\bigcup_{p=1}^{N}(\mathfrak{C}_p \cap \Omega_p) = \emptyset$, then x_0^+ does not exist but the system can still have a divergent jump. Even $\bigcup_{p=1}^{N}(\mathfrak{C}_p \cap \Omega_p)$ is not empty, Algorithm 1 may still not return to a point x_0^+, as we show in the following simple example.

Example 3.5. Consider a SwDAE Δ_σ with the states $x = (x_1, x_2) \in \mathbb{R}^2$ and two modes

$$\Delta_1 : \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\Delta_2 : \begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $\gamma > 0$ is a constant. The following switching signal is chosen:

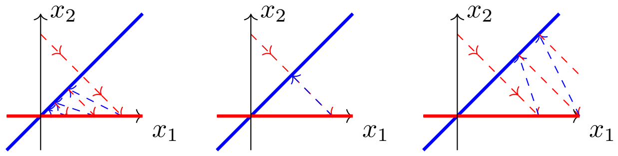
$$\sigma(x) = \begin{cases} 1, & \text{if } x \in \mathfrak{C}_2 \setminus \{0\}, \\ 2, & \text{if } x \in \mathfrak{C}_1 \setminus \{0\}, \\ 1 \text{ or } 2, & \text{if } x \in (\overline{\mathfrak{C}_1} \cap \overline{\mathfrak{C}_2}) \cup \{0\}, \end{cases} \tag{9}$$

where $\overline{\mathfrak{C}_1}$ and $\overline{\mathfrak{C}_2}$ denote, respectively, the complimentary set of the consistency space $\mathfrak{C}_1 = \left\{ x \in \mathbb{R}^2 \mid x_2 = 0 \right\}$ and $\mathfrak{C}_2 = \left\{ x \in \mathbb{R}^2 \mid x_1 - x_2 = 0 \right\}$. To fix $\sigma(x)$ when $x \in (\overline{\mathfrak{C}_1} \cap \overline{\mathfrak{C}_2}) \cup \{0\}$ and to avoid jump sliding modes, we assume that the change the value of $\sigma(x)$ may happen only when $x \in \mathfrak{C}_1 \cup \mathfrak{C}_2$. So the only possible jumping point x_0^+ is the origin because $\bigcup_{p=1}^{2}(\mathfrak{C}_p \cap \Omega_p) = \{0\}$. Given a point $x_0^- = \left[\begin{smallmatrix} 0 \\ c \end{smallmatrix}\right]$ with $c > 0$, we calculate the jump by Algorithm 1 and in the $2k$-th iteration,

$$\hat{x}_0^+ = (\Pi_2\Pi_1)^k \left[\begin{smallmatrix} c \\ 0 \end{smallmatrix}\right] = \left(\begin{bmatrix} \frac{1}{\gamma+1} & \frac{1}{\gamma+1} \\ \frac{1}{\gamma+1} & \frac{1}{\gamma+1} \end{bmatrix} \right)^k \left[\begin{smallmatrix} c \\ 0 \end{smallmatrix}\right]$$

$$= \frac{1}{2} \left(\frac{2}{\gamma+1} \right)^{2^{k-1}} \left[\begin{smallmatrix} c \\ c \end{smallmatrix}\right],$$

where the consistency projectors $\Pi_1 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $\Pi_2 = \begin{bmatrix} \frac{1}{\gamma+1} & \frac{\gamma}{\gamma+1} \\ \frac{1}{\gamma+1} & \frac{\gamma}{\gamma+1} \end{bmatrix}$. Clearly, Algorithm 1 returns to $x_0^+ = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ if and only if $\gamma > 1$ because $\lim_{k\to\infty} \frac{1}{2}\left(\frac{2}{\gamma+1}\right)^{2^{k-1}} = 0$ for $\gamma > 1$. Observe that for $\gamma = 1$, we have in the $2k+1$-th iteration that $\hat{x}_0 = \Pi_1 \begin{bmatrix} \frac{c}{2} \\ \frac{c}{2} \end{bmatrix} = \left[\begin{smallmatrix} c \\ 0 \end{smallmatrix}\right] \neq \begin{bmatrix} \frac{c}{2} \\ \frac{c}{2} \end{bmatrix}$, thus the jump oscillates between two points $\begin{bmatrix} \frac{c}{2} \\ \frac{c}{2} \end{bmatrix}$ and $\left[\begin{smallmatrix} c \\ 0 \end{smallmatrix}\right]$; for $0 < \gamma < 1$, the jump goes to infinity, they are both divergent jumps in the sense of Definition 3.1.



(a) $\gamma > 1$ (b) $\gamma = 1$ (c) $0 < \gamma < 1$

Fig. 2. Red and blue dashed lines with arrows: Jumps of Δ_1 and Δ_2, Red and blue lines: $\mathfrak{C}_1$ and $\mathfrak{C}_2$.

3.2 Jump-flow solutions and jump-flow sliding modes

With the help of Definition 3.1, we define the impulse-free jump-flow solutions for state-dependent SwDAEs as follows:

Definition 3.6 (jump-flow solutions)**.** Consider a SwDAE Δ_σ with a state-dependent switching signal: $\sigma(x) = p$ if $x \in \Omega_p \subseteq \mathbb{R}^n$, for $p \in \mathcal{N}$. Let $t_1 \ldots, t_{k+1}$ be the switching time of $\sigma(t) = \sigma(x(t))$ on an interval $\mathbb{I} = [t_0, t_{k+1})$. A piece-wise $\mathcal{C}^1$-curve $x : \mathbb{I} \to \mathbb{R}^n$ is called a jump-flow solution of Δ_σ if for all $0 \leq i \leq k$, the jump $x(t_i^-) \to x(t_i^+)$ is a (convergent) jump of Δ_σ and the jump $x(t_{k+1}^-) \to x(t_{k+1}^+)$ is a (convergent or divergent) jump in the sense of Definition 3.1 and the curve $x(\cdot)$ is a $\mathcal{C}^1$-solution of $\Delta_{\sigma(x(t_i^+))}$ on $[t_i, t_{i+1})$ such that $x(t_i) = x(t_i^+)$.

It has been shown in the previous subsection that on a switching surface S_{pq} of two modes Δ_p and Δ_q, a jump sliding mode can be present for an inconsistent point $x_0 \notin \mathfrak{C}_p \cup \mathfrak{C}_q$. While for any consistent point $x \in S_{pq} \cap \mathfrak{C}_p \cap \mathfrak{C}_q$, both $\dot{x} = A_p^{\mathrm{df}}x$ and $\dot{x} = A_q^{\mathrm{df}}x$ should be satisfied, thus it is also possible that a (flow) sliding mode can happen if $f_p^{\mathrm{df}}(x) = A_p^{\mathrm{df}}x$ and $f_q^{\mathrm{df}}(x) = A_q^{\mathrm{df}}x$ both point towards S_{pq} according to the classical ODE switched systems theory [13]. A problem arises if $x \in (S_{pq} \cap \mathfrak{C}_p) \setminus \mathfrak{C}_p$, i.e., x is consistent for one mode Δ_q but is *not* for another mode Δ_p. In that case, the dynamics should respect both the rule $\dot{x}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_p^{\mathrm{df}}x(t)$ and the jump rule $\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = A_q^{\mathrm{jp}}x(\tau)$ simultaneously. Recall that the parameterization variable τ introduced in Definition 3.1 is, in general, not a time variable (see also [5]). It is not clear the flow dynamics and the jump dynamics can be related as they are parametrized by different variables.

Now if the vector fields $f_p^{\mathrm{df}}(x(t)) = A_p^{\mathrm{df}}x(t)$ and $f_q^{\mathrm{jp}}(x(\tau)) = A_q^{\mathrm{jp}}x(\tau)$ both point towards S_{pq} at $x \in S_{pq}$, i.e., there exists $0 \le \alpha \le 1$ such that

$$\alpha f_p^{\mathrm{df}}(x) + (1-\alpha) f_q^{\mathrm{jp}}(x) \in T_x S_{pq} \tag{10}$$

for $x \in S_{pq}$, we *define* that the system follows a *jump-flow sliding modes* $\frac{\mathrm{d}x}{\mathrm{d}t} \in \alpha f_p^{\mathrm{df}}(x) + (1-\alpha) f_q^{\mathrm{jp}}(x)$, $\alpha \in [0,1]$. An intuition for such a definition comes from the singular perturbation approximations of DAEs [12, 5], τ and t are related via a small parameter ϵ by $\frac{\mathrm{d}\tau}{\mathrm{d}t} = \epsilon$, then the jump rule becomes $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{1}{\epsilon} f_q^{\mathrm{jp}}(x(t))$, there always exist a convex combination of $f_p^{\mathrm{df}}(x)$ and $\frac{1}{\epsilon} f_q^{\mathrm{jp}}(x)$ belongs to $T_x S_{pq}$ if and only if (10) holds. Indeed, let $\beta := \frac{\alpha}{\epsilon(1-\alpha)+\alpha}$ (so $0 \le \beta \le 1$), it is clear that $\beta f_p^{\mathrm{df}}(x) + (1-\beta)\epsilon f_q^{\mathrm{jp}}(x)$ is proportional to $\alpha f_p^{\mathrm{df}}(x) + (1-\alpha) f_q^{\mathrm{jp}}(x)$ and is thus in $T_x S_{pq}$.

Example 3.7. Consider a SwDAE Δ_σ on $\mathbb{R}^2$ with two modes

$$\Delta_1 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$
$$\Delta_2 : \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Clearly, Δ_1 is an ODE, i.e., an index-0 DAE and Δ_2 is an index-1 DAE. We show two different switching signals, the first is chosen as

$$\sigma_1(x) = \begin{cases} 1, & \text{if } x_1 > x_2, \\ 2, & \text{if } x_1 \le x_2, \end{cases} \tag{11}$$

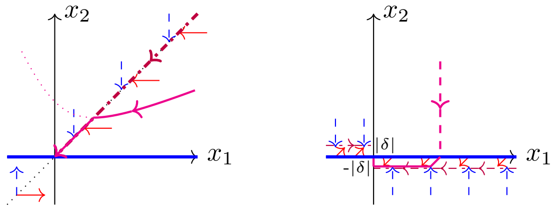
Thus $S_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1 - x_2 = 0 \right\}$ is a switching surface. For each $x \in S_1 \setminus \{0\}$, there exists $0 \le \alpha \le 1$ such that $\alpha f_1^{\mathrm{df}}(x) + (1-\alpha) f_2^{\mathrm{jp}}(x) \in T_x S_1 = \operatorname{im} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, where

$$f_1^{\mathrm{df}}(x) = \begin{bmatrix} -x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix} \text{ and } f_2^{\mathrm{jp}}(x) = \begin{bmatrix} 0 \\ -x_2 \end{bmatrix}.$$

So for any $x_0 \in S_1 \setminus 0$, the solution of Δ_σ is a jump-flow sliding modes, i.e., the Filippov solution of $\dot{x} \in \alpha f_1^{\mathrm{df}}(x) + (1-\alpha) f_2^{\mathrm{jp}}(x)$, $\alpha \in [0,1]$ as shown in Fig 3a. The second switching signal is chosen as

$$\sigma_2(x) = \begin{cases} 1, & \text{if } x_2 = 0, \\ 2, & \text{if } x_2 \ne 0. \end{cases} \tag{12}$$

The surface $S_2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_2 = 0 \right\}$ coincides with the consistency space $\mathfrak{C}_2$ of Δ_2. For any point $x \in S_2 \setminus \{0\}$, the flow dynamics $\dot{x} = f_1^{\mathrm{df}}(x) = A_1^{\mathrm{df}} x$ should be respected. Because S_2 is not A_1^{df}-invariant, once the trajectory reaches any point of S_2, it will leave S_2 immediately to $S_2^\delta = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_2 = \delta \right\}$ with



(a) $\sigma = \sigma_1(x)$ (b) $\sigma = \sigma_2(x)$

Fig. 3. Red and blue dashed arrows: Flow directions of Δ_1 and jump direction of Δ_2, blue lines: $\mathfrak{C}_2$, purple line: Jump-flow sliding modes, magenta line: Jump-flow solutions.

an arbitrarily small parameter $\delta > 0$. Then for $x \in S_2^\delta$, the flow dynamic $\dot{x} = f_1^{\mathrm{df}}(x)$ and the jump dynamic $\dot{x} = f_2^{\mathrm{jp}}(x)$ should be respected, and there exists $0 \le \alpha \le 1$ such that $\alpha f_1^{\mathrm{df}}(x) + (1-\alpha) f_2^{\mathrm{jp}}(x) \in T_x S_2^\delta = \operatorname{im} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, thus there exists a jump-flow sliding modes on S_2^δ. For any point $(x_{10}, \delta) \in S_2^\delta$, the trajectory slides to $(0, \delta)$ and eventually heading towards $(0,0)$ by the rule (12).

4 Stability and stabilization of state-dependent SwDAEs

4.1 Stability analysis under arbitrary state-dependent switching signal

Consider a SwDAE Δ_σ and fix a state-dependent switching signal $\sigma(x)$, suppose that for any initial point $x_0 \in \mathbb{R}^n$, the jump-flow solution $x : [0, +\infty) \to \mathbb{R}^n$ of Δ_σ is well-defined. The SwDAE Δ_σ is called *stable* if for any $\epsilon > 0$, there exists $\delta > 0$ such that $||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon$, $\forall t > 0$ holds for all jump-flow solutions; Δ_σ is called *asymptotically stable* if it is stable and and all jump-flow solutions converge to zero.

Remark 4.1. Impulsive behaviors caused by state jumps can be present for SwDAEs and they are usually viewed as unstable phenomena. Stability results for time-dependent SwDAEs, such as those in [14, 15, 7], typically exclude these impulsive behaviors from solutions using what's known as impulse-free conditions formed via the consistency projector. Exploring the impulse-freeness of state-dependent switched DAEs could indeed be an interesting research topic. However, in this paper, our primary focus revolves around state-dependent stabilization strategies, we will not dive into the discussion of the impulse-free conditions for stat-dependent SwDAEs. Additionally, to simply eliminate impulses, one might assume that all modes of the SwDAE have an index-1.

It has been proved in [14, 15, 7] that a switching DAE is asymptotically stable under arbitrary *time-dependent* switching signals if there exists a common Lyapunov-function $V(x)$ for all the flow dynamics $\dot{x} = A_p^{\mathrm{df}} x$ on their own consistency spaces $\mathfrak{C}_p$, and that

$$\forall p, q \in \mathcal{N}\ \forall x \in \mathfrak{C}_q : V(\Pi_p x) \leq V(x),$$

which is equivalent to (see [6, 7])

$$\forall p, q \in \mathcal{N}\ \forall x \in \mathfrak{C}_q : \frac{\partial V(x)}{\partial x} f_p^{\mathrm{jp}}(x) \leq 0, \qquad (13)$$

where $f_p^{\mathrm{jp}}(x) = A_p^{\mathrm{jp}} x$. However, the above results are *no longer true* for state-dependent SwDAEs. Take $\lambda = 1$ for Example 3.5, we have $A_1^{\mathrm{df}} = \left[\begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix}\right]$, $A_1^{\mathrm{jp}} = \left[\begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}\right]$ and $A_2^{\mathrm{df}} = \left[\begin{smallmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{smallmatrix}\right]$, $A_2^{\mathrm{jp}} = \left[\begin{smallmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{smallmatrix}\right]$. Clearly, $V(x) = (x_1 + x_2)^2$ is a common Lyapunov function for $\dot{x} = A_1^{\mathrm{df}} x$ on $\mathfrak{C}_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0 \right\}$ and for $\dot{x} = A_2^{\mathrm{df}} x$ on $\mathfrak{C}_2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 = 0 \right\}$, and $\frac{\partial V(x)}{\partial x} A_1^{\mathrm{jp}} x = \frac{\partial V(x)}{\partial x} A_2^{jp} x \equiv 0$ implies that (13) holds (for all $(x_1, x_2) \in \mathbb{R}^2$). However, Δ_σ has divergent jumps and thus is not asymptotically stable. Observe that $V(x) = (x_1 - x_2)^2 + x_2^2$ is another common Lyapunov function for the two flow dynamics (on their own consistency space), and $\frac{\partial V(x)}{\partial x} A_1^{\mathrm{jp}} x = 4(x_1 - x_2)x_2 - 2x_2^2 < 0$ for all $(x_1, x_2) \in \mathfrak{C}_2 \setminus \{0\}$ and $\frac{\partial V(x)}{\partial x} A_2^{\mathrm{jp}} x = -2(x_1 - x_2)^2 + x_2(x_1 - 2) < 0$ for all $(x_1, x_2) \in \mathfrak{C}_1 \setminus \{0\}$. The reasons that (13) can not guarantee a convergent jump in the state-dependent switching case are two-folds: for one thing, $V(x)$ should be *strictly decreasing* along each $f_p^{\mathrm{jp}}(x)$ as a state-dependent jump can be stable but not asymptotically stable; for another, the jumps governed by $f_p^{\mathrm{jp}}(x)$ may start from *any point outside* $\mathfrak{C}_p$ while in the time-dependent case, it can only start from $\mathfrak{C}_q$ for $q \notin p$. In fact, we have the following results in the state-dependent case.

Theorem 4.2 (common Lyapunov function). *Consider a SwDAE Δ_σ under arbitrary state-dependent switching signal $\sigma(x) = p$ for $x \in \Omega_p$. Assume that the jump-flow solutions are well-defined, then Δ_σ is asymptotically stable if there exists a positive-definite (Lyapunov) function $V : \mathbb{R}^n \to [0, \infty)$ such that the level set $\mathcal{L}_a := \{x \in \mathbb{R}^n \mid V(x) \leq a\}$ is compact for $a \geq 0$ and*

$$\forall p \in \mathcal{N}\ \forall x \in \mathfrak{C}_p \setminus \{0\} : \quad \frac{\partial V(x)}{\partial x} f_p^{\mathrm{df}}(x) < 0, \qquad (14)$$

$$\forall p \in \mathcal{N}\ \forall x \notin \mathfrak{C}_p : \quad \frac{\partial V(x)}{\partial x} f_p^{\mathrm{jp}}(x) < 0, \qquad (15)$$

where $f_p^{\mathrm{df}}(x) = A_p^{\mathrm{df}} x$ and $f_p^{\mathrm{jp}}(x) = A_p^{\mathrm{jp}} x$.

Proof. The proof follows a similar line as that in [7, 15]. The only differences is the possible presence of jump sliding modes and jump-flow sliding modes, but their asymptotical stability are also guaranteed by (14) and (15) because by definitions, the solutions of the differential inclusion $\dot{x} \in F(x)$, where $F(x)$ are convex combinations of $f_p^{\mathrm{jp}}(x)$'s and/or $f_p^{\mathrm{df}}(x)$'s, are always asymptotical stable by (14) and (15). $\square$

Corollary 4.3. *The system Δ_σ of Theorem 4.2 is asymptotically stable under arbitrary state-dependent switching signal if there exists a positive-definite matrix $L = L^\top > 0$ and positive scalars $\kappa_p > 0$ such that $\forall p \in \mathcal{N}$:*

$$C_p^\top ((A_p^{\mathrm{df}})^\top L + L A_p^{\mathrm{df}}) C_p < 0, \qquad (16)$$

$$(A_p^{\mathrm{jp}})^\top L + L A_p^{\mathrm{jp}} + \kappa_p B_p^\top B_p \leq 0, \qquad (17)$$

where C_p is any full column rank matrix such that $\operatorname{im} C_p = \mathfrak{C}_p$ and B_p is any full row rank matrix such that $\ker B_p = \mathfrak{C}_p$.

Proof. By choosing a quadratic Lyapunov function $V(x) = x^\top L x$, the inequality (14) is equivalent to $x^\top ((A_p^{\mathrm{df}})^\top L + L A_p^{\mathrm{df}}) x < 0$ for all $x \in \operatorname{im} C_p \setminus \{0\}$, then by Finsler's lemma, that is equivalent to (16). Moreover, the inequality (15) is equivalent to $x^\top ((A_p^{\mathrm{jp}})^\top L + L A_p^{\mathrm{jp}}) x < 0$ for all $x \in \mathbb{R}^n$ such that $x^\top B_p^\top B_p x \neq 0$, the latter is equivalent to $x^\top B_p^\top B_p x > 0$ because $x^\top B_p^\top B_p x$ is always non-negative as B_p is of full row rank. The inequality (17) is actually *equivalent* to (15) with a quadratic Lyapunov function, the conclusion is although slightly different but is close to the (lossless) S-lemma. If (17) holds, then $x^\top ((A_p^{\mathrm{jp}})^\top P + P A_p^{\mathrm{jp}}) x \leq -\kappa_p x^\top B_p^\top B_p x < 0$ for all $x \notin \mathfrak{C}_p$. The proof of the converse is much harder, which can be done following the same line as proving the (lossless) S-lemma, see e.g., [20]. $\square$

4.2 Stable convex combinations

Now we generalise the stable convex combinations results of switched ODEs via state-dependent switches [13] to the DAE cases. Consider a SwDAE Δ_σ, given by (1), we define two linear subspaces $\mathfrak{C}_\cap$ and $\bar{\mathfrak{C}}_\cap$ by

$$\mathfrak{C}_\cap := \mathfrak{C}_1 \cap \ldots \cap \mathfrak{C}_N \quad \text{and} \quad \mathfrak{C}_\cap \oplus \bar{\mathfrak{C}}_\cap = \mathbb{R}^n.$$

Then define, respectively, two convex combinations: $\mathbf{A}^{\mathrm{df}}$ of the flow matrices A_p^{df} and $\mathbf{A}^{\mathrm{jp}}$ of the jump matrices A_p^{jp}:

$$\mathbf{A}^{\mathrm{df}} := \sum_{p=1}^{N} \alpha_p A_p^{\mathrm{df}} \quad \text{and} \quad \mathbf{A}^{\mathrm{jp}} := \sum_{p=1}^{N} \beta_p A_p^{\mathrm{jp}},$$

where $\alpha_p, \beta_p \in [0, 1]$, $\sum_{p=1}^{N} \alpha_p = 1$ and $\sum_{p=1}^{N} \beta_p = 1$. Choose two full column rank matrices C and $\bar{C}$ such that $\operatorname{im} C = \mathfrak{C}_\cap$ and $\operatorname{im} \bar{C} = \bar{\mathfrak{C}}_\cap$, define a coordinates transformation matrix

$$T = [C, \bar{C}]^{-1},$$

then the linear map $\mathbf{A}^{\mathrm{jp}}$ has the following form in the new coordinates

$$T\mathbf{A}^{\mathrm{jp}}T^{-1} = \begin{bmatrix} 0 & \mathbf{A}_2^{\mathrm{jp}} \\ 0 & \mathbf{A}_4^{\mathrm{jp}} \end{bmatrix}.$$

The first columns of the above matrix are zeros. Indeed, we have that $\mathbf{A}^{\mathrm{jp}}C = \sum_{p=1}^{N} \beta_p A_p^{\mathrm{jp}} C = 0$ as $\ker A_p^{\mathrm{jp}} = \mathfrak{C}_p$, hence $T\mathbf{A}^{\mathrm{jp}}T^{-1}TC = T\mathbf{A}^{\mathrm{jp}}T^{-1}\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] = 0$.

Lemma 4.4. *(i) A different choice of $\bar{C}$ such that $\operatorname{im}\bar{C} = \bar{\mathfrak{C}}_\cap$ does not change the eigenvalues of $\mathbf{A}_4^{\mathrm{jp}}$ (when β_p's are fixed).*

(ii) If $\mathbf{A}_4^{\mathrm{jp}}$ has no zero eigenvalues, i.e., $\mathbf{A}_4^{\mathrm{jp}}$ is invertible, then there exists a choice of $\bar{C}$ such that $\mathbf{A}_2^{\mathrm{jp}} \equiv 0$, i.e.,

$$T\mathbf{A}^{\mathrm{jp}}T^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{A}_4^{\mathrm{jp}} \end{bmatrix}. \tag{18}$$

Proof. (i) Let $\bar{C}'$ be another full column matrix such that $\mathfrak{C} \oplus \operatorname{im}\bar{C}' = \mathbb{R}^n$. Then $\bar{C}' = \bar{C}M + CN$ for an invertible matrix M and a matrix N. Thus the coordinates transformation becomes $T' = \left[\begin{smallmatrix} C & \bar{C}' \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & -NM^{-1} \\ 0 & M^{-1} \end{smallmatrix}\right] T$. It follows that $\mathbf{A}_4^{\mathrm{jp}}$ under $T'x$-coordinates becomes $M^{-1}\mathbf{A}_4^{\mathrm{jp}}M$, so the eigenvalues of $\mathbf{A}_4^{\mathrm{jp}}$ are preserved.

(ii) If $\mathbf{A}_4^{\mathrm{jp}}$ of $T\mathbf{A}^{\mathrm{jp}}T^{-1} = \left[\begin{smallmatrix} 0 & \mathbf{A}_2^{\mathrm{jp}} \\ 0 & \mathbf{A}_4^{\mathrm{jp}} \end{smallmatrix}\right]$ is invertible, then choose $\bar{C}$ such that the basis of $\operatorname{im}\bar{C}$ consists of the independent eigenvectors of $\left[\begin{smallmatrix} 0 & \mathbf{A}_2^{\mathrm{jp}} \\ 0 & \mathbf{A}_4^{\mathrm{jp}} \end{smallmatrix}\right]$ corresponding to its non-zero eigenvalues (i.e., the eigenvalues of $\mathbf{A}_4^{\mathrm{jp}}$), then $\operatorname{im}\bar{C}$ becomes $\mathbf{A}^{\mathrm{jp}}$-invariant. Now it is clear that $\mathbf{A}^{\mathrm{jp}}$ has the form (18) in Tx-coordinates by the $\mathbf{A}^{\mathrm{jp}}$-invariance of $\operatorname{im}\bar{C}$. □

Fix the matrix T such that (18) holds, then denote

$$T\mathbf{A}^{\mathrm{df}}T^{-1} = \begin{bmatrix} \mathbf{A}_1^{\mathrm{df}} & \mathbf{A}_2^{\mathrm{df}} \\ \mathbf{A}_3^{\mathrm{df}} & \mathbf{A}_4^{\mathrm{df}} \end{bmatrix} \tag{19}$$

Theorem 4.5. *Consider a SwDAE Δ_σ, given by (1). If there exist, respectively, two convex combinations of A_p^{jp} and A_p^{df} such that both $\mathbf{A}_4^{\mathrm{jp}}$ of (18) and $\mathbf{A}_1^{\mathrm{df}}$ of (19) are Hurwitz* [1] *then there exists a state-dependent switching signal $\sigma = \sigma(x)$ such that Δ_σ is asymptotically stable with possible jump-flow sliding modes being present in the jump-flow solutions.*

Moreover, if the following condition is additionally satisfied,

(IN) : $\forall p \in \mathcal{N}$: *$\mathfrak{C}_\cap$ is A_p^{df}-invariant, i.e., $A_p^{\mathrm{df}}\mathfrak{C}_\cap \subseteq \mathfrak{C}_\cap$,*

then there exists a state-dependent signal $\sigma = \sigma(x)$ asymptotically stabilize Δ_σ without any sliding-mode being present in the jump-flow solutions.

Proof. Since $\mathbf{A}_4^{\mathrm{jp}}$ is Hurwitz and thus it is invertible, there always exists an invertible matrix T such that (18) holds. If both $\mathbf{A}_1^{\mathrm{df}}$ and $\mathbf{A}_4^{\mathrm{jp}}$ are Hurwitz, then there exist two positive-definite matrices $\hat{L} = \hat{L}^\top$ and $\bar{L} = \bar{L}^\top$ such that

$$(\mathbf{A}_1^{\mathrm{df}})^\top \hat{L} + \hat{L}\mathbf{A}_1^{\mathrm{df}} < 0 \quad \text{and} \quad (\mathbf{A}_4^{\mathrm{jp}})^\top \bar{L} + \bar{L}\mathbf{A}_4^{\mathrm{jp}} < 0$$

Define a positive-definite matrix $L := T^\top \left[\begin{smallmatrix} \hat{L} & 0 \\ 0 & \bar{L} \end{smallmatrix}\right] T$. Denote the coordinates $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = Tx$, then the subspace $\mathfrak{C}_\cap = \{Cx_1 \mid x_1 \in \mathbb{R}^{n_1}\} = \{(x_1, x_2) \in \mathbb{R}^n \mid x_2 = 0\}$. It follows that

$$\begin{aligned}
&(Cx_1)^\top (\mathbf{A}^{\mathrm{df}})^\top L + L\mathbf{A}^{\mathrm{df}})Cx_1 \\
&= x_1^\top (TC)^\top ((T\mathbf{A}^{\mathrm{df}}T^{-1})^\top T^{-\top}LT^{-1} + \\
&\qquad\qquad T^{-\top}LT^{-1}T\mathbf{A}^{\mathrm{df}}T^{-1})TCx_1 \\
&= \left[\begin{smallmatrix} x_1 & 0 \end{smallmatrix}\right] \left(\left[\begin{smallmatrix} \mathbf{A}_1^{\mathrm{df}} & \mathbf{A}_2^{\mathrm{df}} \\ \mathbf{A}_3^{\mathrm{df}} & \mathbf{A}_4^{\mathrm{df}} \end{smallmatrix}\right]^\top \left[\begin{smallmatrix} \hat{L} & 0 \\ 0 & \bar{L} \end{smallmatrix}\right] + \left[\begin{smallmatrix} \hat{L} & 0 \\ 0 & \bar{L} \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbf{A}_1^{\mathrm{df}} & \mathbf{A}_2^{\mathrm{df}} \\ \mathbf{A}_3^{\mathrm{df}} & \mathbf{A}_4^{\mathrm{df}} \end{smallmatrix}\right] \right) \left[\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right] \\
&= x_1^\top \left((\mathbf{A}_1^{\mathrm{df}})^\top \hat{L} + \hat{L}\mathbf{A}_1^{\mathrm{df}} \right) x_1 < 0, \ \forall x_1 \neq 0,
\end{aligned}$$

and

$$\begin{aligned}
&x^\top \left((\mathbf{A}^{\mathrm{jp}})^\top L + L\mathbf{A}^{\mathrm{jp}} \right) x \\
&= (Tx)^\top \left((T\mathbf{A}^{\mathrm{jp}}T^{-1})^\top T^{-\top}LT^{-1} + T^{-\top}LT^{-1}T\mathbf{A}^{\mathrm{jp}}T^{-1} \right) Tx \\
&= \left[\begin{smallmatrix} x_1^\top & x_2^\top \end{smallmatrix}\right] \left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathbf{A}_4^{\mathrm{jp}} \end{smallmatrix}\right]^\top \left[\begin{smallmatrix} \hat{L} & 0 \\ 0 & \bar{L} \end{smallmatrix}\right] + \left[\begin{smallmatrix} \hat{L} & 0 \\ 0 & \bar{L} \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathbf{A}_4^{\mathrm{jp}} \end{smallmatrix}\right] \right) \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \\
&= x_2^\top \left((\mathbf{A}_4^{\mathrm{jp}})^\top \bar{L} + \bar{L}\mathbf{A}_4^{\mathrm{jp}} \right) x_2 < 0, \ \forall x_2 \neq 0.
\end{aligned}$$

The above two inequalities are, respectively, equivalent to

$$\begin{aligned}
&\sum_{p=1}^{N} \alpha_p \left(x^\top ((A_p^{\mathrm{df}})^\top L + LA_p^{\mathrm{df}})x \right) < 0, \quad \forall x \in \mathfrak{C}_\cap \setminus \{0\}, \\
&\sum_{p=1}^{N} \beta_p \left(x^\top ((A_p^{\mathrm{jp}})^\top L + LA_p^{\mathrm{jp}})x \right) < 0, \quad \forall x \notin \mathfrak{C}_\cap.
\end{aligned} \tag{20}$$

[1] In the case that one of the matrices $\mathbf{A}_1^{\mathrm{df}}$ and $\mathbf{A}_4^{\mathrm{jp}}$ is empty, only the other one is required to be Hurwitz. In particular, if $\mathbf{A}_4^{\mathrm{jp}}$ is empty, the results reduce to the case of switched ODEs [13].

Then we define the sets

$$\Omega_p^{\mathrm{df}} := \left\{ x \in \mathfrak{C}_\cap \mid x^\top ((A_p^{\mathrm{df}})^\top L + L A_p^{\mathrm{df}}) x < 0 \right\}, \quad p \in \mathcal{N},$$
$$\Omega_p^{\mathrm{jp}} := \left\{ x \notin \mathfrak{C}_\cap \mid x^\top ((A_p^{\mathrm{jp}})^\top L + L A_p^{\mathrm{jp}}) x < 0 \right\}, \quad p \in \mathcal{N}.$$

It is seen that all Ω_p^{df}'s and all Ω_p^{jp}'s are open and conic, and we have $\bigcup_{p=1}^N \Omega_p^{\mathrm{df}} = \mathfrak{C}_\cap \setminus \{0\}$ and $\bigcup_{p=1}^N \Omega_p^{\mathrm{jp}} = \mathbb{R}^n \setminus \mathfrak{C}_\cap$ by (20). Moreover, $\forall x \in \Omega_p^{\mathrm{df}} : \frac{\partial V(x)}{\partial x} A_p^{\mathrm{df}} x < 0$ and $\forall x \in \Omega_p^{\mathrm{jp}} : \frac{\partial V(x)}{\partial x} A_p^{\mathrm{jp}} x < 0$, where $V(x) := x^\top L x$ is defined for all $x \in \mathbb{R}^n$.

In order to make $V(x)$ decrease along the jump-flow solutions of Δ_σ, it is reasonable to active the mode Δ_p in the sets Ω_p^{df} and Ω_p^{jp}. Given an initial point $x_0^- \notin \mathfrak{C}_\cap$, the first step is to use *only* state-dependent jumps to drive x_0^- to any point $x_0^+ \in \mathfrak{C}_\cap$. To that end and to avoid jump sliding modes, we use a hysteresis switching strategy as in [29, 13] but for the jump dynamics. Namely, $\sigma(x)$ can be taken as:

(R1): Let $\sigma(x_0^-) = p$ if $x_0^- \in \Omega_p^{\mathrm{jp}}$. For $\tau \geq 0$, keep $\sigma(x(\tau)) = p$ and change $\sigma(x(\tau)) = q$ once $x(\tau^+) \in \Omega_q^{\mathrm{jp}} \setminus \Omega_p^{\mathrm{jp}}$ for some $q \in \mathcal{N}$. Repeat the latter switching rule unless $x(\tau) \in \mathfrak{C}_\cap$.

The Lyapunov function $V(x) = x^\top L x$ decreases along the jump solutions of Δ_σ under **(R1)**, i.e., $\frac{\mathrm{d}V(x)}{\mathrm{d}\tau} < 0$, $\forall x \in \mathbb{R}^n \setminus \mathfrak{C}_\cap$, which indicates that the jump solutions will eventually reach some point $x_0^+ \in \mathfrak{C}_\cap$ (because in the fixed coordinates, $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = Px = [C, \bar{C}]^{-1} x$, the jump solution $x_2(\tau) \to 0$ under **(R1)** by standard Lyapunov arguments). Then there are two possible cases depending on whether the condition **(IN)** is satisfied.

If **(IN)** is satisfied, then any (flow) solution of Δ_σ starting from $x_0^+ \in \mathfrak{C}_\cap$ stays in $\mathfrak{C}_\cap$. To drive x_0^+ to the origin, we take the hysteresis rule:

(R2): Let $\sigma(x_0^+) = p$ if $x_0^+ \in \Omega_p^{\mathrm{df}}$. For $t \geq 0$, keep $\sigma(x(t)) = p$ and change $\sigma(x(t)) = q$ once $x(t^+) \in \Omega_q^{\mathrm{df}} \setminus \Omega_p^{\mathrm{df}}$ for some $q \in \mathcal{N}$. Repeat the procedure.

If **(IN)** is not satisfied, then it is possible that $x(t)$ tends to leave $\mathfrak{C}_\cap$ when taking the rule **(R2)**. In that case, we show that there must exist a (stable) jump-flow sliding mode near $\mathfrak{C}_\cap$. We only illustrate the case that $\dim \mathfrak{C}_\cap = n-1$, the general case can be proved by extending the proof below with the help of the Jordan forms of Hurwitz matrices. Suppose without loss of generality that for a certain $p \in \mathcal{N}$, $A_p^{\mathrm{df}} \mathfrak{C}_\cap \not\subseteq \mathfrak{C}_\cap$. Then we have $A_{p3}^{\mathrm{df}} \in \mathbb{R}^{1 \times (n-1)}$ is not zero, where $T^{-1} A_p^{\mathrm{df}} T = \begin{bmatrix} A_{p1}^{\mathrm{df}} & A_{p2}^{\mathrm{df}} \\ A_{p3}^{\mathrm{df}} & A_{p4}^{\mathrm{df}} \end{bmatrix}$. Recall that in $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = Tx$-coordinates, $\mathfrak{C}_\cap = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0 \right\}$. Starting from a point $(x_1, 0) \in \Omega_p^{\mathrm{df}} \subseteq \mathfrak{C}_\cap$, any (flow)-solution under **(R2)** leaves immediately from $\mathfrak{C}_\cap$ and reach a surface $S_{pq}^\delta = \left\{ \tilde{x} = (x_1, x_2) \in \Omega_q^{\mathrm{jp}} \mid h(\tilde{x}) = x_2 - \delta = 0 \right\}$ for some $q \in \mathcal{N}$ and an arbitrarily small parameter $\delta \in \mathbb{R}$. Moreover, we have $\mathrm{sign}(\delta) = \mathrm{sign}(A_{p3}^{\mathrm{df}} x_1)$. Observe that as $\delta \to 0$, for any point $\tilde{x} = (x_1, \delta) \in S_{pq}^\delta$, both the flow dynamics $\frac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t} = T A_p^{\mathrm{df}} T^{-1} \tilde{x} = f_p^{\mathrm{df}}(\tilde{x})$ and the jump rule $\frac{\mathrm{d}\tilde{x}(\tau)}{\mathrm{d}\tau} = T A_q^{\mathrm{jp}} T^{-1} \tilde{x} = f_q^{\mathrm{jp}}(\tilde{x})$ are respected. It follows that $\frac{\partial h(\tilde{x})}{\partial \tilde{x}} f_p^{\mathrm{df}}(\tilde{x}) = A_{p3}^{\mathrm{df}} x_1$ and $\frac{\partial h(\tilde{x})}{\partial \tilde{x}} f_q^{\mathrm{jp}}(\tilde{x}) = A_{q4}^{\mathrm{jp}} \delta$, where $T A_q^{\mathrm{jp}} T^{-1} = \begin{bmatrix} 0 & A_{q2}^{\mathrm{jp}} \\ 0 & A_{q4}^{\mathrm{jp}} \end{bmatrix}$. It is seen that $\mathrm{sign}(A_{q4}^{\mathrm{jp}} \delta) = -\mathrm{sign}(\delta)$ because $(x_1, \delta) \in \Omega_q^{\mathrm{jp}}$. Therefore both f_p^{df} and f_q^{jp} point towards S_{pq}^δ, which results in a jump-flow sliding mode $\dot{x} \in \alpha f_p^{\mathrm{df}}(x) + (1-\alpha) f_q^{\mathrm{jp}}(x)$, $\alpha \in [0,1]$. Moreover, for all $x \in S_{pq}$, we have $\dot{V}(x) = \frac{\partial V(x)}{\partial x}(\alpha f_p^{\mathrm{df}}(x) + (1-\alpha) f_q^{\mathrm{jp}}(x)) = \alpha x_1^\top ((A_{p1}^{\mathrm{jp}})^\top \hat{L} + \hat{L} A_{p1}^{\mathrm{jp}}) x_1 + (1-\alpha) \frac{\partial V(x)}{\partial x} f_q^{\mathrm{jp}}(x) + \alpha g(x_1, \delta)$, where $g(x_1, \delta) = x_1^\top ((A_{p3}^{\mathrm{jp}})^\top \hat{L} + \hat{L} A_{p3}^{\mathrm{jp}}) \delta + \delta ((A_{p2}^{\mathrm{jp}})^\top \bar{L} + \bar{L} A_{p2}^{\mathrm{jp}}) x_1 + \delta ((A_{p4}^{\mathrm{jp}})^\top \bar{L} + \bar{L} A_{p4}^{\mathrm{jp}}) \delta$. Notice that $\alpha x_1^\top ((A_{p1}^{\mathrm{jp}})^\top \hat{L} + \hat{L} A_{p1}^{\mathrm{jp}}) x_1 < 0$ as $(x_1, 0) \in \Omega_p^{\mathrm{df}}$ and $\frac{\partial V(x)}{\partial x} f_q^{\mathrm{jp}}(x) < 0$ as $x \in S_{pq} \subseteq \Omega_q^{\mathrm{jp}}$. So $\dot{V}(x) < 0$ as δ can be taken arbitrarily small. Hence $V(x)$ decreases along any jump-flow solutions under the rules **(R1)** and **(R2)**. $\square$

If it is required that the sliding modes (in particular, jump-flow sliding modes) are excluded from the jump-flow solutions, then the condition **(IN)** has to be satisfied. In order to show that **(IN)** is *not* a very restrictive condition, we compare **(IN)** with the projector assumption **(PA)** made in [18], the latter has already been shown to be a weaker assumption than supposing the consistency projectors Π_p pair-wisely commute ([16, 17]). Set $\Pi_\cap := \prod_{p=1}^N \Pi_p$, then

$$\textbf{(PA):}\ \forall p \in \mathcal{N} : \mathrm{im}\, \Pi_\cap \subseteq \mathrm{im}\, \Pi_p, \quad \ker \Pi_\cap \supseteq \ker \Pi_p.$$

Lemma 4.6. **(PA)** *implies* **(IN)** *but not conversely, i.e., in general,* **(IN)** *does not imply* **(PA)**.

Proof. Firstly, **(PA)** is equivalent to (21) by Lemma 2 of [18].

$$\forall p \in \mathcal{N} : \Pi_p \Pi_\cap = \Pi_\cap = \Pi_\cap \Pi_p. \tag{21}$$

It follows that $\forall p \in \mathcal{N}$, Π_p commutes with $\Pi_\cap$, i.e, $P_p^{-1} \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] P_p \Pi_\cap = \Pi_\cap P_p^{-1} \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] P_p \Rightarrow \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] P_p \Pi_\cap P_p^{-1} = P_p \Pi_\cap P_p^{-1} \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$. As a consequence, we have

$$\left[\begin{smallmatrix} A_{p1} & 0 \\ 0 & 0 \end{smallmatrix}\right] P_p \Pi_\cap P_p^{-1} = P_p \Pi_\cap P_p^{-1} \left[\begin{smallmatrix} A_{p1} & 0 \\ 0 & 0 \end{smallmatrix}\right] \Rightarrow$$
$$P_p^{-1} \left[\begin{smallmatrix} A_{p1} & 0 \\ 0 & 0 \end{smallmatrix}\right] P_p \Pi_\cap = \Pi_\cap P_p^{-1} \left[\begin{smallmatrix} A_{p1} & 0 \\ 0 & 0 \end{smallmatrix}\right] P_p,$$

i.e., $A_p\Pi_\cap = \Pi_\cap A_p$. Secondly, if **(PA)** holds, then we show $\operatorname{im}\Pi_\cap = \mathfrak{C}_\cap$. Notice that $\Pi_\cap$ is also a projector if **(PA)** satisfied by Corollary 3 of [18]. By choosing suitable coordinates to rectify $\operatorname{im}\Pi_\cap$ and $\ker\Pi_\cap$ (note that $\operatorname{im}\Pi_\cap \oplus \ker\Pi_\cap = \mathbb{R}^n$), we can assume without loss of generality that $\Pi_\cap = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then in the rectified coordinates, all Π_p's are in block diagonal form by (21), i.e., $\Pi_p = \begin{bmatrix} \Pi_{p1} & 0 \\ 0 & \Pi_{p4} \end{bmatrix}$. Thus by the definition of $\Pi_\cap$, we get that all Π_{p1}'s are invertible as $\prod_{p=1}^{N} \Pi_{p1} = I$ and $\prod_{p=1}^{N} \Pi_{p4} = 0$. The latter implies $\bigcap_{p=1}^{N} \operatorname{im}\Pi_{p4} = \{0\}$. Indeed, suppose that $\bigcap_{p=1}^{N} \operatorname{im}\Pi_{p4} \neq \{0\}$, then we take any nonzero point $x_{04} \in \bigcap_{p=1}^{N} \operatorname{im}\Pi_{p4}$, it follows that $\prod_{p=1}^{N} \Pi_{p4}x_{04} = x_{04} \neq 0$ (note that Π_{p4}'s are also projectors), which contradicts $\prod_{p=1}^{N} \Pi_{p4} = 0$. Therefore,

$$\operatorname{im}\Pi_\cap = \operatorname{im}\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] = \begin{bmatrix} \bigcap_{p=1}^{N} \operatorname{im}\Pi_{p1} & 0 \\ 0 & \bigcap_{p=1}^{N} \operatorname{im}\Pi_{p4} \end{bmatrix} = \mathfrak{C}_\cap$$

(recall that $\operatorname{im}\Pi_p = \mathfrak{C}_p$). Finally, by $A_p\Pi_\cap = \Pi_\cap A_p$ and $\operatorname{im}\Pi_\cap = \mathfrak{C}_\cap$, we have $A_p\mathfrak{C}_\cap \subseteq \mathfrak{C}_\cap$.

Conversely, we show a simple example for which **(IN)** holds but not **(PA)**. Take $\Xi_1 : \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ and $\Xi_2 : \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$. We have $\Pi_1 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\Pi_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. So $\mathfrak{C}_\cap = \operatorname{im}\Pi_1 \cap \operatorname{im}\Pi_2 = \{0\}$ implies **(IN)** holds. However, easy calculations show that **(PA)** is not satisfied.

□

Example 4.7. Consider a SwDAE Δ_σ with generalized states $\xi = (x, y, z) \in \mathbb{R}^3$ and two modes:

$$\Delta_1 : \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\Delta_2 : \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & -1 & \gamma \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

where $\gamma \geq 0$ is a constant. Then direct calculations give

$$A_1^{\mathrm{jp}} = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right], \quad A_2^{\mathrm{jp}} = \left[\begin{smallmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right],$$

$$A_1^{\mathrm{df}} = \left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right], \quad A_2^{\mathrm{df}} = \left[\begin{smallmatrix} 0 & 0 & 0 \\ -1 & 1 & -\gamma \\ 0 & 0 & -1 \end{smallmatrix}\right].$$

The consistency spaces are given by

$$\mathfrak{C}_1 = \left\{\xi \in \mathbb{R}^3 \,\middle|\, y = 0\right\}, \quad \mathfrak{C}_2 = \left\{\xi \in \mathbb{R}^3 \,\middle|\, x = 0\right\}.$$

Both Δ_1 and Δ_2 are unstable as the flow matrix A_p^{df}, $p = 1, 2$, restricted to the consistency space $\mathfrak{C}_p$, $p = 1, 2$, has unstable eigenvalues. Now choose

$$T = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$$

to have

$$T\mathbf{A}^{\mathrm{jp}}T^{-1} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathbf{A}_4^{\mathrm{jp}} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & -\beta & \beta-1 \\ 0 & \beta & \beta-1 \end{smallmatrix}\right],$$

$$T\mathbf{A}^{\mathrm{df}}T^{-1} = \left[\begin{smallmatrix} \mathbf{A}_1^{\mathrm{df}} & \mathbf{A}_2^{\mathrm{df}} \\ \mathbf{A}_3^{\mathrm{df}} & \mathbf{A}_4^{\mathrm{df}} \end{smallmatrix}\right] = \alpha \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right] + (1-\alpha) \left[\begin{smallmatrix} -\gamma-1 & \gamma+2 & -1 \\ -\gamma & \gamma+1 & -1 \\ 0 & 0 & 0 \end{smallmatrix}\right].$$

Thus by choosing e.g., $\alpha = 0$ and $\beta = 0.5$, we have that both $\mathbf{A}_4^{\mathrm{jp}} = \left[\begin{smallmatrix} -0.5 & -0.5 \\ 0.5 & -0.5 \end{smallmatrix}\right]$ and $\mathbf{A}_1^{\mathrm{df}} = -\gamma - 1$ are Hurwitz. We now use the switching rules **(R1)** and **(R2)** above to stabilize the SwDAE Δ_σ. Set $L = T^\top I_3 T = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right]$, define

$$\begin{aligned} \Omega_1^{\mathrm{jp}} &= \left\{\xi \notin \mathfrak{C}_\cap \,\middle|\, \xi^\top((A_1^{\mathrm{jp}})^\top L + LA_1^{\mathrm{jp}})\xi < 0\right\} \\ &= \left\{\xi \notin \mathfrak{C}_\cap \,\middle|\, -y^2 + xy < 0\right\} \\ \Omega_2^{\mathrm{jp}} &= \left\{\xi \notin \mathfrak{C}_\cap \,\middle|\, \xi^\top((A_2^{\mathrm{jp}})^\top L + LA_2^{\mathrm{jp}})\xi < 0\right\} \\ &= \left\{\xi \notin \mathfrak{C}_\cap \,\middle|\, -x^2 - 2xy - yz < 0\right\} \end{aligned}$$

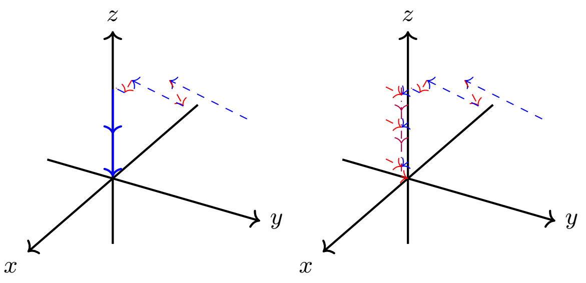
and

$$\Omega_1^{\mathrm{df}} = \{0\}, \quad \Omega_2^{\mathrm{df}} = \mathfrak{C}_\cap.$$

A jump-flow solution under the rule **(R1)** and **(R2)** for $\gamma = 0$ and $\gamma = 1$ is shown in Figure 4. The rule **(R1)** renders the jumps to z-axis (i.e., $\mathfrak{C}_\cap$). Then the rule **(R2)** means if the solutions is on $\mathfrak{C}_\cap$, then only Δ_2 is activated. Observe that $\mathfrak{C}_\cap$ is A_1^{df}-invariant but it is A_2^{df}-invariant only if $\gamma = 0$. So the condition **(IN)** is only satisfied for $\gamma = 0$ but not for $\gamma = 1$. Therefore, in the case of $\gamma = 1$, the flow solution of Δ_2 tend to leave z-axis but the rule **(R1)** keep s driving the solution into z-axis, which results in a flow-jump sliding modes around z-axis. Figure 5 shows the projections the jumps , Ω_1^{jp} and Ω_2^{jp} to the $x - y$ plane.

Now we apply the results of Theorem 4.5 to stabilize an unstable circuit.

Example 4.8. Consider the switching circuit shown in Figure 6, which consists of an inductor with inductance L, a capacitor with capacitance C, a current controlled voltage source v_s, a voltage controlled current source i_s



(a) $\gamma = 0$ (b) $\gamma = 1$

Fig. 4. Red dashed lines and blue dashed lines: Jumps of Δ_1 and Δ_2, blue lines: Flow of Δ_2, purple line: Jump-flow sliding modes.

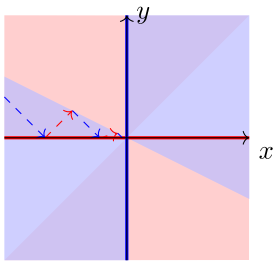


Fig. 5. Red $\cup$ purple region: Ω_1^{jp}, blue $\cup$ purple region: Ω_2^{jp}, purple region: $\Omega_1^{\mathrm{jp}} \cap \Omega_2^{\mathrm{jp}}$; red and blue dashed lines with arrows: Jumps of Δ_1 and Δ_2, red and blue lines: $\mathfrak{C}_1$ and $\mathfrak{C}_2$.

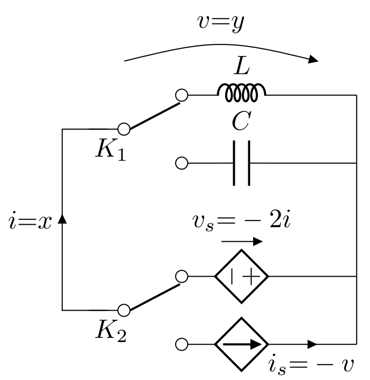


Fig. 6. A switching electric circuit

and two switches K_1 and K_2. The voltage and current of the controlled sources are set as follows:

$$v_s = -2i, \qquad i_s = -v.$$

Depending on the states (up or down) of the two switches, the circuit can be modeled by a SwDAE Δ_σ with four DAE modes Δ_i, $i = 1, 2, 3, 4$, considering the characteristics of the inductor and the capacitor, and Kirchhoff's law.

K_1 \ K_2	Up	Down
Up	Δ_1	Δ_2
Down	Δ_3	Δ_4

The four modes are, respectively, given by

$$\Delta_1 : \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\Delta_2 : \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\Delta_3 : \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\Delta_4 : \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

All modes are deliberately set to be *unstable* via the controlled sources, we now use the results of Theorem 4.5 to stabilize the circuit. By calculation, we have $\mathfrak{C}_\cap = \{0\}$, thus $T = I_2$, $\mathbf{A}_4^{\mathrm{jp}} = \mathbf{A}^{\mathrm{jp}}$ and $\mathbf{A}_1^{\mathrm{df}}$ is empty. Moreover, the jump matrices $A_1^{\mathrm{jp}} = \left[\begin{smallmatrix} 0 & 0 \\ 2 & -1 \end{smallmatrix}\right]$, $A_2^{\mathrm{jp}} = \left[\begin{smallmatrix} 0 & 0 \\ 1 & -1 \end{smallmatrix}\right]$, $A_3^{\mathrm{jp}} = \left[\begin{smallmatrix} -1 & 0.5 \\ 0 & 0 \end{smallmatrix}\right]$, $A_4^{\mathrm{jp}} = \left[\begin{smallmatrix} -1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, have the stable convex combination:

$$\mathbf{A}^{\mathrm{jp}} = 0A_1^{\mathrm{jp}} + \beta A_2^{\mathrm{jp}} + (1-\beta)A_3^{\mathrm{jp}} + 0A_4^{\mathrm{jp}} = \left[\begin{smallmatrix} \beta-1 & 0.5-0.5\beta \\ \beta & -\beta \end{smallmatrix}\right]$$

which is Hurwitz for any $0 < \beta < 1$. Thus it is possible to switch between modes 2 and 3 to stabilize the system via their *jump solutions* only. Indeed, it is seen that $(\mathbf{A}^{\mathrm{jp}})^\top L + L\mathbf{A}^{\mathrm{jp}} < 0$ for $L = I_2$, define

$$\begin{aligned} \Omega_2^{\mathrm{jp}} &= \left\{ (x, y) \neq 0 \,\middle|\, x^\top((A_2^{\mathrm{jp}})^\top L + LA_2^{\mathrm{jp}})x < 0 \right\} \\ &= \left\{ (x, y) \,\middle|\, xy - y^2 < 0 \right\} \end{aligned}$$

and

$$\begin{aligned} \Omega_3^{\mathrm{jp}} &= \left\{ (x, y) \neq 0 \,\middle|\, x^\top((A_3^{\mathrm{jp}})^\top L + LA_3^{\mathrm{jp}})x < 0 \right\} \\ &= \left\{ (x, y) \,\middle|\, xy - 2x^2 < 0 \right\}. \end{aligned}$$

By using the hysteresis switching jump rule **(R1)**, we activate mode Δ_p, $p = 2, 3$, in the region Ω_p^{jp}, $p = 2, 3$, respectively, to steer and keep the jump solution in the set $\Omega_2^{\mathrm{jp}} \cap \Omega_3^{\mathrm{jp}}$ and eventually drive any initial point into the origin. We show the solutions for two different initial points in Figure 7.

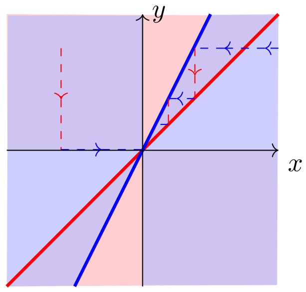


Fig. 7. Red ∪ purple region: Ω_2^{jp}, blue ∪ purple region: Ω_3^{jp}, purple region: $\Omega_2^{\text{jp}} \cap \Omega_3^{\text{jp}}$; red and blue dashed lines with arrows: Jumps of Δ_2 and Δ_3, red and blue lines: $\mathfrak{C}_2$ and $\mathfrak{C}_3$.

4.3 Stabilization via state-dependent switching rule

For linear switched ODEs without Hurwitz convex combinations, it is still possible to find state-dependent switching signals to stabilize the system, e.g., using the "min-max–switching" strategy shown in [13] (see also [33] for the nonlinear case). We now generalize this strategy to linear SwDAEs.

Theorem 4.9. *Consider a SwDAE Δ_σ, given by (1), and the flow matrices A_p^{df} and the jump matrices A_p^{jp} defined by (5) for each mode Δ_p. Suppose that there exist positive-definite matrices $L_p^{\text{jp}} = (L_p^{\text{jp}})^\top$ and $L_p^{\text{jp}} = (L_p^{\text{df}})^\top$, $p = 1, \dots, N$, such that*

(A1): $x^\top((A_p^{\text{jp}})^\top L_p^{\text{jp}} + L_p^{\text{jp}} A_p^{\text{jp}})x < 0$ *whenever* $\forall q \in \mathcal{N} : x^\top L_p^{\text{jp}} x \le x^\top L_q^{\text{jp}} x$ *and* $x \notin \mathfrak{C}_\cap$*;*

(A2): $x^\top((A_p^{\text{df}})^\top L_p^{\text{df}} + L_p^{\text{df}} A_p^{\text{df}})x < 0$ *whenever* $\forall q \in \mathcal{N} : x^\top L_p^{\text{df}} x \le x^\top L_q^{\text{df}} x$ *and* $x \in \mathfrak{C}_\cap \setminus \{0\}$*.*

Then the switching rule

$$\textbf{(MMS):}\quad \sigma(x) = \begin{cases} \arg\min\limits_{p\in\mathcal{N}}(x^\top L_p^{\text{jp}} x) & \text{if } x \notin \mathfrak{C}_\cap \\ \arg\min\limits_{p\in\mathcal{N}}(x^\top L_p^{\text{df}} x) & \text{if } x \in \mathfrak{C}_\cap \end{cases}$$

asymptotically stabilizes Δ_σ.

Moreover, if $x^\top L_p^{\text{jp}} x \le x^\top L_q^{\text{jp}} x$ of **(A1)** *(or $x^\top L_p^{\text{df}} x \le x^\top L_q^{\text{df}} x$ of* **(A2)***) becomes $x^\top L_p^{\text{jp}} x \ge x^\top L_q^{\text{jp}} x$ (or $x^\top L_p^{\text{df}} x \ge x^\top L_q^{\text{df}} x$), then by changing $\arg\min\limits_{p\in\mathcal{N}}(x^\top L_p^{\text{jp}} x)$ (or $\arg\min\limits_{p\in\mathcal{N}}(x^\top L_p^{\text{df}} x)$) into $\arg\max\limits_{p\in\mathcal{N}}(x^\top L_p^{\text{jp}} x)$ (or $\arg\max\limits_{p\in\mathcal{N}}(x^\top L_p^{\text{df}} x)$), the rule* **(MMS)** *still asymptotically stabilizes Δ_σ under the changed conditions* **(A1)** *and* **(A2)***.*

Proof. We first prove that any initial point $x_0^- \notin \mathfrak{C}_\cap$ will jump into a consistent point $x_0^+ \in \mathfrak{C}_\cap$ under the rule **(MMS)** by condition **(A1)**. Recall that the jump dynamics are driven by $\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} = A_\sigma^{\text{df}} J(\tau)$. The Lyapunov function $V_\sigma^{\text{jp}}(x) = x^\top L_\sigma^{jp} x$, by construction, is continuous (also piece-wise smooth) for all $x \notin \mathfrak{C}_\cap$. Moreover, for each $p \in \mathcal{N}$, we have $\frac{\mathrm{d}V_p^{\text{jp}}(x)}{\mathrm{d}\tau} < 0$ for all τ such that the p-th mode is active, so $V_\sigma^{\text{jp}}(x)$ strictly decreases for all $x \notin \mathfrak{C}_\cap$ if no jump sliding modes are present. The same conclusion holds when there exist *jump* sliding modes on the switching surface (by a similar proof as that in section 3.4.2 of [13]). Observe that $\frac{\mathrm{d}V_\sigma^{\text{jp}}(x)}{\mathrm{d}\tau} = x^\top((A_\sigma^{\text{jp}})^\top L_p^{\text{jp}} + L_p^{\text{jp}} A_\sigma^{\text{jp}})x = 0$ for all $x \in \mathfrak{C}_\cap$ as $\forall p \in \mathcal{N} : \ker A_p^{\text{jp}} = \mathfrak{C}_p$. Following a similar line as proving LaSalle's invariance principle (see e.g., Theorem 4.4 of [12]), it can be shown that any solution $J(\tau)$ of $\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} = A_\sigma^{\text{df}} J(\tau)$ starting from x_0^- converges to $\mathfrak{C}_\cap$, which means that x_0^- jumps into a point $x_0^+ \in \mathfrak{C}_\cap$.

If condition **(IN)** is satisfied, then any solution $x(t)$ starting from $x_0^+ \in \mathfrak{C}_\cap$ is a flow solution and stays in $\mathfrak{C}_\cap$ for all x. Condition **(A2)** guarantees that $V_\sigma^{\text{df}} = x^\top L_\sigma^{\text{df}} x$ decreases along the flow solutions of $\dot{x} = A_\sigma^{\text{df}} x$ (which coincides with those of Δ_σ) under the rule **(MMS)** no matter the (flow) sliding modes are present or not as in the case of switched ODEs (see section 3.4.2 of [13]). It follows that the flow solution $x(t)$ starting from $x_0^+ \in \mathfrak{C}_\cap$ asymptotically converges to the origin as $t \to \infty$.

If condition **(IN)** is not satisfied, then it is possible that there exists a jump-flow sliding mode on a surface S_{pq}^δ near $S_{pq} = \{x \in \mathfrak{C}_\cap \mid x^\top(L_p^{\text{df}} - L_q^{\text{jp}})x = 0\}$, i.e., the solution of $\dot{x} = A_p^{\text{df}} x$ escapes $\mathfrak{C}_\cap$ to S_{pq}^δ (see also the proof of Theorem 4.5 above), and then both the rule $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_p^{\text{df}} x$ and $\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = A_q^{\text{jp}} x$ are respected. The sliding mode is characterized by $x^\top((A_p^{\text{df}})^\top(L_p^{\text{df}} - L_q^{\text{jp}}) + (L_p^{\text{df}} - L_q^{\text{jp}})A_p^{\text{df}})x \ge 0$ and $x^\top((A_q^{\text{jp}})^\top(L_p^{\text{df}} - L_q^{\text{jp}}) + (L_p^{\text{df}} - L_q^{\text{jp}})A_q^{\text{jp}})x \le 0$ for $x \in S_{pq}^\delta$. Then it is possible to prove that both $V_p^{\text{df}}(x)$ and $V_q^{\text{jp}}(x)$ decrease along the Filippov solution of $\dot{x} = \alpha A_p^{\text{df}} x + (1-\alpha) A_q^{\text{df}} x$. We only show the case for V_p^{df}: $\dot{V}_p(x) = \alpha x^\top((A_p^{\text{df}})^\top L_p^{\text{df}} + L_p^{\text{df}} A_p^{\text{df}})x + (1-\alpha)x^\top((A_q^{\text{jp}})^\top L_p^{\text{df}} + L_p^{\text{df}} A_q^{\text{jp}})x \le \alpha x^\top((A_p^{\text{df}})^\top L_p^{\text{df}} + L_p^{\text{df}} A_p^{\text{df}})x + (1-\alpha)x^\top((A_q^{\text{jp}})^\top L_q^{\text{jp}} + L_q^{\text{jp}} A_q^{\text{jp}})x$. The first term $\alpha x^\top((A_p^{\text{df}})^\top L_p + L_p A_p^{\text{df}})x < 0$ for all $x \in S_{pq}^\delta$ by **(A1)** (recall that δ is arbitrarily small); $x^\top((A_q^{\text{jp}})^\top L_q^{jp} + L_q^{\text{jp}} A_q^{\text{jp}})x < 0$ for all $x \in S_{pq}^\delta$ by **(A2)**. Therefore, the jump-flow sliding mode is also asymptotically stable.

□

It is also possible to use the S-lemma to write the conditions **(A1)** and **(A2)** in matrix inequalities forms as shown in **(A1)'** and **(A2)'** below. However, similar to the switched ODEs case [13], it is required to exclude the sliding modes for the rules derived from the inequalities **(A1)'** and **(A2)'** when the scalars ϵ_{pq} or τ_{pq} are non-positive.

Corollary 4.10. *Consider a SwDAE Δ_σ, the switching rule* (**MMS**) *asymptotically stabilizes Δ_σ if there exist positive-definite matrices $L_p^{\mathrm{jp}} = (L_p^{\mathrm{jp}})^\top > 0$, $L_p^{\mathrm{df}} = (L_p^{\mathrm{df}})^\top > 0$, non-negative scalars $\epsilon_{pq} \geq 0$, $\tau_{pq} \geq 0$ and positive scalar $\kappa_p > 0$ such that $\forall p \in \mathcal{N}$:*

(A1)': $(A_p^{\mathrm{jp}})^\top L_p^{\mathrm{jp}} + L_p^{\mathrm{jp}} A_p^{\mathrm{jp}} - \sum_{q=1}^{N} \epsilon_{pq}(L_p^{\mathrm{jp}} - L_q^{\mathrm{jp}}) + \kappa_p B^\top B \leq 0$*;*

(A2)': $C^\top((A_p^{\mathrm{df}})^\top L_p^{\mathrm{df}} + L_p^{\mathrm{df}} A_p^{\mathrm{df}} - \sum_{q=1}^{N} \tau_{pq}(L_p^{\mathrm{df}} - L_q^{\mathrm{df}}))C < 0$,

where C is any full column rank matrix such that $\operatorname{im} C = \mathfrak{C}_\cap$ *and B is any full row rank matrix such that* $\ker B = \mathfrak{C}_\cap$.

Moreover, suppose that sliding modes does not exist in any solution of Δ_σ, if **(A1)'** *(or* **(A2)'***) holds for non-positive scalars $\epsilon_{pq} \leq 0$ (or $\tau_{pq} \leq 0$), then by changing $\arg\min_{p\in\mathcal{N}}(x^\top L_p^{\mathrm{jp}} x)$ (or $\arg\min_{p\in\mathcal{N}}(x^\top L_p^{\mathrm{df}} x)$) into $\arg\max_{p\in\mathcal{N}}(x^\top L_p^{\mathrm{jp}} x)$ (or $\arg\max_{p\in\mathcal{N}}(x^\top L_p^{\mathrm{df}} x)$), the rule* (**MMS**) *still asymptotically stabilizes Δ_σ under the changed conditions* **(A1)'** *and* **(A2)'**.

Remark 4.11. In the case of $\mathbf{A}_4^{\mathrm{jp}}$ is invertible, i.e., there exists an invertible matrix T such that (18) holds, **(A1)'** may be rewrote as the following inequality in Tx-coordinates:

$$\begin{bmatrix} 0 & 0 \\ 0 & A_{4p}^{\mathrm{jp}} \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ 0 & \bar{L}_p \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \bar{L}_p \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & A_{4p}^{\mathrm{jp}} \end{bmatrix} \\ - \sum_{q=1}^{N} \epsilon_{pq} \begin{bmatrix} 0 & 0 \\ 0 & \bar{L}_p - \bar{L}_q \end{bmatrix} + \kappa_p \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \leq 0$$

for $L_p^{\mathrm{jp}} = T^\top \begin{bmatrix} I & 0 \\ 0 & \bar{L}_p \end{bmatrix} T$, thus **(A1)'** holds if and only if there exist $\bar{L}_p = \bar{L}_p^\top > 0$ such that

$$\textbf{(A1)''}: \quad (A_{4p}^{\mathrm{jp}})^\top \bar{L}_p + \bar{L}_p A_{4p}^{\mathrm{jp}} - \sum_{q=1}^{N} \epsilon_{pq}(\bar{L}_p - \bar{L}_q) < 0.$$

Similarly, define $L_p^{\mathrm{df}} := T^\top \begin{bmatrix} \hat{L}_p & 0 \\ 0 & I \end{bmatrix} T$, $\hat{L}_p = \hat{L}_p^\top > 0$, the condition **(A2)'** can be rewrote as

$$\textbf{(A2)''}: \quad (A_{1p}^{\mathrm{df}})^\top \hat{L}_p + \hat{L}_p A_{1p}^{\mathrm{jp}} - \sum_{q=1}^{N} \tau_{pq}(\hat{L}_p - \hat{L}_q) < 0.$$

Example 4.12. Consider a SwDAE Δ_σ with generalized states $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and two *unstable* modes

$$\Delta_1 : \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

$$\Delta_2 : \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

The consistency spaces are given by

$$\mathfrak{C}_1 = \{x \in \mathbb{R}^3 \mid x_3 = 0\}, \quad \mathfrak{C}_2 = \{x \in \mathbb{R}^3 \mid x_4 = 0\}.$$

Calculate the jump and flow matrices through (5) to get

$$A_1^{\mathrm{jp}} = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad A_2^{\mathrm{jp}} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$A_1^{\mathrm{df}} = \begin{bmatrix} 0 & -1 & 0 & -2 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad A_2^{\mathrm{df}} = \begin{bmatrix} 0 & -2 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find a matrix T such that (18) is satisfied, i.e.,

$$T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T\mathbf{A}^{\mathrm{jp}}T^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{A}_4^{\mathrm{jp}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 2\beta-2 \\ 0 & 0 & 2\beta & \beta-1 \end{bmatrix}.$$

It follows that

$$T\mathbf{A}^{\mathrm{jp}}T^{-1} = \begin{bmatrix} \mathbf{A}_1^{\mathrm{df}} & \mathbf{A}_2^{\mathrm{df}} \\ \mathbf{A}_3^{\mathrm{df}} & \mathbf{A}_4^{\mathrm{df}} \end{bmatrix} = \alpha \begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 & -2 & 1 & 0 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is seen that although the convex combination $\mathbf{A}_4^{\mathrm{jp}}$ can be Hurwitz for some $0 < \beta < 1$, it does *not* exist $0 \leq \alpha \leq 1$ such that $\mathbf{A}_1^{\mathrm{df}} = \alpha \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ is Hurwitz. So it is not possible to use the results in section 4.2 to stabilize the SwDAE.

Nevertheless, it is found that for

$$L_1^{\mathrm{jp}} = T^\top \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2.2 & 0 \\ 1 & 0 & 0 & 1.2 \end{bmatrix},$$
$$L_2^{\mathrm{jp}} = T^\top \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 1.2 \end{bmatrix} T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1.2 & 0 \\ 1 & 0 & 0 & 2.2 \end{bmatrix},$$

the conditions **(A1)** of Theorem 4.9 are satisfied for

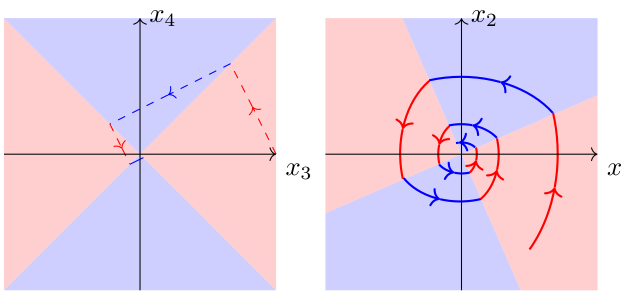
$x^\top L_p^{\mathrm{jp}} x \geq x^\top L_p^{\mathrm{jp}} x$. Moreover, the matrices

$$L_1^{\mathrm{df}} = T^\top \begin{bmatrix} 1.32 & -0.21 & 0 & 0 \\ -0.21 & 0.85 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T = \begin{bmatrix} 1.32 & -0.21 & -0.21 & 1.32 \\ -0.21 & 0.85 & 0.85 & -0.21 \\ -0.21 & 0.85 & 1.85 & -0.21 \\ 1.32 & -0.21 & -0.21 & 2.32 \end{bmatrix},$$
$$L_2^{\mathrm{df}} = T^\top \begin{bmatrix} 0.85 & 0.21 & 0 & 0 \\ 0.21 & 1.32 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T = \begin{bmatrix} 0.85 & 0.21 & 0.21 & 0.85 \\ 0.21 & 1.32 & 1.32 & 0.21 \\ 0.21 & 1.32 & 2.32 & 0.21 \\ 0.85 & 0.21 & 0.21 & 1.85 \end{bmatrix},$$

yield **(A2)** of Theorem 4.9 for $x^\top L_p^{\mathrm{jp}} x \geq x^\top L_p^{\mathrm{jp}} x$. Thus the switching rule

$$\sigma(x) = \begin{cases} \arg \max\limits_{p \in \{1,2\}} (x^\top L_p^{\mathrm{jp}} x) & \text{if } x_3 \neq 0 \vee x_4 \neq 0 \\ \arg \max\limits_{p \in \{1,2\}} (x^\top L_p^{\mathrm{df}} x) & \text{if } x_3 = x_4 = 0 \end{cases}$$

stabilizes the SwDAE. The following figure shows the jump-flow solution under the above switching rule for an initial point $x_0 \notin \mathfrak{C}_1 \cap \mathfrak{C}_2$. Remark that the linear subspace $\mathfrak{C}_\cap = \mathfrak{C}_1 \cap \mathfrak{C}_2$ is both A_1^{df}-invariant and A_2^{df}-invariant, so the condition **(IN)** is satisfied and there exist no jump-flow sliding modes.



(a) Jump solution projected to $x_3 - x_4$ plane

(b) Flow solution existing on $x_1 - x_2$ plane

Fig. 8. Red dashed lines and blue dashed lines: Jumps of Δ_1 and Δ_2, red and blue lines: Flows of Δ_1 and Δ_2.

5 Conclusions and perspectives

In this paper, a novel solution framework is proposed for linear state-dependent SwDAEs, enabling the generalization of classical stability and stabilization results from switched ODEs to their DAE counterparts. The key innovation lies in treating the reinitialization-induced jumps during switching as ODE-governed dynamics, thereby addressing the inherent conflicts between consistency projection and state-dependent switching rules. This approach paves the way for a comprehensive analysis of three distinct sliding modes: those triggered by jump dynamics, flow dynamics, and a combination of both. Building upon this solution framework, two well-known switched ODEs stabilization strategies – the Hurwitz convex combination and the min-max switching rule – is investigated in the context of DAEs. The approach behind our stabilization rule involves exclusively employing state-dependent jumps to guide the initial point towards the intersection of all consistency spaces $\mathfrak{C}_\cap$. Subsequently, by switching between the flow dynamics of each mode restricted to $\mathfrak{C}_\cap$, our methodology ensures the convergence of the entire jump-flow solution.

The proposed solution framework presents a fresh perspective for exploring linear SwDAEs, paving the way for potential extensions to nonlinear switched SwDAEs or those in the discrete-time case. Furthermore, beyond our state-dependent switching strategy, there exists promising potential in exploring alternative stabilization techniques that mix the utilization of jumps and flow dynamics. Investigating various switching rules could yield valuable insights. Exploring the utilization of multiple Lyapunov functions for SwDAEs is also an interesting topic for further research. Finally, as highlighted in the introduction, the close relationship between complementarity systems and state-dependent SwDAEs suggests a compelling direction for future work—adapting and applying our results to complementarity systems could significantly advance the field.

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