Normandie Université

THESE

Pour obtenir le diplôme de doctorat

Spécialité Mathématiques

Préparée au sein de l'Institut National des Sciences Appliquées de Rouen Normandie

Geometric Analysis of Differential-Algebraic Equations and Control Systems: Linear, Nonlinear and Linearizable

Présentée et soutenue par
Yahao CHEN

Thèse soutenue publiquement le 13 Juin 2019 devant le jury composé de		
M. Claude MOOG	Directeur de Recherche, LS2N, Laboratoire des Sciences du Numérique de Nantes, CNRS	Rapporteur
M. Stephan TRENN	Professeur, University of Groningen	Rapporteur
M. Jean-Pierre BARBOT	Professeur, ENSEA	Examinateur
M. Philippe JOUAN	Maître de conférences, Université de Rouen Normandie	Examinateur
Mme. Françoise LAMNABHI-LAGARRIGUE	Directeur de Recherche, Laboratoire des Signaux et Systèmes, CNRS	Examinateur
M. Pierre ROUCHON	Professeur, MINES ParisTech	Examinateur
M. Witold RESPONDEK	Professeur, INSA de Rouen Normandie	Directeur de thèse

Thèse dirigée par M. Witold RESPONDEK,
Laboratoire de Mathématiques de l'INSA de Rouen Normandie - LMI (EA 3226)

INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES ROUEN

LMI - EA 3226

Acknowledgments

Firstly, I would like to express my sincere gratitude to my advisor Prof. Witold Respondek for his patience, immense knowledge, encouragement and suggestions. His guidance helped me in all the years of my Ph.D and will continue to inspire and help me for my future work and life.

I am extremely grateful to the rest of my thesis committee: the reviewers: Prof. Claude MOOG and Prof. Stephan TRENN; and the examiners: Prof. Françoise LAMNABHI-LAGARRIGUE, Prof. Pierre ROUCHON and Prof. Jean-Pierre BARBOT, M. Philippe JOUAN for their precious time, insightful comments and remarks, and useful suggestions to broad my research field.

It is my honor to work as a member of "Laboratoire de Mathématiques de l'INSA Rouen Normandie (LMI)". Special thanks to Prof. Nicolas FORCADEL and Prof. Christian GOUT, the directors of LMI, for their welcome, availability and support. I am grateful to Ms. Brigitte DIARRA, the secretary of LMI, to Dr. Imene KHAMES, to M.Timothée Schmoderer. Thank you for your kindness, suggestions, help, listening and encouragement.

My research would have been impossible without the scholarship of China Scholarship Council. I am profoundly grateful to my country and the government of China for giving me the opportunity to study abroad.

A special thanks to my girl friend Qingqing for her accompany, patient, support and to all my friends in our little group "Fan Ka" for their free coffee, interesting discussions and funny jokes to keep my Ph.D life full of fun. I also want to thank the bands and singers of my song list, it is your voices keeping me moving forward everyday.

Last but not the least, I would like to thank my family: my parents and my sister for supporting me spiritually throughout writing this thesis and my life in general.

“It’s better to burn out than to fade away.”

—Neil Young *«Hey Hey, My My»*

Contents

Résumé:

Dans la première partie de cette thèse, nous étudions les équations différentielles algébriques (en abrégé EDA) linéaires et les systèmes de contrôles linéaires associés (en abrégé SCEDA). Les problèmes traités et les résultats obtenus sont résumés comme suit.

1. Relations géométriques entre les EDA linéaires et les systèmes de contrôles génériques SCEDO. Nous introduisons une méthode, appelée explicitation, pour associer un SCEDO à n'importe quel EDA linéaire. L'explicitation d'une EDA est une classe des SCEDO, précisément un SCEDO défini, à un changement de coordonnées près, une transformation de bouclage près et une injection de sortie près. Puis nous comparons les « suites de Wong » d'une EDA avec les espaces invariants de son explicitation. Nous prouvons que la forme canonique de Kronecker **FCK** d'une EDA linéaire et la forme canonique de Morse **FCM** d'un SCEDO, ont une correspondance une à une et que leur invariants sont liés. De plus, nous définissons l'équivalence interne de deux EDA et montrons sa particularité par rapport à l'équivalence externe en examinant les relations avec la régularité interne, i.e., l'existence et l'unicité de solutions.

2. Transformation d'un SCEDA linéaire vers sa forme canonique via la méthode d'explicitation avec des variables de driving. Nous étudions les relations entre la forme canonique par bouclage **FCFB** d'un SCEDA proposée dans la littérature et la forme canonique de Morse pour les SCEDO. Premièrement, dans le but de relier SCEDA avec les SCEDO, nous utilisons une méthodes appelée explicitation (avec des variables de driving). Cette méthode attache à une classe de SCEDO avec deux types d'entrées (le contrôle original et le vecteur des variables de driving) à un SCEDA donné. D'autre part, pour un SCEDO linéaire classique (sans variable de driving) nous proposons une forme de Morse triangulaire **FMT** pour modifier la construction de la **FCM**. Basé sur la **FMT** nous proposons une forme étendue **FMT** et une forme étendue de **FCM** pour les SCEDO avec deux types d'entrées. Finalement, un algorithme est donné pour transformer un SCEDA dans sa **FCFB**. Cet algorithme est construit sur la **FCM** d'un SCEDO donné par la procédure d'explicitation. Un exemple numérique illustre la structure et l'efficacité de l'algorithme.

Pour les EDA non linéaires et les SCEDA (quasi linéaires) nous étudions les problèmes suivants:

3. Explicitations, analyse externe et interne et formes normales des EDA non linéaires.

Nous généralisons les deux procédures d'explicitation (avec ou sans variables de driving) dans le cas des EDA non linéaires. L'objectifs de ces deux méthodes est d'associer un SCEDO non linéaire à une EDA non linéaire telle que nous puissions l'analyser à l'aide de la théorie des EDO non linéaires. Nous comparons les différences de l'équivalence interne et externe des EDA non linéaires en étudiant leur relations avec l'existence et l'unicité d'une solution (régularité interne). Puis nous montrons que l'analyse interne des EDA non linéaires est liée à la dynamique nulle en théorie classique du contrôle non linéaire. De plus, nous montrons les relations des EDAS de forme purement semi-explicite avec les 2 procédures d'explicitations. Finalement, une généralisation de la forme de Weierstrass non linéaire **FW** basée sur la dynamique nulle d'un SCEDO non linéaire donné par la méthode d'explicitation est proposée.

4. Linéarisation par bouclage et sous variété controllable invariante des EDA non liéaires. Nous étudions la linéarisation par bouclage des EDA non linéaires (de forme quasi-linéaire) sous l'action de deux sortes de bouclage, i.e., l'équivalence par bouclage externe et l'équivalence par bouclage interne. Des conditions nécessaires et suffisantes sont donnés à l'aide de l'explicitation (avec variables de driving). Nous montrons que la linéarisation par bouclage d'un SCEDA est liée à l'involutivité des distributions, qui forment deux suites, attachées à un SCEDO donné par la procédure d'explicitation. De plus, nous étudions la sous variété invariante controllable maximale d'un système d'EDA et, si celle-ci existe, une forme normale sous l'action du bouclage externe est déduite sous des hypothèses de rang constant. Cette forme normale explicite le rôle de différentes variables des SCEDA non linéaires.

En outre, pour les EDA non linéaires (de forme semi-explicite), nous étudions:

5. Linéarisation interne et externe des EDA semi explicites. Nous étudions deux sortes de linéarisation (interne et externe) pour les EDA non linéaires de forme semi-explicite. La différence entre linéarisation interne et externe est illustré par un exemple de système mécanique. De plus, nous définissons plusieurs niveaux d'équivalence externe pour deux EDA de forme semi-explicite. L'explicitation proposée nous permet de traiter une EDA semi-explicite comme un système de contrôle défini à un bouclage près (une classe de systèmes de contrôle). Puis des conditions nécessaires et suffisantes exprimées par l'explicitation caractérisent l'equivalence externe niveau-3 d'une EDA semi-explicite avec une EDA linéaire semi-explicite d'une forme partciulière. Finalement, nous montrons par un exemple que la linéarisation par bouclage de niveau-2 peut être réalisée si l'une de ses explicitations est linéarisable entrée-sortie de niveau-2.

Abstract:

In the first part of this thesis, we study linear differential-algebraic equations (shortly, DAEs) and linear control systems given by DAEs (shortly, DAECSs). The discussed problems and obtained results are summarized as follows.

1. Geometric connections between linear DAEs and linear ODE control systems ODECSs. We propose a procedure, named explicitation, to associate a linear ODECS to any linear DAE. The explicitation of a DAE is a class of ODECSs, or more precisely, an ODECS defined up to a coordinates change, a feedback transformation and an output injection. Then we compare the "Wong sequences" of a DAE with invariant subspaces of its explicitation. We prove that the basic canonical forms, the Kronecker canonical form **KCF** of linear DAEs and the Morse canonical form **MCF** of ODECSs, have a perfect correspondence and their invariants (indices and subspaces) are related. Furthermore, we define the internal equivalence of two DAEs and show its difference with the external equivalence by discussing their relations with internal regularity, i.e., the existence and uniqueness of solutions.

2. Transform a linear DAECS into its feedback canonical form via the explicitation with driving variables. We study connections between the feedback canonical form **FBCF** of DAE control systems DAECSs proposed in the literature and the famous Morse canonical form **MCF** of ODECSs. First, in order to connect DAECSs with ODECSs, we use a procedure named explicitation (with driving variables). This procedure attaches a class of ODECSs with two kinds of inputs (the original control input and the vector of driving variables) to a given DAECS. On the other hand, for classical linear ODECSs (without driving variables), we propose a Morse triangular form **MTF** to modify the construction of the classical **MCF**. Based on the **MTF**, we propose an extended **MTF** and an extended **MCF** for ODECSs with two kinds of inputs. Finally, an algorithm is proposed to transform a given DAECS into its **FBCF**. This algorithm is based on the extended **MCF** of an ODECS given by the explication procedure. Finally, a numerical example is given to show the structure and efficiency of the proposed algorithm.

For nonlinear DAEs and DAECSs (of quasi-linear form), we study the following problems:

3. Explicitations, external and internal analysis, and normal forms of nonlinear DAEs. We generalize the two explicitation procedures (with or without driving variable) proposed in the linear case for nonlinear DAEs of quasi-linear form. The purpose of these two explicitation procedures is to associate a nonlinear ODECS to any nonlinear DAE such that we can use the classical nonlinear ODE control theory to analyze nonlinear DAEs. We discuss differences of internal and external equivalence of nonlinear DAEs by showing their relations with the existence and uniqueness of solutions (internal regularity). Then we show that the internal analysis of nonlinear DAEs is closely related to the zero dynamics in the classical nonlinear control theory. Moreover, we show relations of DAEs of pure semi-explicit form with the two explicitation procedures. Furthermore, a nonlinear

generalization of the Weierstrass form **WF** is proposed based on the zero dynamics of a nonlinear ODECS given by the explicitation procedure.

4. Feedback linearization and controlled invariant submanifolds of nonlinear DAECSs. We study feedback linearizability of nonlinear DAECSs (of quasi-linear form) under two kinds of feedback equivalence, namely, the external feedback equivalence and internal feedback equivalence. Necessary and sufficient conditions are given with the help of the explicitation (with driving variable) procedure. It is proved that feedback linearizability of a DAECS is closely related to the involutivity of distributions, forming two sequences, associated to an ODECS given by the explicitation procedure. Moreover, we investigate the maximal controlled invariant submanifold for DAE systems and two normal forms under external feedback equivalence are derived, under constant rank or involutivity hypothesis, assuming only the existence of the invariant submanifold. These normal forms facilitate understanding of the role of various variables in nonlinear DAECSs.

Furthermore, for nonlinear DAEs (of semi-explicit form), we study:

5. Internal and external linearization of semi-explicit DAEs. We study two kinds of linearization (internal and external) for nonlinear DAEs of semi-explicit SE form. The difference between external and internal linearization is illustrated by an example of mechanical system. Moreover, we define different levels of external equivalence for two SE DAEs. The proposed explicitation procedure allows us to treat a given SE DAE as a control system defined up to a feedback transformation (a class of control systems). Then sufficient and necessary conditions, expressed via explicitation, are given to describe when a given SE DAE is level-3 external equivalent to a linear SE DAE of some specific form. Finally, we show by an example that the level-2 external linearization can be achieved if one of its explicitation is level-2 input-output linearizable.

Chapter 1

Introduction

In this thesis, we are interested in systems given by differential-algebraic equations DAEs, which are also called implicit, singular, generalized, or descriptor systems. In particular, we will study DAE systems of the following different forms, the differences between these forms come from their structures, nonlinearities and levels of implicitation. More precisely, we consider the following linear DAE systems:

- A *linear differential-algebraic equation*, shortly a linear DAE, is of the form

$$\Delta : E\dot{x} = Hx. \tag{1.1}$$

- A *linear differential-algebraic equation control system*, shortly a linear DAECS, is of the form

$$\Delta^u : E\dot{x} = Hx + Lu. \tag{1.2}$$

- A *linear semi-explicit differential-algebraic equation*, shortly a linear SE DAE, is of the form

$$\Delta^{se} : \begin{cases} R\dot{x} = Ax \\ \ 0 \ = Cx. \end{cases} \tag{1.3}$$

In the above equations, $E \in \mathbb{R}^{l\times n}$, $H \in \mathbb{R}^{l\times n}$, $L \in \mathbb{R}^{l\times m}$, $R \in \mathbb{R}^{r\times n}$, $A \in \mathbb{R}^{r\times n}$, $C \in \mathbb{R}^{p\times n}$, and R is of full row rank. The variable $x \in \mathbb{R}^n$ is called the "generalized" state (also called semi-state, see e.g. [127],[165]) and $u \in \mathbb{R}^m$ is a predefined control input. Correspondingly, we consider the following nonlinear DAE systems.

- A *nonlinear differential-algebraic equation*, shortly a nonlinear DAE, is of the form

$$\Xi : E(x)\dot{x} = F(x). \tag{1.4}$$

- A *nonlinear differential-algebraic equation control system*, shortly a nonlinear DAECS, is of the form

$$\Xi^u : E(x)\dot{x} = F(x) + G(x)u. \tag{1.5}$$

- A *nonlinear semi-explicit differential-algebraic equation*, shortly a nonlinear SE DAE, is of the form

$$\Xi^{se}: \begin{cases} R(x)\dot{x} = a(x) \\ 0 \quad = c(x). \end{cases} \tag{1.6}$$

In the above nonlinear systems, $x \in X$ is the "generalized" state and X is an open subset of $\mathbb{R}^n$ (or more generally, X is a differential manifold of dimension n), and $u \in \mathbb{R}^m$ is a predefined control input and $\mathbb{R}^m$ is called the input space. The matrix-valued functions $E(x)$, $R(x)$, $F(x)$, $G(x)$, $a(x)$ and $c(x)$ above are smooth and of appropriate sizes, and $R(x)$ is usually assumed to be of full row rank. Throughout the thesis, the word smooth will always mean $\mathscr{C}^\infty$-smooth.

Note that the matrices E, R and the matrix-valued functions $E(x)$, $R(x)$ are not necessarily invertible, which is the reason that the DAE systems are different from ordinary differential equation ODE systems. In fact, if E is invertible, the DAE Δ, given by (1.1), can be expressed as $\dot{x} = E^{-1}Hx$, where E^{-1} is the inverse of E, which is an ODE. Also note that for linear systems Δ^u, given by (1.2), and nonlinear systems Ξ^u, given by (1.5), we emphasize the difference between the variable x and the variable u. Notice that, although there may exist free variables among the components of the "generalized" state x, we will not call these free variables control inputs as we do for the components of u.

The above linear and nonlinear DAE systems can be seen as special cases of the following DAE of the general form

$$\Theta^u : F(x, \dot{x}, u) = 0, \tag{1.7}$$

where $(x, \dot{x}) \in TX$, the tangent bundle of X, and $F : TX \times \mathcal{U} \to \mathbb{R}^l$ is smooth. Denote by $\mathbf{Class}(\Theta^u)$ the class of systems of the same form as Θ^u and use the same notation $\mathbf{Class}(\cdot)$ for Δ, Δ^u, Δ^{se}, Ξ, Ξ^u, Ξ^{se}. Then, apparently, we have

$$\begin{array}{ccccccc} \mathbf{Class}(\Delta^{se}) & \subseteq & \mathbf{Class}(\Delta) & \subseteq & \mathbf{Class}(\Delta^u) & \searrow^{\subseteq} & \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \mathbf{Class}(\Theta^u). \\ \mathbf{Class}(\Xi^{se}) & \subseteq & \mathbf{Class}(\Xi) & \subseteq & \mathbf{Class}(\Xi^u) & \nearrow_{\subseteq} & \end{array}$$

The above diagram illustrates the relations between different classes of DAEs. We will not study general DAEs of form (1.7), but it is worth to mention that the following references, which discuss problems concerning such DAEs, gave inspirations for the present thesis. The discussions on geometric interpretations of the existence and uniqueness of solutions can be seen in [162, 158], numerical methods of analyzing the solutions of DAEs can be consulted in [28, 38, 157], the discontinuities of DAE solutions are considered in [179], various definitions of DAE indices are given in [78, 77, 37, 123], some index reduction methods are shown in [76, 140, 6, 122, 142], connections between DAEs and infinite-dimensional differential geometry (or, differential flatness, see [68, 71]) are shown in [70, 125, 58, 156, 169, 69] etc.

1.1 Motivation and a short survey of DAE models in practical systems

The motivation of studying linear and nonlinear DAEs is their frequent presence in mathematical models of some practical systems. In particular, DAEs are a proper tool to model the following classes of systems.

Constrained mechanical systems: Since there is plenty of examples for such systems, e.g., the textbooks [159],[28] and thesis [177],[175], and some examples will be used in the next chapters of the present thesis, we will use a part of this section to discuss them. In general, the dynamics of a mechanical system, usually given by *the Euler-Lagrange equations*, are of the following form [57]:

$$M(p)\ddot{p} + V(\dot{p}, p) + G(p) = \tau + N^T(p)\lambda_n + H^T(p)\lambda_h, \tag{1.8}$$

where p is the vector of position variables, $M(p)$ is a matrix-valued function which is associated with mass (or inertia) and $V(\dot{p}, p)$ is a vector function which characterizes the Coriolis and centrifugal forces, $G(p)$ represents the gravity force and τ is a vector of external torque, where λ_h and λ_n are Lagrange multipliers corresponding to the *holonomic constraints* and *nonholomic constraints* (we will introduce their definitions below), respectively, $N(p)$ and $H(p)$ are matrix-valued functions of appropriate sizes. DAEs appear frequently in the models of mechanical systems subject to *holonomic constraints* and *nonholomic constraints* [199]. Whether constraints are holonomic or nonholonomic can be determined by the Frobenius theorem [82]. In particular, *holonomic constraints* are constraints depend on positions only and are usually of the following form:

$$C(p) = \begin{bmatrix} c_1(p) \\ \vdots \\ c_k(p) \end{bmatrix} = 0, \tag{1.9}$$

where $C(p)$ is a vector of scalar functions $c_i(p)$, $i = 1, \ldots, k$ and the matrix $N(p)$ in (1.8) satisfies $N(p) = \frac{\partial C(p)}{\partial p}$. The following examples are mechanical systems of form (1.8) subject to *holonomic constraints* of form (1.9) (we also indicate the form of these DAE systems using the notations Δ, Δ^{se}, Δ^u, Ξ, Ξ^{se}, Ξ^u, given by equations (1.1)-(1.6)):

- Some classical mechanisms: Form Ξ^{se}: the horizontal beam with specified shape supporting a load distributed along it in Example 7 of [35], the planar pendulum given in [158],[28] and Example 4.1.13 of [177], the lolly in Example 4.1.13 of [177], the slider crank (2-link robotic arm with end joint sliding on horizontal surface) in Example 4.1.5 of [177], the two-link, flexible joint, planar robotic arm in Example 5 of [35] or see [34],[100],[199], the 3-link robotic arm in [113], the constrained robot systems (a robot arm with inertial load or contacting with a rigid surface, two robots arm with a common inertial load) in [141], the Huygens oscillation center in [71],[183]; Form Ξ^u: the cart-pendulum system given in Example 1.1 of [142], the planar crane shown in [72],[68],[25], the constrained cylinder robot given in [41]; Form Δ^u: the linear mass-spring-system given in [15].

- Redundant mechanisms: Form Ξ^u: the redundant parallel robotic arm in [138], the double four joint mechanism in Example 4.1.16 of [177].
- Mechanisms with constrained trajectory: Form Ξ^u: the 3-link planar robotic arm with the trajectory end-point constrained on different surfaces in [198], the air-craft performing a circular loop in Example 6 of [35] or see [23],[108].
- Mechanical systems with prescribed trajectory: Note that such systems can be characterized by equation (1.8) and equation (1.10) below,

$$C(q(t)) - \gamma(t) = 0, \tag{1.10}$$

  where $r(t)$ is a curve describing the desired trajectory. For example, the aircrafts of pre-described trajectory given in [24], the system consisting of two-mass connected by a spring in Example 1 and Example 3 of [25], the reentry (space) vehicle on a spherical earth with given trajectories in [28].

The *nonholomic constriants,* linear with respect to velocity, have the following form

$$H(p)\dot{p} = 0, \tag{1.11}$$

where $H(p)$ is a matrix-valued function of an appropriate size. Nonholomic constraints usually characterize the sliding and rolling motions of mechanical systems. The following examples are systems subject to nonholonomic constraints or both holonomic and nonholonomic constraints, formulated by equations (1.8), (1.9), and (1.11):

- Form Ξ: The 3-link planar robotic arm with a free joint in [89],[3], two spheres systems (the rolling motion of one small sphere on the surface of a large sphere) in [199], similarly the rolling ball in (9.18) of [159], the skateboard rolling on a horizontal surface in [177] or a simpler case: the (single) sharp-edged skate in (9.3) of [159], the rolling disk in [99], three rolling cylinders in (9.32) [159]; Form Ξ^u: the system of mass, spring and double pendulum, called the roll-ring model in (9.13) of [159].

Electrical circuits (or power) systems: Such kinds of systems are often described by the mix of differential equations of the form

$$\begin{aligned} C(v_c)\dot{v}_c &= i_c \\ L(i_l)\dot{i}_l &= v_l \end{aligned}$$

coming from the characteristics of the devices (e.g., capacitors, inductors) and algebraic equations of the form

$$\begin{aligned} Ai &= 0 \\ v &= A^T e \end{aligned}$$

coming from the Kirchhoff's laws, where c, l, i and v stand for capacitors, inductors, current and voltage, respectively, and where e are the *node potentials*. Some simple examples of electrical circuits are, e.g., the simple nonlinear RC (only one capacitor, one resistor and one voltage source) circuit in [162], the linear RLC circuit (composed only of resistors, capacitors and voltage sources), and the nonlinear RLC circuits with differential or operational amplifiers in [28], the electrical network consisting of nonlinear resistors, inductors and current-sources and linear controlled sources, see Fig.20 of [171], the resistive nonlinear n ports shown in [51],[171], the parallel AC/DC power system shown in [107], the discretized transmission line (linear) given in [73],[15]. Actually, electrical circuits systems can be classified into different models, e.g., NTA, ANA, MNA, tree-based, hybrid and multi-port models as shown in the textbook [165], survey paper [166] and thesis [11]. Note that except for the MNA and some hybrid models, by assuming that $C(v_c)$ and $L(i_l)$ are invertible (or of full row rank) matrices, the models mentioned above are all of the semi-explicit form Ξ^{se} or Δ^{se} [166].

Chemical processes: Broad discussions on models of chemical processes from DAE point of view can be seen in the survey paper [60] and book [121]. Here we only mention a few examples from the vast documented literature: the distillation column in Example 8 of [35], various reactors (two phase reactor, reactor with fast and slow reaction, reactor with fast heat transfer though a jacket and cascade of reactors with negligible pressure drop) in [121],[119],[120], the catalytic reactors in [81] and the Phase-Locked Loop Circuit in [175].

Some other systems: Biological systems of heartbeat and nerve impulse in [200], the water hammer modeling for water network in [110],[109], the model of cyber-physical systems under attack in [155], the economic system in [134], the fluid dynamics in [84], etc.

1.2 Linear ODE systems and linear DAE systems

A linear ODE control system, shortly ODECS, is of the following form:

$$\Lambda : \begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{cases} \tag{1.12}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input and $y \in \mathbb{R}^p$ is the output. A linear ODECS of form (1.12) is denoted by $\Lambda_{n,m,p} = (A, B, C, D)$; if $D = 0$, we denote it by $\Lambda_{n,m,p} = (A, B, C)$. Note that we use x to denote both the "generalized" state of DAE systems and the state of ODE systems, but their differences should be pointed out as follows. The states of an ODECS are the variables that enter the system dynamically (and inputs are the variables that enter the system statically), but the "generalized" states of a DAE may include some free (static) variables. More precisely, the "generalized" states are the predefined states when modeling the system. Unless the analysis of the system shows some properties of the variables in the "generalized" states, we do not know the statuses of those variables, i.e., some of them may perform as free inputs, some are usual states, and some of them could be constrained by implicit algebraic constraints. Linear ODECSs draw attentions from researchers of mathematical control theory since the state space representations of systems were introduced in 1950s (see e.g. Kalman's papers [103],[106]). After decades of evolutions, linear ODE control theory has been well-developed by efforts of the control theory community. For a good review of linear control theory, including the history of its development, we refer to textbooks [167],[193],[102],[170],[43],[30]. In particular, we are interested in the geometric analysis of linear ODECSs, some pioneering works of that are from Basile, Marro, Wonham and Morse [9, 7, 194, 148, 147, 192, 8] and other interesting contributions are [190],[187],[188].

The geometric tools used to analyze the structure of ODECSs include various *invariant subspaces* of linear ODECSs (see e.g. [9]). In the matrix theory (see e.g. [75]), for a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, a subspace $\mathbb{V}$ of $\mathbb{R}^n$ is called A-invariant if $A\mathbb{V} \subseteq \mathbb{V}$. Invariant subspaces of ODECSs generalize the concept of A-invariance to control systems. For instance, for $\Lambda_{n,m,p} = (A, B, C)$, a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called (A, B)-controlled invariant if $\mathcal{V}$ satisfies

$$A\mathcal{V} \subseteq \mathcal{V} + \mathrm{Im}B$$

and a subspace $\mathcal{W} \subseteq \mathbb{R}^n$ is called (C, A)-conditioned invariant if $\mathcal{W}$ satisfies

$$A(\mathcal{W} \cap \ker C) \subseteq \mathcal{W}.$$

Denote by $\mathcal{V}^*$ the largest (A, B)-controlled invariant subspace contained in $\ker C$ and by $\mathcal{W}^*$ the smallest (C, A)-conditioned invariant subspace containing $\mathrm{Im}B$. An important application of the notion of invariant subspaces is to derive normal forms and canonical forms for ODECSs, e.g., the Kalman decomposition [105] provides a decomposed normal form for linear ODECSs based on the controllability and observability subspaces, the Brunovský canonical form [31] is a canonical form for controllable pairs (A, B) under feedback, the Morse canonical form **MCF** [146] is a fully decoupled canonical form for ODECSs with system matrices (A, B, C). In [146], transformations of an ODECS, given by a triple (A, B, C), into its **MCF** are constructed via the controlled and conditioned invariant subspaces. Also the structure invariants of the **MCF**, called the Morse indices, appearing in the transformations are calculated via those invariant subspaces. Molinari [145] generalized the **MCF** to ODECSs described by a quadruple of system matrices (A, B, C, D), based on the generalized (also called strong or weak) controllability and observability subspaces [144]. These subspaces are actually the controlled and conditioned invariant subspaces generated by the quadruple (A, B, C, D). A more general normal form of ODECSs is given in [1], whose authors decomposed an ODECS into nine parts (based on a full consideration of controllability and observability subspaces, controlled and conditioned invariant subspaces) and discuss their relations with system zeros.

In Chapter 1 and Chapter 2 of this thesis, we will review the precise definitions and calculation algorithms for conditioned and controlled invariant subspaces. Moreover, we will modify the construction procedure of the **MCF**, given in [146],[145], by proposing a Morse triangular form **MTF**. The proposed **MTF** makes the transformation from an ODECS into its **MCF** transparent and geometrical.

On the other hand, we consider linear DAEs of the form Δ and linear DAECSs of the form Δ^u, given by (1.1) and (1.2), respectively. Fundamental achievement and general discussions on linear DAEs and DAECSs can be consulted in textbooks [59],[36] and survey paper [127]. Early results on linear DAEs can be traced back to two famous canonical forms of the matrix pencil $sE - H$ given by Weierstrass [186] and Kronecker [117]. The Weierstrass form **WF** is a canonical form for *regular* matrix pencil and the Kronecker canonical form **KCF** handles the general (regular and non-regular) case. The word *regular* above means that E and H are square and $\det(sE - H) \neq 0$, $\forall s \in \mathbb{C}$, see e.g. [75]. Luenberger proposed a shuffle algorithm in [133] to test if a given DAE is regular. Structure analysis of DAECSs from polynomial system matrices point of view was introduced by Gantmatcher [75] and Rosenbrock [167]. After Rosenbrock introduced the restrictive system equivalence for two DAECSs (with output) in [168], there appeared various definition of the equivalence relations (e.g., external, strong, completely system, constant, input-output equivalence) for DAECSs, see the surveys in [64],[118]. In particular, there is a definition of external equivalence in [189],[118] via the behavior characterization, see also [2]. Notice that the external equivalence of [2] is different from two others mentioned above. Moreover, we emphasize here, that in Chapter 1 and Chapter 2 of this thesis, we will give our definition of external equivalence, which is actually the same as the strict and restricted equivalence defined in [75] and [168], respectively, and different from the ones in [189],[118] and [2].

The following literature discusses the normal forms and canonical forms of linear DAE systems under some predefined equivalence. The authors of [85] proposed a feedback canonical from for controllable and regular DAECSs. Several canonical forms for regular systems based on their controllability and impulse controllability are given in [80]. Moreover, in [153], a canonical form of general DAECSs was discussed. A more subtle and detailed feedback canonical form for general DAECSs was given in [131] by considering a group of transformations including state (proportional) feedback (P) and state derivative (proportional-derivative) feedback. Then the P and PD feedback canonical form was extended to DAECSs with output in [124] by considering additional transformations including coordinates changes in the output space and output injections. Furthermore, the canonical form of general DAECSs with output under the restricted system equivalence was discussed in [182]. More recently, a normal form based on the impulse-controllability and impulse-observability of DAECSs was proposed in [181], and a quasi-Weierstrass and a quasi-Kronecker normal form of DAEs were given in [16] and [20], respectively, using a geometrical way to make the transformations for a DAE to its **WF** and **KCF** more transparent.

As in the ODE case, fundamental geometrical tools for deriving canonical and normal forms of DAE systems are invariant subspaces of DAEs', i.e., the limits of the so-called Wong sequences and their augmented version (see e.g. [128],[17]). The Wong sequences of a DAE Δ, given by (1.1), are the subspace sequences $\mathscr{V}_i$ and $\mathscr{W}_i$ below, first given in [191] to discuss the existence and uniqueness of solutions for linear DAEs:

$$\begin{aligned} \mathscr{V}_0 &= \mathbb{R}^n, & \mathscr{V}_{i+1} &= H^{-1}E\mathscr{V}_i, \quad i \in \mathbb{N}, \\ \mathscr{W}_0 &= \{0\}, & \mathscr{W}_{i+1} &= E^{-1}H\mathscr{W}_i, \quad i \in \mathbb{N}. \end{aligned}$$

The augmented version of the Wong sequences are just the extensions of the above sequences in order to characterize feedback invariants of DAECSs. A deeper generalization of the Wong sequences to DAECSs with outputs are given, e.g., in [135],[152],[124]. These invariant subspaces of DAEs are powerful tools for the following problems of DAE systems: structure analysis [136], regularization and pole placement [126],[54],[18], disturbance decoupling [65],[14], controllability [137],[152],[17] and observability [19] analysis, etc. In the present thesis, we are particularly interested in the **KCF** for linear DAEs [117] and the feedback canonical form **FBCF** for linear DAECSs [131]. Relations of the structure invariants of the **KCF** (called the Kronecker indices) with the Wong sequences are shown in [130],[20],[21]. Moreover, relations of the indices of the **FBCF** with the augmented Wong sequences are shown in [131],[18].

In view of the similarities and mutual correspondence (e.g., the Morse indices of ODECS and the Kronecker indices of DAEs, the invariant subspaces of ODECSs and the Wong sequences of DAECSs, etc) that we have just described in the review about the history of the geometric aspects of linear ODE and DAE systems, it is natural to think about relations of the two classes of systems. This leads to our studies of Chapter 2, in which a given DAE is associated with a class of ODECSs and of Chapter 3, in which the connections of ODECSs and DAECSs are built.

1.3 Preliminaries on geometric theory of nonlinear ODE control systems

In this section, we will review some concepts of the classical geometric control theory (see the list of notations from differential geometry at the end of the thesis) for nonlinear ODECSs of the following control-affine form:

$$\Sigma : \begin{cases} \dot{x} = f(x) + \sum\limits_{i=1}^{m} g_i(x)u_i = f(x) + g(x)u \\ y = h(x), \end{cases} \tag{1.13}$$

where $x \in X$ is the state, X is an open subset of $\mathbb{R}^n$, $u \in \mathbb{R}^m$ is the input, $h(x)$ is a smooth $\mathbb{R}^p$-valued function on X, and where $f, g_1, \ldots, g_m$ are smooth vector fields on X. Nonlinear ODECS (1.13) will be denoted by $\Sigma_{n,m,p} = (f, g, h)$ or, simply, Σ. If we only discuss (1.13) but without output, we denote it by $\Sigma_{n,m} = (f, g)$. Two ODECSs $\Sigma_{n,m} = (f, g)$ and $\tilde{\Sigma}_{n,m} = (\tilde{f}, \tilde{g})$ defined on X and $\tilde{X}$ are called feedback equivalent if there exists a diffeomorphism $\psi : X \to \tilde{X}$, an $\mathbb{R}^m$-valued function $\alpha(x)$ and an invertible $m \times m$ matrix-valued function $\beta(x)$ satisfying

$$\begin{aligned} \tilde{f}(\psi(x)) &= \tfrac{\partial \psi(x)}{\partial x} \left(f + g\alpha\right)(x), \\ \tilde{g}(\psi(x)) &= \tfrac{\partial \psi(x)}{\partial x} (g\beta)(x). \end{aligned}$$

Obviously, $(x(t), u(t))$ is a solution of $\Sigma_{n,m}$ if and only if $(\tilde{x}(t), \tilde{u}(t))$, where $\tilde{x} = \psi(x(t))$ and $u(t) = \alpha(x(t)) + \beta(x(t))\tilde{u}(t)$, is a solution of $\tilde{\Sigma}_{n,m}$. Nonlinear ODE control theory has been well-developed for decades, fundamental theory and insightful results on nonlinear ODECSs using geometric methods can be found in the textbooks of Isidori [92] and

Nijmeijer and Van der Schaft [151]. Here we only make a review of a few notions which will be used in Chapters 4, 5 and 6.

The zero dynamics: The notion of zero dynamics was introduced in [95] in order to generalize the concept of transmission zeros of linear control systems. For an ODECS Σ, the zero dynamics algorithm is given as follows.

Step 0: Fix a nominal point x^0, around which we will work locally. Suppose that $h(x^0) = 0$. Set $N_0 = h^{-1}(0)$. Step k: assume for some neighborhood V_{k-1} of a point x^0, $N_{k-1} \cap V_{k-1}$ is a smooth submanifold and denote by N^c_{k-1} the connected components of $N_{k-1} \cap V_{k-1}$ such that $x^0 \in N^c_{k-1}$. Let

$$N_k = \left\{ x \in N^c_{k-1} : f(x) \in T_x N^c_{k-1} + \mathrm{span}\{g_1(x), \ldots, g_m(x)\} \right\}.$$

A smooth submanifold N is called controlled invariant if there exists a $\mathbb{R}^m$-valued function $\alpha(x)$ on N such that $\forall\, x \in N$, $f(x) + g(x)\alpha(x) \in T_x N$. An output zeroing submanifold of Σ is a locally controlled invariant submanifold $N \subseteq X$ satisfying $\forall\, x \in N$, $h(x) = 0$. If x^0 is a regular point of the zero dynamics algorithm, i.e., at every step k, $N_{k-1} \cap V_{k-1}$ is a smooth submanifold (around x^0), then the zero dynamics algorithm converges in $k^* < n$ steps and $N^* = N^c_{k^*}$ is a locally maximal output zeroing submanifold.

Static feedback linearization and relative degree: An ODECS $\Sigma_{n,m} = (f, g)$ is linearizable by static feedback if it is feedback equivalent to a linear controllable ODECS Λ, given by (1.12) but without output, i.e., $\dot{x} = Ax + Bu$. The problem of static feedback linearization for ODECS with single input was formulated and solved by Brockett [29] (for feedback of the form $u = \alpha + \tilde{u}$). Then, Jakubczyk and Respondek [98] and, independently, Hunt and Su [87] gave necessary and sufficient conditions to solve the static feedback linearization problem for multi-input ODECSs. Consider the following sequence of distributions:

$$G_1 := \mathrm{span}\{g_1, \ldots, g_m\}, \quad G_{i+1} = G_i + [f, G_i].$$

The system Σ is locally feedback linearizable if and only if for all $i \geq 1$, the distributions G_i are constant dimensional, involutive and $G_n = TX$.

The concept of relative degree is introduced to solve the problem of input-output decoupling (see [148] for linear ODECSs and [173],[174],[92] for nonlinear ODECSs). A square control system $\Lambda_{n,m,m} = (f, g, h)$ has a (vector) relative degree $(\rho_1, \ldots, \rho_m)$, shortly *r.d.*, at a point x^0 if (i) $L_{g_j} L^k_f h_i(x) = 0$, for all $1 \leq i, j \leq m$, $k < \rho_i - 1$, around x^0; (ii) The $m \times m$ decoupling matrix: $D(x^0) = \left(L_{g_j} L_f^{\rho_i - 1} h_i(x^0) \right)$ is invertible. Note that a nonlinear ODECS without output, given by (f, g), is feedback linearizable in a neighborhood U, if and only if there exist dummy outputs $y_1 = h_1(x), \ldots, y_m = h_m(x)$, where $h_i(x)$ are scalar functions defined on U, such that the ODECS with output, given by (f, g, h), where $h = (h_1, \ldots, h_m)$, has relative degree $(\rho_1, \ldots, \rho_m)$ at x^0 and $\rho_1 + \cdots + \rho_m = n$ [92].

Controlled and conditioned invariant distributions: Recall the following sequences of distributions S_i and codistributions P_i for ODECS Σ:

$$\begin{cases} S_1 & := \mathrm{span}\{g_1, \ldots, g_m\} \\ S_{i+1} & := S_i + [f, S_i \cap \ker dh] + \sum_{j=1}^{m} [g_j, S_i \cap \ker dh], \\ S^* & := \sum\limits_{i \geq 1} S_i, \end{cases}$$

$$\begin{cases} P_1 & := \mathrm{span}\{dh_1, \ldots, dh_p\} \\ P_{i+1} & := P_i + L_f(P_i \cap G^\perp) + \sum_{j=1}^{m} L_{g_j}(P_i \cap G^\perp), \\ P^* & := \sum\limits_{i \geq 1} P_i. \end{cases}$$

Define also $V_i := P_i^\perp$, $V^* := (P^*)^\perp$. The distribution V^* generalizes the largest controlled invariant subspace in $\ker C$ and the distribution S^* generalizes the smallest conditioned invariant subspace including ImB for linear ODECSs (see [9]). These distributions play an important role in the problems of linearization and decoupling of nonlinear control systems, see e.g., [94, 86, 93, 150, 114, 115, 139, 61].

Some other results on linearization: A survey about linearization problems for nonlinear ODECSs can be found in [53] and [180]. In particular, the problem of linearization of the input-output map of the system Σ is considered in [96],[91], in which, *the structure algorithm* (generalizing the linear version given by [172]) is used to construct transformations to linearize the input-output map. Then the results of [96] are modified and used in [50] in order to linearize ODE control systems with output. In [139], sufficient and necessary conditions are given for the problem of when a system Σ is equivalent to a prime form (for the definition of prime form, see [146]) by a group of transformations consisting of diffeomorphisms, feedback transformations, and coordinates changes in the output space. Based on the results of [139], the problem of using generalized output transformations to bring an ODECS into prime form is solved in [4]. The results of [139] and [4] can be interpreted as using some transformations in the output space to achieve a desired relative degree. Another way of achieving desired relative degree is using dynamic feedback. Since the linearization by dynamic feedback will not be discussed in this thesis but could be a nature direction for the future works, we only mention here a few references, see e.g, [39, 40, 5, 149, 66], or see Section 8.2 of [151] and Section 5.4 of [92].

1.4 Geometric aspects of nonlinear DAE systems

In this section, we review some geometric methods in the existing literature for DAE systems , including DAE Ξ, DAECS Ξ^u and DAE Θ of the general form, given by (1.4), (1.5) and (1.7), respectively. Compared to a large variety of results on nonlinear ODE systems using geometric methods, much less results are available for nonlinear DAE systems.

The early efforts of using differential-geometric methods to analyze DAEs are the works of Rheinboldt [164] and Reich [161], which regard a DAE as an implicit description of a vector field on a manifold. Also Chua [52] considered the DAE $\Xi = (E, F)$ and defined the pair $(E(x), F(x))$ as a generalized vector field. In [161] and [162], the concept of *regularity* in the linear DAE case was generalized for nonlinear DAEs to characterize the existence and uniqueness of DAE solutions. All the pioneering papers as [164],[161],[162], see also [160] [158],[165], lead to a geometrical reduction method. This reduction method is, roughly speaking, based on some constant rank and smoothness assumptions: one can construct a sequence of submanifolds from a given DAE, and if the constructed sequence converges after finitely many steps of iterations, then solutions of the DAE correspond to the solutions of an ODE evolving on a smooth submanifold (called the constrained submanifold or invariant manifold). The use of such a reduction method in the control context can be consulted in [116],[119],[197],[129],[196] in order to get a state space representation of a given DAECS. In Chapter 4 of this thesis, we will restate this reduction procedure as a nonlinear generalization of the shuffle algorithm given in [133] for checking the regularity of linear DAEs. We will also give our notion of regularity, called *internal regularity* for nonlinear DAEs of form Ξ. A main difference between the results of Chapter 4 and the former mentioned papers is the distinction between internal and external analysis of DAEs. Roughly speaking, the word "internal" means that we consider the DAE on its constrained submanifold only, i.e., where the solutions exist, the word "external" means that we consider the DAE in a whole neighborhood and for most points in that neighborhood there may not exist solutions. Although for a point which is not on the constrained submanifold, there are no solutions, the external analysis matters if we want to steer solutions from that point towards the constrained submanifold (in finite time or asymptotically). So the form of the DAE in this case matters not only on the constraint set but in a neighborhood as well.

The study of external forms for DAE systems under pre-defined equivalence can be seen in [111] for the feedback linearization problem of a class of nonlinear DAECSs, the equivalence relation considered there is actually "external". A nonlinear generalization of the Kronecker canonical form is shown in [169], which is an "external" normal form as well. Recently, Berger [12] generalized the notions of controlled invariant manifold and zero dynamics from nonlinear ODE control theory for nonlinear DAECSs. A normal form, called the zero dynamic form of nonlinear DAEs, is given in [13], based on which, the author of [13] discussed invertibility of nonlinear DAEs. In the present thesis, the study of external forms of DAE control systems leads to a nonlinear generalization of the Weierstrass form for nonlinear DAEs given in Chapter 4, and the maximal controlled invariant submanifold form for nonlinear DAECSs proposed in Chapter 5. Moreover, we will also consider external linearization problems. As in Chapter 5, we will discuss when a nonlinear DAECS is externally feedback equivalent to a linear completely controllable DAECS and in Chapter 5 we discuss when a SE DAE is externally equivalent to a linear SE DAE of some special form.

The results of Chapter 2 and 3 are inspirations for the ones of Chapter 4, 5 and 6, e.g. connections of linear ODECSs and DAEs inspire to study connections of nonlinear DAE systems and nonlinear ODE systems, the linear Weierstrass form is an inspiration for its nonlinear generalization, the Wong sequences and the augmented Wong sequences for linear DAEs are inspirations for their nonlinear generalizations as well.

Chapter 2

Geometric Analysis of Differential-Algebraic Equations via Linear Control Theory

Abstract: We consider linear differential-algebraic equations DAEs of the form $E\dot{x} = Hx$ and the Kronecker canonical form **KCF** of the corresponding matrix pencils $sE - H$. We also consider linear control systems and their Morse canonical form **MCF** [146],[145]. For a linear DAE, a procedure named explicitation is proposed, which attaches to any linear DAE a linear control system defined up to a coordinates change, a feedback transformation and an output injection. Then we compare subspaces associated to a DAE in a geometric way with those associated (also in a geometric way) to a control system, namely, we compare the the Wong sequences of DAEs and invariant subspaces of control systems. We prove that the **KCF** of linear DAEs and the **MCF** of control systems have a perfect correspondence and that their indices are related. In this way, we connect the geometric analysis of linear DAEs with the classical geometric linear control theory. Finally, we propose a concept named internal equivalence for DAEs and discuss its relation with internal regularity, i.e., the existence and uniqueness of solutions.

2.1 Introduction

Consider a linear differential-algebraic equation DAE of the form

$$\Delta : E\dot{x} = Hx, \tag{2.1}$$

where $x \in \mathscr{X} \cong \mathbb{R}^n$ is called the "generalized" state, $E \in \mathbb{R}^{l\times n}$ and $H \in \mathbb{R}^{l\times n}$. Throughout, a linear DAE of form (2.1) will be denoted by $\Delta_{l,n} = (E, H)$ or, shortly, Δ and the corresponding matrix pencil of Δ by $sE - H$, which is a polynomial matrix of degree one.

Terminologies as "singular", "implicit", "generalized" are frequently used to describe a DAE due to its difference from an ordinary differential equations ODE. Since the structure of DAE Δ is totally determined by the corresponding matrix pencil $sE - H$, it is useful to find a simplified form (a normal form or canonical form) for $sE - H$. Under predefined equivalence (see ex-equivalence of Definition 2.2.1), canonical forms as the Weierstrass form **WF** [186] for regular matrix pencils and the Kronecker canonical form [117] (for details see **KCF** in Appendix and [75]) for more general matrix pencils have been proposed. Note that in the present chapter, we will not distinguish the difference between the **KCF** of a matrix pencil $sE - H$ and the **KCF** of a DAE Δ, since although **KCF** is introduced for matrix pencils, it is immediate to put the **KCF** of $sE - H$ into the corresponding form for the DAE Δ.

Geometric analysis of linear and nonlinear DAEs can be found in [124, 127, 128, 135, 136, 158, 161, 162]. We highlight an important concept named the Wong sequences ($\mathscr{V}_i$ and $\mathscr{W}_i$ of Definition 2.4.1) for linear DAEs, which were first introduced in [191]. Connections between the Wong sequences with the **WCF** and the **KCF** have been recently established in, respectively, [16] and [20, 21]. In particular, invariant properties for the limits of the Wong sequences ($\mathscr{V}^*$ and $\mathscr{W}^*$ in Definition 2.4.3) were used to obtain a triangular quasi-Kronecker form in [20, 21]. Moreover, [20, 21] show that some of the Kronecker indices can be calculated via the Wong sequences and the remaining ones can be derived from a modified version of the Wong sequences.

On the other hand, consider a linear time-invariant control system of the following form

$$\Lambda : \begin{cases} \dot{z} = Az + Bu \\ y = Cz + Du, \end{cases} \tag{2.2}$$

where $z \in \mathscr{X} = \mathbb{R}^q$ is the system state, $u \in \mathscr{U} = \mathbb{R}^m$ represents the input and $y \in \mathscr{Y} = \mathbb{R}^p$ is the output. System matrices A, B, C, D above are constant and of appropriate sizes. We also consider the prolongation of Λ of the following form

$$\mathbf{\Lambda} : \begin{cases} \dot{z} = Az + Bu \\ \dot{u} = v \\ y = Cz + Du \end{cases} \Leftrightarrow \begin{cases} \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}v \\ y = \mathbf{C}\mathbf{z}, \end{cases} \tag{2.3}$$

where

$$\mathbf{z} = \begin{bmatrix} z \\ u \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} C & D \end{bmatrix}.$$

Denote a control system of form (2.2) by $\Lambda_{q,m,p} = (A, B, C, D)$ or, simply, Λ and denote the prolonged system (2.3) by $\mathbf{\Lambda}_{n,m,p} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$, or shortly $\mathbf{\Lambda}$, where $n = q + m$. Notice that there is a one to one correspondence between $\mathcal{C}^\infty$-solutions of (2.2) and (2.3) (or a one-one correspondence between $\mathcal{C}^1$-solutions $(z(t), u(t))$ of (2.2) and $\mathcal{C}^1$-solutions $\mathbf{z}(t)$, given by $\mathcal{C}^0$-controls v, of (2.3)).

Two kinds of invariant subspaces have been studied for analyzing the structure of linear control systems, see e.g. [193, 9]. More specifically, the largest $(\mathbf{A}, \mathbf{B})$-invariant subspace contained in $\ker \mathbf{C}$ (denoted $\boldsymbol{\mathcal{V}}^*$ in Definition 2.4.5), which is related with disturbance decoupling problems, and the smallest $(\mathbf{C}, \mathbf{A})$-conditioned invariant subspace containing $\operatorname{Im} \mathbf{B}$ (denoted $\boldsymbol{\mathcal{W}}^*$ in Definition 2.4.5) which is related to controllability subspaces. With the help of these invariant subspaces, any control system can be brought (see [146],[145]) into its Morse canonical form (for details, see **MCF** in Appendix) under the action of a group of transformations consisting of coordinates changes, feedback, and output injection. The **MCF** consists of four decoupled subsystems MCF^1, MCF^2, MCF^3, MCF^4, to which there correspond four sets of structure invariants (the Morse indices ε_i', ρ_i', σ_i', η_i' in the **MCF**) and these structure invariants are computable using $\mathcal{V}^*$ and $\mathcal{W}^*$. Note that in [146], only the triple (A, B, C) is considered while in [145], the general case of 4-tuple (A, B, C, D), with nonzero matrix D, is studied.

The first aim of the present chapter is to find a way to relate linear DAEs with linear control systems and find their geometric connections. In fact, we will show in the next section that to any linear DAE, we can attach a class of linear control systems defined up to a coordinates change, a feedback transformation and an output injection. We call this attachment the explicitation of a DAE. The second purpose of this chapter is to distinguish two kinds of equivalences in linear DAEs theory, namely, internal equivalence and external equivalence. We will give the formal definition of external equivalence in Definition 2.2.1. Note that our notion of ex-equivalence of DAEs is different from the one introduced in [189],[118], where "systems are defined to be externally equivalent if their behaviors are the same". Actually, the external equivalence (also named strict equivalence in [75]) is widely considered in the linear DAEs literature. For example, the **KCF** of a DAE is actually a canonical form under external equivalence, which is simply defined by all linear nonsingular transformations in the whole "generalized" state space of the DAE. However, since solutions of a DAE exist only on a constrained (invariant) subspace, sometimes we only need to perform the analysis on that constrained subspace. This point of view motivates to introduce the notion of internal equivalence and to find normal forms not on the whole space but only on that constrained subspace.

The chapter is organized as follows. In Section 2.2, we introduce the notations, define the external equivalence of two DAEs, and also the Morse equivalence of two control systems. In Section 2.3, we explain how to associate to any DAE a class of control systems. In Section 2.4, we describe geometric relations of DAEs and the attached control systems. In Section 2.5, we show that there exists a perfect correspondence between the **KCF** and the **MCF**, and that their indices have direct relations. In Section 2.6, we introduce the notion of internal equivalence for DAEs and then discuss the internal regularity. Section 2.7 contains the proofs of our results and Section 2.8 contains the conclusions of this chapter. Finally, in the Appendix we recall two basic canonical forms: the Kronecker canonical form **KCF** for DAEs and the Morse canonical form **MCF** for control systems.

2.2 Preliminaries

We use the following notations in the present chapter.

$\mathbb{N}$	the set of natural numbers with zero and $\mathbb{N}^+ = \mathbb{N}\backslash\{0\}$
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}^{n\times m}$	the set of real valued matrices with n rows and m columns
$\mathbb{R}[s]$	the polynomial ring over $\mathbb{R}$ with indeterminate s
$Gl\,(n,\mathbb{R})$	the group of nonsigular matrices of $\mathbb{R}^{n\times n}$
$\operatorname{rank} A$	the rank of a linear map A

$\operatorname{rank}_{\mathbb{R}[s]}(sE-H)$	the rank of a polynomial matrix $sE-H$ over $\mathbb{R}[s]$
$\ker A$	the kernal of a linear map A
$\dim \mathscr{A}$	the dimension of a linear space $\mathscr{A}$
$\operatorname{Im} A$	the image of a linear map A
$\mathscr{A}/\mathscr{B}$	the quotient of a vector space $\mathscr{A}$ by a subspace $\mathscr{B} \subseteq \mathscr{A}$
I_n	the identity matrix of size $n \times n$ for $n \in \mathbb{N}^+$
$0_{n\times m}$	the zero matrix of size $n \times m$ for $n, m \in \mathbb{N}^+$
A^T	the transpose of a matrix A
A^{-1}	the inverse of a matrix A
$A\mathscr{B}$	$\{Ax \mid x \in \mathscr{B}\}$, the image of $\mathscr{B}$ under a linear map A
$A^{-1}\mathscr{B}$	$\{x \mid Ax \in \mathscr{B}\}$, the preimage of $\mathscr{B}$ under a linear map A
$A^{-T}\mathscr{B}$	$(A^T)^{-1}\mathscr{B}$
$\mathscr{A}^\perp$	$\{x \mid \forall a \in \mathscr{A} : x^T a = 0\}$, the orthorgonal complement of $\mathscr{A}$ in $\mathbb{R}^n$

Consider a DAE $\Delta_{l,n} = (E, H)$, given by (2.1), denoted shortly by Δ, and the corresponding matrix pencil $sE - H$. A *solution* , or *trajectory*, $x(t)$ of Δ is any $\mathcal{C}^1$-differentiable map $x : \mathbb{R} \to \mathbb{R}^n$ satisfying $E\dot{x}(t) = Hx(t)$. A trajectory starting from a point $x(0) = x^0$ is denoted by $x(t, x^0)$.

Definition 2.2.1. Two DAEs $\Delta_{l,n} = (E, H)$ and $\tilde{\Delta}_{l,n} = (\tilde{E}, \tilde{H})$ are called externally equivalent, shortly ex-equivalent, if there exist $Q \in Gl(l, \mathbb{R})$ and $P \in Gl(n, \mathbb{R})$ such that

$$\tilde{E} = QEP^{-1} \text{ and } \tilde{H} = QHP^{-1}.$$

We denote ex-equivalence of two DAEs as $\Delta \overset{ex}{\sim} \tilde{\Delta}$, and ex-equivalence of the two corresponding matrix pencils as $sE - H \overset{ex}{\sim} s\tilde{E} - \tilde{H}$.

If the "generalized" states of Δ and $\tilde{\Delta}$ are x and $\tilde{x}$, respectively, then $\tilde{x} = Px$ is, clearly, just a coordinate transformation. The following remark points out the relation of the ex-equivalence and solutions of DAEs.

Remark 2.2.2. Ex-equivalence preserves trajectories, more precisely, if $\Delta \overset{ex}{\sim} \tilde{\Delta}$ via (Q, P), then any trajectory $x(t)$ of Δ satisfying $x(0) = x^0$, is mapped via P into a trajectory $\tilde{x}(t)$ of $\tilde{\Delta}$ passing through $\tilde{x}^0 = Px^0$. Moreover, if $x(t)$ is a trajectory of Δ, then $E\dot{x}(t) - Hx(t) = 0$ and, obviously $Q(E\dot{x}(t) - Hx(t)) = 0$ implying that $x(t)$ is also a trajectory of $QE\dot{x} = QHx$. The converse, however, is not true: even if two DAEs have the same trajectories, they are not necessarily ex-equivalent, since the trajectories of DAEs are contained in a subspace $\mathscr{M}^* \subseteq \mathbb{R}^n$ (see Definition 2.6.1 of Section 2.6).

Definition 2.2.3. (Morse equivalence and Morse transformation) Two linear control systems $\Lambda_{q,m,p} = (A, B, C, D)$ and $\tilde{\Lambda}_{q,m,p} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are called Morse equivalent, denoted by $\Lambda \overset{M}{\sim} \tilde{\Lambda}$, if there exist $T_s \in Gl(q, \mathbb{R})$, $T_i \in Gl(m, \mathbb{R})$, $T_o \in Gl(p, \mathbb{R})$, $F \in \mathbb{R}^{m\times q}$, $K \in \mathbb{R}^{q\times p}$ such that

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} T_s & T_s K \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix}. \tag{2.4}$$

Any 5-tuple $M_{tran} = (T_s, T_i, T_o, F, K)$, is called a Morse transformation.

Remark 2.2.4. (i) Apparently, in the above definition of a Morse transformation, T_s, T_i, T_o are coordinate transformations in the, respectively, state space $\mathscr{X}$, input space $\mathscr{U}$, and output space $\mathscr{Y}$, and F defines a state feedback and K defines an output injection. Moreover, if we consider two control systems without outputs, denoted by $\Lambda_{q,m} = (A, B)$ and $\tilde{\Lambda}_{q,m} = (\tilde{A}, \tilde{B})$, then the Morse equivalence reduces to the feedback equivalence, i.e., the corresponding system matrices satisfy $\tilde{A} = T_s(A + BF)T_s^{-1}$ and $\tilde{B} = T_sBT_i^{-1}$.

(ii) The feedback transformation $A \mapsto A + BF$ preserves all trajectories (although changes their parametrization with respect to controls). On the other hand, the output injection $A \mapsto A + KC$, $B \mapsto B + KD$ preserves only those trajectories $x(t)$ that satisfy $y(t) = Cx(t) + Du(t) = 0$. Finally, $A \mapsto T_sAT_s^{-1}$ maps trajectories into trajectories while $B \mapsto BT_i^{-1}$ re-parametrizes controls and $C \mapsto T_oC$ and $D \mapsto T_oD$ re-parametrize outputs.

2.3 Implicitation of linear control systems and explicitation of linear DAEs

It is easy to see that, if for a linear control system Λ, given by (2.2), we require the output $y = Cz + Du$ to be identically zero, then Λ can be seen as a DAE. We call such an output zeroing procedure the *implicitation* of a control system, which can be formalized as follows.

Definition 2.3.1. For a linear control system $\Lambda_{q,m,p} = (A, B, C, D)$ on $\mathscr{X} = \mathbb{R}^q$ with inputs in $\mathscr{U} = \mathbb{R}^m$ and outputs in $\mathscr{Y} = \mathbb{R}^p$, by setting the output y of Λ to be zero, that is

$$Impl(\Lambda) : \begin{cases} \dot{z} = Az + Bu \\ 0 = Cz + Du, \end{cases}$$

we define the following DAE with "generalized" states in $\mathbb{R}^{q+m}$:

$$\Delta^{Impl} : \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}. \tag{2.5}$$

We call the procedure of output zeroing above the implicitation procedure, and the DAE given by (2.5) will be called the implicitation of Λ and denoted by $\Delta^{Impl}_{q+p,q+m} = Impl(\Lambda)$ or, shortly, $\Delta^{Impl} = Impl(\Lambda)$.

The converse procedure, of associating a control systems to a given DAE, is less straightforward, since the variables are expressed implicitly in DAEs. In order to understand the different roles of the variables in a DAE, take, for example, the nilpotent pencil $N_\sigma(s)$ of the **KCF** of DAEs (see Appendix 2.9), denote the corresponding variables by $x_1, ..., x_\sigma$ and then the DAE is

$$\begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{\sigma-1} \\ \dot{x}_\sigma \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{\sigma-1} \\ x_\sigma \end{bmatrix}.$$

It is easy to see that the last equation $x_\sigma = 0$ is an algebraic constraint which can be seen as the zero output of a control system. The variable x_1 is different from the others because it is free to be given any value and thus it performs like an input. The variables $x_2, ..., x_{\sigma-1}$ are constrained by a differential chain forming an ODE, so they can be seen as states of a control system. Notice that in this case, replacing $\dot{x}_i = x_{i-1}$ by $\dot{x}_i = x_{i-1} + k_i x_\sigma$, for $2 \le i \le \sigma$ and for any $k_i \in \mathbb{R}$ does not change the system because $x_\sigma = 0$, which means that if we want to associate to our DAE a control system, the association is not unique. Below we show a way to attach a class of control systems to a given DAE.

- Consider a DAE $\Delta_{l,n} = (E, H)$, given by (2.1). Denote $\operatorname{rank} E = q$, define $p = l - q$ and $m = n - q$. Choose a map

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in Gl(n, \mathbb{R}),$$

where $P_1 \in \mathbb{R}^{q\times n}$, $P_2 \in \mathbb{R}^{m\times n}$ such that $\ker P_1 = \ker E$.

- Define coordinates transformation

$$\begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} P_1 x \\ P_2 x \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} x = Px.$$

Then from $\ker P_1 = \ker E$, we have $EP^{-1} = \begin{bmatrix} E_0 & 0 \end{bmatrix}$, where $E_0 \in \mathbb{R}^{l\times q}$. Moreover, since P is invertible, it follows that $\operatorname{rank} E_0 = \operatorname{rank} E = q$. Thus via P, Δ is ex-equivalent to

$$\begin{bmatrix} E_0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = H_0 \begin{bmatrix} z \\ u \end{bmatrix},$$

where $H_0 = HP^{-1}$. The variables z are states (dynamical variables, their derivatives $\dot{z}$ are present) and u are controls (enter statically into the system).

- Since $\operatorname{rank} E_0 = q$, there exists $Q_0 \in Gl(l, \mathbb{R})$ such that $Q_0 E_0 = \begin{bmatrix} E_0^1 \\ 0 \end{bmatrix}$, where $E_0^1 \in Gl(q, \mathbb{R})$. Thus via (Q_0, P), Δ is ex-equivalent to

$$\begin{bmatrix} E_0^1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix},$$

where $Q_0 H_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}$, $A_0 \in \mathbb{R}^{q\times q}, B_0 \in \mathbb{R}^{q\times m}, C_0 \in \mathbb{R}^{p\times q}, D_0 \in \mathbb{R}^{p\times m}$.

- Finally, via $Q_1 = \begin{bmatrix} (E_0^1)^{-1} & 0 \\ 0 & I_p \end{bmatrix}$, we bring the above DAE into

$$\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}, \tag{2.6}$$

where $A = (E_0^1)^{-1} A_0$, $B = (E_0^1)^{-1} B_0$, $C = C_0$, $D = D_0$.

- Therefore, the DAE Δ is ex-equivalent (via P and $Q = Q_1Q_0$) to (2.6) and the latter is the control system
$$\Lambda : \begin{cases} \dot{z} = Az + Bu \\ y = Cz + Du, \end{cases}$$
together with the constraint $y = 0$, that is, $\Delta \overset{ex}{\sim} \Delta^{Impl} = Impl(\Lambda)$.

Let us give a few comments on the above construction:

- The map $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ defines state variables $z = P_1x$ as coordinates on the state space $\mathscr{X} = \mathbb{R}^n/\ker E$ isomorphic to $\mathbb{R}^q$ and control variables $u = P_2x$ as coordinates on $\mathscr{U} \cong kerE \cong \mathbb{R}^m$. The output variables y are coordinates on $\mathscr{Y} \cong \mathbb{R}^l/\mathrm{Im}\, E \cong \mathbb{R}^p$ and define the output map via $y = Cz + Du$.

- Choose other coordinates (z', u') given by $z' = P_1'x$ and $u' = P_2'x$ such that $\ker P_1' = \ker E = \ker P_1$, then
$$\begin{cases} z' = T_sz \\ u' = F'z + T_iu, \end{cases} \tag{2.7}$$
where $T_s \in Gl(n, \mathbb{R})$ and $F' \in \mathbb{R}^{m\times n}$, $T_i \in Gl(m, \mathbb{R})$. Clearly, $z' = T_sz$ is another set of coordinates on the state space and $u' = F'z + T_iu$ is a *state feedback transformation.*

- The output y takes values in the quotient space $\mathbb{R}^l/\mathrm{Im}\, E$. Since $y = Cz + Du = 0$, we can add y to the dynamics without changing solutions of the system on the subspace $\{y = 0\}$. Together with a state transformation $z' = T_sz$ and an output transformation $y' = T_oy$, it results in a triangular transformation (output injection) of the system
$$\begin{bmatrix} \dot{z}' \\ y' \end{bmatrix} = \begin{bmatrix} T_s & K' \\ 0 & T_o \end{bmatrix} \begin{bmatrix} \dot{z} \\ y \end{bmatrix} = \begin{bmatrix} T_s & K' \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} \tag{2.8}$$
where $K' \in \mathbb{R}^{n\times p}$, $T_o \in Gl(p, \mathbb{R})$.

In view of the above analysis, the non-uniqueness of the construction leads to a control system defined up to a coordinates change, a feedback transformation and an output injection, which is actually, a class of control systems.

Definition 2.3.2. Given a DAE $\Delta_{l,n} = (E, H)$, there always exist $Q \in Gl(l, \mathbb{R})$ and $P \in Gl(n, \mathbb{R})$ such that
$$QEP^{-1} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \text{ and } QHP^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{2.9}$$
The control system Λ, given by $\Lambda_{q,m,p} = (A, B, C, D)$, is called the (Q, P)-explicitation of Δ. The class of all (Q, P)-explicitations, corresponding to all $Q \in Gl(l, \mathbb{R})$ and $P \in Gl(n, \mathbb{R})$, will be called the explicitation class of Δ and denoted by $Expl(\Xi)$. If a particular control system Λ belongs to the explicitation class $Expl(\Delta)$ of Δ, we will write $\Lambda \in Expl(\Delta)$.

Remark 2.3.3. The implicitation of a given control system Λ is a unique DAE Δ^{Impl}, given by (2.5). The explicitation $Expl(\Delta)$ of a given DAE Δ is, however, a control system defined up to a coordinates change, a feedback transformation, and an output injection, that is, a class of control systems.

Theorem 2.3.4. *(i) Consider a DAE $\Delta = (E, H)$ and a control system $\Lambda = (A, B, C, D)$. Then $\Lambda \in Expl(\Delta)$ if and only if $\Delta \overset{ex}{\sim} \Delta^{Impl}$, where $\Delta^{Impl} = Impl(\Lambda)$. More specifically, Λ is the (Q, P)-explicitation of Δ if and only if $\Delta \overset{ex}{\sim} \Delta^{Impl}$ via (Q, P).*

(ii) Given two DAEs $\Delta = (E, H)$ and $\tilde{\Delta} = (\tilde{E}, \tilde{H})$, choose two control systems $\Lambda \in Expl(\Delta)$ and $\tilde{\Lambda} \in Expl(\tilde{\Delta})$. Then $\Delta \overset{ex}{\sim} \tilde{\Delta}$ if and only if $\Lambda \overset{M}{\sim} \tilde{\Lambda}$.

(iii) Consider two control systems $\Lambda = (A, B, C, D)$ and $\tilde{\Lambda} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. Then $\Lambda \overset{M}{\sim} \tilde{\Lambda}$ if and only if $\Delta^{Impl} \overset{ex}{\sim} \tilde{\Delta}^{Impl}$, where $\Delta^{Impl} = Impl(\Lambda)$ and $\tilde{\Delta}^{Impl} = Impl(\tilde{\Lambda})$.

The proof is given in Section 2.7.

Remark 2.3.5. Theorem 2.3.4 describes relations of DAEs and control systems, which we illustrate in Figure 2.1. We conclude that Morse equivalent control systems (and only such) give, via *implicitation,* ex-equivalent DAEs. Furthermore, *explicitation* is a universal procedure of producing control systems from a DAE and ex-equivalent DAEs produce Morse equivalent control systems.

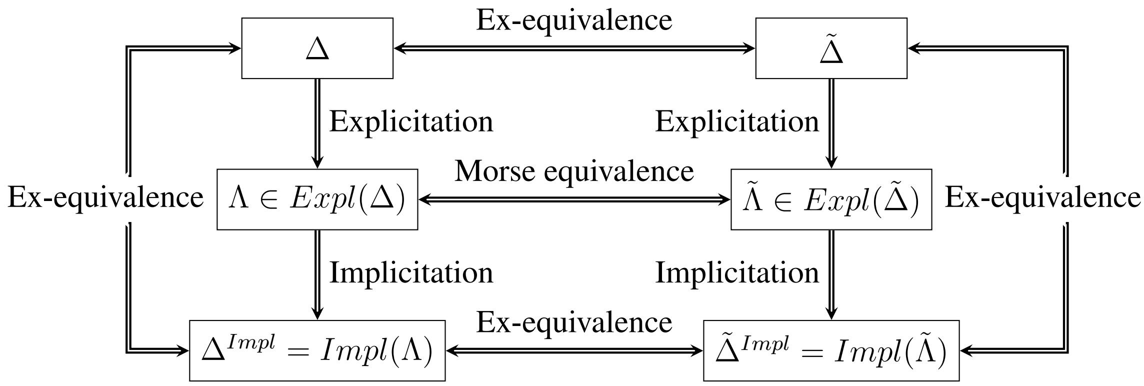


Figure 2.1 – Explicitation of DAEs and implicitation of control systems

2.4 Geometric connections between DAEs and control systems

The Wong sequences [191] of a DAE are defined as follows.

Definition 2.4.1. For a DAE $\Delta_{l,n} = (E, H)$, its Wong sequences are defined by

$$\mathscr{V}_0 = \mathbb{R}^n, \quad \mathscr{V}_{i+1} = H^{-1}E\mathscr{V}_i, \quad i \in \mathbb{N}, \tag{2.10}$$

$$\mathscr{W}_0 = \{0\}, \quad \mathscr{W}_{i+1} = E^{-1}H\mathscr{W}_i, \quad i \in \mathbb{N}. \tag{2.11}$$

Remark 2.4.2. The Wong sequences $\mathcal{V}_i$ and $\mathcal{W}_i$ satisfy

$$\begin{aligned} &\mathscr{V}_0 \supsetneq \mathscr{V}_1 \supsetneq \cdots \supsetneq \mathscr{V}_{k^*} = \mathscr{V}_{k^*+j} = \mathscr{V}^* = H^{-1}E\mathscr{V}^* \supseteq \ker H,\ j \in \mathbb{N}, \\ &\mathscr{W}_0 \subseteq \ker E = \mathscr{W}_1 \subsetneq \cdots \subsetneq \mathscr{W}_{l^*} = \mathscr{W}_{l^*+j} = \mathscr{W}^* = E^{-1}H\mathscr{W}^*,\ j \in \mathbb{N}. \end{aligned} \tag{2.12}$$

We now propose a different definition of the limits of the Wong sequences and review the notions of invariant subspaces in linear control theory.

Definition 2.4.3. For a DAE $\Delta_{l,n} = (E, H)$, a subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is called (H^{-1}, E)-invariant if $\mathscr{V}$ satisfies $\mathscr{V} = H^{-1}E\mathscr{V}$; a subspace $\mathscr{W} \subseteq \mathbb{R}^n$ is called (E^{-1}, H)-invariant if $\mathscr{W}$ satisfies $\mathscr{W} = E^{-1}H\mathscr{W}$. Denote by $\mathscr{V}^*$ the largest (H^{-1}, E)-invariant subspace of $\mathbb{R}^n$ and by $\mathscr{W}^*$ the smallest (E^{-1}, H)-invariant subspace of $\mathbb{R}^n$.

Proposition 2.4.4. *(i) For a DAE $\Delta_{l,n} = (E, H)$, the largest (H^{-1}, E)-invariant subspace $\mathscr{V}^*$ and the smallest (E^{-1}, H)-invariant subspace $\mathscr{W}^*$ exist and are given, respectively, by*

$$\mathscr{V}^* = \mathscr{V}_{k^*} \quad \text{and} \quad \mathscr{W}^* = \mathscr{W}_{l^*},$$

where k^* is the smallest integer such that $\mathscr{V}_{k^*} = \mathscr{V}_{k^*+1}$ and l^* is the smallest interger such that $\mathscr{W}_{l^*} = \mathscr{W}_{l^*+1}$;

(ii) $\mathscr{V}^*$ is also the largest subspace such that $H\mathscr{V}^* \subseteq E\mathscr{V}^*$, however, $\mathscr{W}^*$ is not necessarily the smallest subspace such that $E\mathscr{W}^* \subseteq H\mathscr{W}^*$.

The proof is given in Section 2.7. For invariant subspaces of control systems, we consider two cases depending on whether the control system is strictly proper (D is zero or not). We will use the bold-notation for the strictly proper case $D = 0$, since throughout it applies to prolongation (2.3), which we denote by bold symbols.

Definition 2.4.5. For a control system $\mathbf{\Lambda}_{n,m,p} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$, a subspace $\boldsymbol{\mathcal{V}} \subseteq \mathbb{R}^n$ is called an $(\mathbf{A}, \mathbf{B})$-controlled invariant subspace if $\boldsymbol{\mathcal{V}}$ satisfies

$$\mathbf{A}\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{V}} + \operatorname{Im}\mathbf{B}$$

and a subspace $\boldsymbol{\mathcal{W}} \subseteq \mathbb{R}^n$ is called a $(\mathbf{C}, \mathbf{A})$-conditioned invariant subspace if $\boldsymbol{\mathcal{W}}$ satisfies

$$\mathbf{A}(\boldsymbol{\mathcal{W}} \cap \ker \mathbf{C}) \subseteq \boldsymbol{\mathcal{W}}.$$

Denote by $\boldsymbol{\mathcal{V}}^*$ the largest $(\mathbf{A}, \mathbf{B})$-controlled invariant subspace contained in $\ker \mathbf{C}$ and by $\boldsymbol{\mathcal{W}}^*$ the smallest $(\mathbf{C}, \mathbf{A})$-conditioned invariant subspace containing $\operatorname{Im}\mathbf{B}$.

The following fundamental lemma shows that $\boldsymbol{\mathcal{V}}^*$, $\boldsymbol{\mathcal{W}}^*$ exist and they can be calculated via the sequences of subspaces $\boldsymbol{\mathcal{V}}_i$, $\boldsymbol{\mathcal{W}}_i$ given below.

Lemma 2.4.6. ([193],[9]) *Initialize $\boldsymbol{\mathcal{V}}_0 = \mathbb{R}^n$ and, for $i \in \mathbb{N}$, define inductively*

$$\boldsymbol{\mathcal{V}}_{i+1} = \ker \mathbf{C} \cap \mathbf{A}^{-1}(\boldsymbol{\mathcal{V}}_i + \operatorname{Im}\mathbf{B}). \tag{2.13}$$

Initialize $\boldsymbol{\mathcal{W}}_0 = 0$ *and, for* $i \in \mathbb{N}$, *define inductively*

$$\boldsymbol{\mathcal{W}}_{i+1} = \mathbf{A}(\boldsymbol{\mathcal{W}}_i \cap \ker \mathbf{C}) + \operatorname{Im} \mathbf{B}. \tag{2.14}$$

Then there exist $k^* \leq n$ *and* $l^* \leq n$ *such that*

$$\boldsymbol{\mathcal{V}}_0 \supseteq \ker \mathbf{C} = \boldsymbol{\mathcal{V}}_1 \supsetneq \cdots \supsetneq \boldsymbol{\mathcal{V}}_{k^*} = \boldsymbol{\mathcal{V}}_{k^*+j} = \boldsymbol{\mathcal{V}}^* = \ker \mathbf{C} \cap \mathbf{A}^{-1}(\boldsymbol{\mathcal{V}}^* + \operatorname{Im} \mathbf{B}), \ \ j \in \mathbb{N},$$
$$\boldsymbol{\mathcal{W}}_0 \subseteq \operatorname{Im} \mathbf{B} = \boldsymbol{\mathcal{W}}_1 \subsetneq \cdots \subsetneq \boldsymbol{\mathcal{W}}_{l^*} = \boldsymbol{\mathcal{W}}_{l^*+j} = \boldsymbol{\mathcal{W}}^* = \mathbf{A}(\boldsymbol{\mathcal{W}}^* \cap \ker \mathbf{C}) + \operatorname{Im} \mathbf{B}, \ \ j \in \mathbb{N}.$$

It is well-known (see e.g. [194],[193],[9]) that $\boldsymbol{\mathcal{V}}$ is an $(\mathbf{A}, \mathbf{B})$-controlled invariant subspace if and only if there exists $\mathbf{F} \in \mathbb{R}^{m\times n}$ such that $(\mathbf{A} + \mathbf{BF})\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{V}}$ and $\boldsymbol{\mathcal{W}}$ is a $(\mathbf{C}, \mathbf{A})$-conditioned invariant subspace if and only if there exists $\mathbf{K} \in \mathbb{R}^{n\times p}$ such that $(\mathbf{A} + \mathbf{KC})\boldsymbol{\mathcal{W}} \subseteq \boldsymbol{\mathcal{W}}$. For a control system which is not strictly proper (D is not zero), following Definitions 1–4 of [145], we use a generalization of that characterization of invariant subspaces.

Definition 2.4.7. For $\Lambda_{q,m,p} = (A, B, C, D)$, a subspace $\mathcal{V} \subseteq \mathbb{R}^q$ is called a null-output (A, B)-controlled invariant subspace if there exists $F \in \mathbb{R}^{m\times q}$ such that

$$(A + BF)\mathcal{V} \subseteq \mathcal{V} \quad \text{and} \quad (C + DF)\mathcal{V} = 0,$$

and for any such $\mathcal{V}$, the subspace $\mathcal{U} \subseteq \mathbb{R}^m$ given by

$$\mathcal{U} = (B^{-1}\mathcal{V}) \cap \ker D,$$

is called a null-output (A, B)-controlled invariant input subspace. Denote by $\mathcal{V}^*$ (resp. $\mathcal{U}^*$) the largest null-output (A, B) controlled invariant subspace (resp. input subspace).

A subspace $\mathcal{W} \subseteq \mathbb{R}^q$ is called an unknown-input (C, A)-conditioned invariant subspace if there exists $K \in \mathbb{R}^{q\times p}$ such that

$$(A + KC)\mathcal{W} + (B + KD)\mathscr{U} = \mathcal{W},$$

and for any such $\mathcal{W}$, the subspace $\mathcal{Y} \subseteq \mathbb{R}^p$ given by

$$\mathcal{Y} = C\mathcal{W} + D\mathscr{U},$$

is called an unknown-input (C, A)-conditioned invariant output subspace. Denote by $\mathcal{W}^*$ (resp. $\mathcal{Y}^*$) the smallest unknown-input (C, A)-conditioned invariant subspace (resp. output subspace).

The following lemma [144] shows that $\mathcal{V}^*$, $\mathcal{U}^*$, $\mathcal{W}^*$, $\mathcal{Y}^*$ exist and provides a calculable algorithm to calculate them.

Lemma 2.4.8. *Initialize* $\mathcal{V}_0 = \mathbb{R}^q$, *and for* $i \in \mathbb{N}$, *define inductively*

$$\mathcal{V}_{i+1} = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{V}_i + \operatorname{Im} \begin{bmatrix} B \\ D \end{bmatrix} \right) \tag{2.15}$$

and $\mathcal{U}_i \subseteq \mathscr{U}$ *for* $i \in \mathbb{N}$ *are given by*

$$\mathcal{U}_i = \begin{bmatrix} B \\ D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_i \\ 0 \end{bmatrix}. \tag{2.16}$$

Then $\mathcal{V}^* = \mathcal{V}_q$ *and* $\mathcal{U}^* = \mathcal{U}_q$ *.*

Initialize $\mathcal{W}_0 = \{0\}$*, and for* $i \in \mathbb{N}$*, define inductively*

$$\mathcal{W}_{i+1} = \begin{bmatrix} A & B \end{bmatrix} \left(\begin{bmatrix} \mathcal{W}_i \\ \mathscr{U} \end{bmatrix} \cap \ker \begin{bmatrix} C & D \end{bmatrix} \right) \tag{2.17}$$

and $\mathcal{Y}_i \subseteq \mathscr{Y}$ *for* $i \in \mathbb{N}$ *are given by*

$$\mathcal{Y}_i = \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} \mathcal{W}_i \\ \mathscr{U} \end{bmatrix}. \tag{2.18}$$

Then $\mathcal{W}^* = \mathcal{W}_q$ *and* $\mathcal{Y}^* = \mathcal{Y}_q$*.*

Remark 2.4.9. (i) Lemma 2.4.8 generalizes the results of Lemma 2.4.6 and, if $D = 0$, Lemma 2.4.8 reduces to Lemma 2.4.6;

(ii) Even if Λ is not strictly proper (if $D \neq 0$), the prolonged system $\mathbf{\Lambda}$ always is; throughout we will use $\mathcal{V}^*$, $\mathcal{U}^*$, $\mathcal{W}^*$ and $\mathcal{Y}^*$ for Λ, and $\boldsymbol{\mathcal{V}}^*$ and $\boldsymbol{\mathcal{W}}^*$ for $\mathbf{\Lambda}$.

Throughout this chapter, for ease of notation, we will write $\mathscr{V}_i(\Delta)$ to indicate that $\mathscr{V}_i$ is calculated for Δ, similarly for all the other subspaces defined in this section. Now we give the main results of this section.

Proposition 2.4.10. (Geometric subspaces relations) *Given a DAE* $\Delta_{l,n} = (E, H)$*, a* (Q, P)*-explicitation* $\Lambda = (A, B, C, D) \in Expl(\Delta)$*, and the prolongation* $\mathbf{\Lambda} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ *of* Λ*, consider the limits of the Wong sequences* $\mathscr{V}^*$ *and* $\mathscr{W}^*$ *of* Δ *and of* $\Delta^{Impl} = Impl(\Lambda)$*, given by Definition 2.4.3, the invariant subspaces* $\mathcal{V}^*$ *and* $\mathcal{W}^*$ *of* Λ*, given by Definition 2.4.7, and the invariant subspaces* $\boldsymbol{\mathcal{V}}^*$ *and* $\boldsymbol{\mathcal{W}}^*$ *of* $\mathbf{\Lambda}$*, given by Definition 2.4.5. Then the following hold*

(i) $P\mathscr{V}^*(\Delta) = \mathscr{V}^*(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}^*(\mathbf{\Lambda}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}^*(\Lambda) \\ 0 \end{bmatrix}$,

(ii) $P\mathscr{W}^*(\Delta) = \mathscr{W}^*(\Delta^{Impl}) = \boldsymbol{\mathcal{W}}^*(\mathbf{\Lambda}) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}^*(\Lambda) \\ 0 \end{bmatrix}$.

The proof is given in Section 2.7.

Remark 2.4.11. (i) The limits $\mathscr{V}^*$ and $\mathscr{W}^*$ of the Wong sequences coincide for Δ and $\tilde{\Delta}$ that are ex-equivalent via (P, Q), where $P = I_n$ and Q is arbitrary, and do not depend on Q. On the other hand, the system Λ, being a (Q, P)-explicitation of Δ, depends on both P and Q (and so does its prolongation $\mathbf{\Lambda}$), but the invariant subspaces $\mathcal{V}^*(\Lambda)$ and $\mathcal{W}^*(\Lambda)$ depend on P only.

(ii) Some particular relations between the Wong sequences of DAEs and the invariant subspaces of control systems is given in Theorem 5 of [56], which can be seen as a corollary of Proposition 2.4.10.

Now we will study various dualities of geometric subspaces by analyzing the dual system. The duality of the subspaces $\boldsymbol{\mathcal{V}}^*$ and $\boldsymbol{\mathcal{W}}^*$ is well-known and studied in [194],[146], [9]. Similarly, properties of the subspaces $\mathcal{V}^*, \mathcal{W}^*, \mathcal{U}^*, \mathcal{Y}^*$ for the dual system of a control system are analyzed in [144] and [145]. In [20], it is proved that the Wong sequences of the transposed matrix pencils have relations with the original matrix pencils. In the following, we will show that all these results can be connected by the *explicitization* of DAEs. Together with Δ we consider its dual $\Delta^d_{n,l} = (E^T, H^T)$ of the form:

$$E^T \dot{x}^d = H^T x^d,$$

where $x^d \in \mathbb{R}^l$ is the "generalized" state of the dual system.

Proposition 2.4.12. *Consider a DAE Δ and its dual Δ^d. Then $\Lambda = (A, B, C, D) \in Expl(\Delta)$ if and only if $\Lambda^d = (A^T, C^T, B^T, D^T) \in Expl(\Delta^d)$.*

Proof. For any invertible matrices Q and P of appropriate sizes that yield (2.9), we have the following equivalence:

$$Q\,(sE - H)\,P^{-1} = \begin{bmatrix} sI_q - A & -C \\ -B & -D \end{bmatrix} \Leftrightarrow P^{-T}\left(sE^T - H^T\right)Q^T = \begin{bmatrix} sI_q - A^T & -C^T \\ -B^T & -D^T \end{bmatrix}.$$

Suppose $\Lambda \in Expl(\Delta)$, then by Theorem 2.3.4(i), there exist $Q \in Gl(l, \mathbb{R})$ and $P \in Gl(n, \mathbb{R})$, such that the left-hand side of the above equivalence holds. Then from the right-hand side we can see $\Lambda^d \in Expl(\Delta^d)$.

Conversely, suppose $\Lambda^d \in Expl(\Delta^d)$. Then there exist $P^{-T} \in Gl(n, \mathbb{R})$ and $Q^T \in Gl(l, \mathbb{R})$ such that right-hand side of the above equivalence holds, then from the left-hand side we can see $\Lambda \in Expl(\Delta)$. □

Proposition 2.4.13. (Subspaces of the dual system) *For $\Delta = (E, H)$ and its dual $\Delta^d = (E^T, H^T)$, consider the subspaces $\mathscr{V}^*$ and $\mathscr{W}^*$ of Definition 2.4.3. For two control systems $\Lambda = (A, B, C, D) \in Expl(\Delta)$ and the dual of Λ, denoted by $\Lambda^d = (A^T, C^T, B^T, D^T)$, consider the subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$ of Definition 2.4.7. Finally, for the prolongation of Λ, denoted by $\mathbf{\Lambda} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ and for the dual of $\mathbf{\Lambda}$, denoted by $\mathbf{\Lambda}^d = (\mathbf{A}^T, \mathbf{C}^T, \mathbf{B}^T)$, consider the subspaces $\boldsymbol{\mathcal{V}}^*$ and $\boldsymbol{\mathcal{W}}^*$ of Definition 2.4.5. Then the following hold:*

(i) $\mathscr{W}^*(\Delta^d) = (E\mathscr{V}^*(\Delta))^{\perp}$, $\mathscr{V}^*(\Delta^d) = (H\mathscr{W}^*(\Delta))^{\perp}$;

(ii) $\mathcal{W}^*(\Lambda^d) = (\mathcal{V}^*(\Lambda))^{\perp}$, $\mathcal{V}^*(\Lambda^d) = (\mathcal{W}^*(\Lambda))^{\perp}$;

(iii) $\boldsymbol{\mathcal{W}}^*(\mathbf{\Lambda}^d) = (\boldsymbol{\mathcal{V}}^*(\mathbf{\Lambda}))^{\perp}$, $\boldsymbol{\mathcal{V}}^*(\mathbf{\Lambda}^d) = (\boldsymbol{\mathcal{W}}^*(\mathbf{\Lambda}))^{\perp}$.

Moreover, assuming one of the items (i), (ii), or (iii) we can conclude the two remaining ones by the relations given in Proposition 2.4.10.

Note that item (i) is proved in [20] by showing that for $i \in \mathbb{N}$,

$$\mathscr{W}_{i+1}(\Delta^d) = (E\mathscr{V}_i(\Delta))^{\perp}, \quad \mathscr{V}_i(\Delta^d) = (H\mathscr{W}_i(\Delta))^{\perp}.$$

Item (iii) is proved in [146] by showing $\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}^d) = (\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}))^{\perp}$, $\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}^d) = (\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}))^{\perp}$. Item (ii) is proved in [145] by showing $\mathcal{W}_i(\Lambda^d) = (\mathcal{V}_i(\Lambda))^{\perp}$, $\mathcal{V}_i(\Lambda^d) = (\mathcal{W}_i(\Lambda))^{\perp}$ as well as observing a supplementary relation $\mathcal{U}_i(\Lambda^d) = (\mathcal{Y}_i(\Lambda))^{\perp}$, $\mathcal{Y}_i(\Lambda^d) = (\mathcal{U}_i(\Lambda))^{\perp}$. Our purpose is to propose a new proof in Section 2.7.4 to show that knowing one of the items (i), (ii) or (iii), we do not need to prove the two others but just to use the relations of Proposition 2.4.10 (between $\mathscr{V}^*$, $\boldsymbol{\mathcal{V}}^*$, $\mathcal{V}^*$ and $\mathscr{W}^*$, $\boldsymbol{\mathcal{W}}^*$, $\mathcal{W}^*$) to simply conclude them. In other words, Proposition 2.4.10 provides a dictionary allowing to go from one of (i), (ii) or (iii) to two remaining ones.

2.5 Relations of the Kronecker indices and the Morse indices

In this section, we discuss relations of the Kronecker indices and the Morse indices see Appendix 2.9. An early result discussing these two sets of indices goes back to [104], where it is observed that the controllability indices of the pair (A, B) and the Kronecker column indices of the matrix pencil $(sI - A, B)$ coincide, which can be seen as a special case of the result in this section. Also in [130], it is shown that the Morse indices of a triple (C, A, B) have direct relations with the Kronecker indices of the matrix pencil (called restricted matrix pencil, see [97]) $N(sI - A)K$, where the rows of N span the annihilator of $\operatorname{Im} B$ and the colunms of K span $\ker C$.

It is known (see Appendix 2.9) that any DAE can be transformed into its **KCF** which is completely determined by the Kronecker indices $\varepsilon_1, ..., \varepsilon_a$, $\rho_1, ..., \rho_b$, $\sigma_1, ..., \sigma_c$, $\eta_1, ..., \eta_d$, the numbers a, b, c, d of blocks and the $(\lambda_1, ..., \lambda_b)$-structure (by this we mean the eigenvalues, together with the dimensions of their eigenspaces). The Kronecker indices (except for ρ_i's and the corresponding eigenvalues λ_i's) can be computed using the Wong sequences as follows. For a DAE $\Delta = (E, H)$, consider the Wong sequences $\mathscr{V}_i$ and $\mathscr{W}_i$ of Definition 2.4.1, define $\mathscr{K}_i = \mathscr{W}_i \cap \mathscr{V}^*$ and $\hat{\mathscr{K}_i} = (E\mathscr{V}_{i-1})^{\perp} \cap (H\mathscr{W}^*)^{\perp}$ for $i \in \mathbb{N}^+$.

Lemma 2.5.1. [20],[21] *For **KCF** of Δ, we have*

(i) $a = \dim(\mathscr{K}_1)$, $d = \dim(\hat{\mathscr{K}}_1)$ *and*

$$\begin{cases} \varepsilon_j = 0, & for \qquad 1 \le j \le a - \omega_0, \\ \varepsilon_j = i, & for \quad a - \omega_{i-1} + 1 \le j \le a - \omega_i, \end{cases} \tag{2.19}$$

$$\begin{cases} \eta_j = 0, & for \qquad 1 \le j \le d - \hat{\omega}_0, \\ \eta_j = i, & for \quad d - \hat{\omega}_{i-1} + 1 \le j \le d - \hat{\omega}_i, \end{cases} \tag{2.20}$$

where $\omega_i = \dim(\mathscr{K}_{i+2}) - \dim(\mathscr{K}_{i+1})$ *and* $\hat{\omega}_i = \dim(\hat{\mathscr{K}}_{i+2}) - \dim(\hat{\mathscr{K}}_{i+1})$, $i \in \mathbb{N}$.

(ii) Define an integer ν *by*

$$\nu = \min\{i \in \mathbb{N} \,|\, \mathscr{V}^* + \mathscr{W}_i = \mathscr{V}^* + \mathscr{W}_{i+1}\}; \tag{2.21}$$

Then either $\nu = 0$*, implying that the nilpotent part* $N(s)$ *is absent, or* $\nu > 0$*, in which case*

$c = \pi_0$ *and*

$$\sigma_j = i, \quad for \quad c - \pi_{i-1} + 1 \leq j \leq c - \pi_i, \quad i = 1, 2, ..., \nu, \tag{2.22}$$

where $\pi_i = \dim(\mathscr{W}_{i+1} + \mathscr{V}^*) - \dim(\mathscr{W}_i + \mathscr{V}^*)$ *for* $i = 0, 1, 2, ..., \nu$ *(in the case of* $\pi_{i-1} = \pi_i$*, the respective index range is empty).*

Any control system $\Lambda = (A, B, C, D)$ can be transformed via a Morse transformation into its Morse canonical form **MCF**, which is determined by the Morse indices $\varepsilon'_1, ..., \varepsilon'_{a'}$, $\rho'_1, ..., \rho'_{b'}$, $\sigma'_1, ..., \sigma'_{c'}$, $\eta'_1, ..., \eta'_{d'}$, the eigenvalues $\lambda_1, .., \lambda_{b'}$ and the numbers $a', b', c', d' \in \mathbb{N}$ of blocks. The following results can be deduced from the results on the Morse indices in [146],[145]. For $\Lambda = (A, B, C, D)$, consider the subspaces $\mathcal{V}_i$, $\mathcal{W}_i$, $\mathcal{U}_i$, $\mathcal{Y}_i$ as in Lemma 2.4.8, define $\mathcal{R}_i = \mathcal{W}_i \cap \mathcal{V}^*$ and $\hat{\mathcal{R}}_i = (\mathcal{V}_i)^\perp \cap (\mathcal{W}^*)^\perp$ for $i \in \mathbb{N}$.

Lemma 2.5.2. *For* ***MCF*** *of* Λ*, we have*

(i) $a' = \dim(\mathcal{U}^*)$*,* $d' = \dim(\mathcal{Y}^*)$ *and*

$$\begin{cases} \varepsilon'_j = 0 & for \qquad 1 \leq j \leq a' - \omega'_0, \\ \varepsilon'_j = i & for \quad a' - \omega'_{i-1} + 1 \leq j \leq a' - \omega'_i, \end{cases} \tag{2.23}$$

$$\begin{cases} \eta'_j = 0 & for \qquad 1 \leq j \leq d' - \hat{\omega}'_0, \\ \eta'_j = i & for \quad d' - \hat{\omega}'_{i-1} + 1 \leq j \leq d' - \hat{\omega}', \end{cases} \tag{2.24}$$

where $\omega'_i = \dim(\mathcal{R}_{i+1}) - \dim(\mathcal{R}_i)$ *and* $\hat{\omega}'_i = \dim(\hat{\mathcal{R}}_{i+1}) - \dim(\hat{\mathcal{R}}_i)$*,* $i \in \mathbb{N}$*.*

(ii) Define an integer ν' *by*

$$\nu' = min\{i \in \mathbb{N} \,|\, \mathcal{V}^* + \mathcal{W}_i = \mathcal{V}^* + \mathcal{W}_{i+1}\};$$

Then $c' = \dim(\mathscr{U}) - \dim(\mathcal{U}^*)$*,* $\delta = c' - \pi'_0$ *and*

$$\begin{cases} \sigma'_j = 0 & for \quad 1 \leq j \leq \delta, \\ \sigma'_j = i & for \quad c' - \pi'_{i-1} + 1 \leq j \leq c' - \pi'_i, \quad i = 1, 2, ..., \nu', \end{cases} \tag{2.25}$$

where $\pi'_i = \dim(\mathcal{W}_{i+1} + \mathcal{V}^*) - \dim(\mathcal{W}_i + \mathcal{V}^*)$ *for* $i = 0, 1, 2, ..., \nu'$ *(in case of* $\pi'_{i-1} = \pi'_i$ *the respective index range is empty).*

Note that for $\Lambda = (A, B, C, D)$, the above index $\delta = \operatorname{rank} D$. Formal similarities between the statements of Lemma 2.5.1 and 2.5.2 suggest possible relations between the Kronecker and the Morse indices. In fact, we have the following result.

Proposition 2.5.3. (Indices relations) *For a DAE $\Delta_{l,n} = (E, H)$, consider its Kronecker indices*

$$(\varepsilon_1, ..., \varepsilon_a),\ (\rho_1, ..., \rho_b),\ (\sigma_1, ..., \sigma_c),\ (\eta_1, ..., \eta_d)\ with\ a, b, c, d \in \mathbb{N},$$

*of the **KCF**, and for a control system $\Lambda_{q,m,p} = (A, B, C, D) \in Expl(\Delta)$, consider its Morse indices*

$$(\varepsilon'_1, ..., \varepsilon'_{a'}),\ (\rho'_1, ..., \rho'_{b'}),\ (\sigma'_1, ..., \sigma'_{c'}),\ (\eta'_1, ..., \eta'_{d'})\ with\ a', b', c', d' \in \mathbb{N},$$

*of the **MCF**. Then the following holds:*

(i) $a = a'$, $\varepsilon_1 = \varepsilon'_1, \cdots, \varepsilon_a = \varepsilon'_{a'}$, and $d = d'$, $\eta_1 = \eta'_1, \cdots, \eta_d = \eta'_{d'}$;

*(ii) $N(s)$ of the **KCF** is present if and only if the subsystem MCF^3 of the **MCF** is present. Moreover, if they are present, then their indices satisfy*

$$c = c',\ \ \sigma_1 = \sigma'_1 + 1, \cdots, \sigma_c = \sigma'_{c'} + 1;$$

*(iii) The invariant factors of $J(s)$ in the **KCF** of Δ coincide with those of MCF^2 in the **MCF** of Λ. Furthermore, the corresponding indices satisfy*

$$b = b',\ \ \rho_1 = \rho'_1, ..., \rho_b = \rho'_{b'}.$$

The proof is given in Section 2.7. Notice that in item (ii) of Proposition 2.5.3, the indices σ_i and σ'_i do not coincide, the reason is that the nilpotent indices $\sigma_1, \ldots, \sigma_c$ of $N(s)$ can not be zero (the minimum nilpotent index is 1 and if σ_i is 1, then $N(s)$ contains the 1×1 matrix pencil $0s - 1$), but the controllability and observability indices $\sigma'_1, \ldots, \sigma'_{c'}$ of MCF^3 can be zero (if $\sigma'_i = 0$, then the output y^3 of MCF^3 contains the static relation $y^3_i = u^3_i$). It is easy to see from Proposition 2.5.3 that, given a DAE, there exists a perfect correspondence between the **KCF** of the DAE and the **MCF** of its explicitation systems. More specifically, the four parts of the **KCF** correspond to the four subsystems of the **MCF**: the bidiagonal pencil $L(s)$ to the controllable but unobservable part MCF^1, the Jordan pencil $J(s)$ to the uncontrollable and unobservable part MCF^2, the nilpotent pencil $N(s)$ to the prime part MCF^3 and the "pertranspose" pencil $L^p(s)$ to the observable but uncontrollable part MCF^4.

2.6 Internal equivalence and regularity of DAEs

An important difference between DAEs and ODEs is that DAEs are not always solvable and solutions of DAEs exist on a subspace of the "generalized" state space only due to the presence of algebraic constrains. In the following, we show that the existence and uniqueness of solutions of DAEs can be clearly explained using the *explicitation* procedure and the notion of internal equivalence (see Definition 2.6.8 below).

Definition 2.6.1. A linear subspace $\mathscr{M}$ of $\mathbb{R}^n$, is called an invariant subspace of $\Delta_{l,n} = (E, H)$ if for any $x^0 \in \mathscr{M}$, there exists a solution $x(t, x^0)$ of Δ such that $x(0, x^0) = x^0$ and $x(t, x^0) \in \mathscr{M}$ for all $t \in \mathbb{R}$. An invariant subspace $\mathscr{M}^*$ of $\Delta_{l,n} = (E, H)$ is called the maximal invariant subspace if for any other invariant subspace $\mathscr{M}$ of $\mathbb{R}^n$, we have $\mathscr{M} \subseteq \mathscr{M}^*$.

Remark 2.6.2. Note that due to the existence of free variables among the "generalized" states, solutions of Δ are not unique. Thus it is possible that one solution of Δ starting at $x^0 \in \mathscr{M}$ stays in $\mathscr{M}$ but other solutions starting at x^0 may escape from $\mathscr{M}$ (either immediately or in finite time).

It is clear that the sum $\mathscr{M}_1 + \mathscr{M}_2$ of two invariant subspaces of Δ is also invariant. Therefore, $\mathscr{M}^*$ exists and is, actually, the sum of all invariant subspaces. If $\mathscr{M}$ is an invariant subspace of $\Delta_{l,n}$, then solutions pass through any $x^0 \in \mathscr{M}$ and it is natural to restrict Δ to $\mathscr{M}$, in particular, to the largest invariant subspace $\mathscr{M}^*$. Moreover, we would like the restriction to be as simple as possible. We achieve the above goals by introducing, respectively, the notion of *restriction* and that of *reduction*. We will define the restriction of a DAE Δ to a linear subspace $\mathscr{R}$ (invariant or not) as follows.

Definition 2.6.3. (Restriction) Consider a linear DAE $\Delta_{l,n} = (E, H)$. Let $\mathscr{R}$ be a subspace of $\mathbb{R}^n$. The restriction of Δ to $\mathscr{R}$, called $\mathscr{R}$-restriction of Δ and denoted $\Delta|_{\mathscr{R}}$ is a linear DAE $\Delta|_{\mathscr{R}} = (E|_{\mathscr{R}}, H|_{\mathscr{R}})$, where $E|_{\mathscr{R}}$ and $H|_{\mathscr{R}}$ are, respectively, the restrictions of the linear maps E and H to the linear subspace $\mathscr{R}$.

Throughout, we consider general DAEs $\Delta_{l,n} = (E, H)$ with no assumptions on the ranks of E and H. In particular, if the map $[E\ H]$ is not of full row rank, then $\Delta_{l,n}$ contains redundant equations. But even if we assume that $[E\ H]$ is of full row rank, then this property, in general, is not any longer true for the restriction $[E|_{\mathscr{R}}\ H_{\mathscr{R}}]$, which may contain redundant equations. To get rid of redundant equations (in particular, of trivial algebraic equations $0 = 0$), we propose the notion of full row rank reduction.

Definition 2.6.4. (Reduction) For a DAE $\Delta_{l,n} = (E, H)$, assume $\operatorname{rank}[E\ H] = l^* \leq l$. Then there exists $Q \in Gl(l, \mathbb{R}^n)$ such that

$$Q\begin{bmatrix} E & H \end{bmatrix} = \begin{bmatrix} E^{red} & H^{red} \\ 0 & 0 \end{bmatrix},$$

where $\operatorname{rank}[E^{red}\ H^{red}] = l^*$ and the full row rank reduction, shortly reduction, of $\Delta_{l,n}$, denoted by Δ^{red}, is a DAE $\Delta^{red}_{l^*,n} = \Delta^{red} = (E^{red}, H^{red})$.

Remark 2.6.5. Clearly, the choice of Q is not unique and thus the reduction of Δ is not unique. Nevertheless, since Q preserves the solutions, each reduction Δ^{red} has the same solutions as the original DAE Δ.

For an invariant subspace $\mathscr{M}$, we consider the $\mathscr{M}$-restriction $\Delta|_{\mathscr{M}}$ of Δ, and then we construct a reduction of $\Delta|_{\mathscr{M}}$ and denote it by $\Delta|^{red}_{\mathscr{M}} = (E|^{red}_{\mathscr{M}}, H|^{red}_{\mathscr{M}})$. Notice that the order matters: to construct $\Delta|^{red}_{\mathscr{M}}$, we first restrict and then reduce while reducing first and then restricting will, in general, not give $\Delta|^{red}_{\mathscr{M}}$ but another DAE $\Delta^{red}|_{\mathscr{M}}$.

Proposition 2.6.6. *Consider a linear DAE $\Delta_{l,n} = (E, H)$. Let $\mathscr{M}$ be a subspace of $\mathbb{R}^n$. The following are equivalent*

(i) $\mathscr{M}$ is an invariant subspace of $\Delta_{l,n}$;

(ii) $H\mathscr{M} \subseteq E\mathscr{M}$;

(iii) For a (and thus any) reduction $\Delta|^{red}_{\mathscr{M}} = (E|^{red}_{\mathscr{M}}, H|^{red}_{\mathscr{M}})$ of $\Delta|_{\mathscr{M}}$, the map $E|^{red}_{\mathscr{M}}$ is of full row rank, i.e., $\operatorname{rank} E|^{red}_{\mathscr{M}} = \operatorname{rank}\,[E|^{red}_{\mathscr{M}}\ H|^{red}_{\mathscr{M}}]$.

Proof. (i)⇔(ii): Theorem 4 of [12], for $B = 0$, implies that $\mathscr{M}$ is an invariant subspace if and only if $H\mathscr{M} \subseteq E\mathscr{M}$.

(ii)⇔(iii): For $\Delta_{l,n} = (E, H)$, choose a full column rank matrix $P_1 \in \mathbb{R}^{n\times n_1}$ such that $\operatorname{Im} P_1 = E\mathscr{M}$, where $n_1 = \dim \mathscr{M}$. Find any $P_2 \in \mathbb{R}^{n\times n_2}$ such that the matrix $[P_1\ P_2]$ is invertible, where $n_2 = n - n_1$. Choose new coordinates $z = Px$, where $P = [P_1\ P_2]^{-1}$, then we have

$$\Delta : EP^{-1}P\dot{x} = HP^{-1}Px \Rightarrow [E_1\ E_2]\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = [H_1\ H_2]\begin{bmatrix}z_1\\ z_2\end{bmatrix},$$

where $E_1 = EP_1$, $E_2 = EP_2$, $H_1 = HP_1$, $H_2 = HP_2$, and $z = (z_1, z_2)$. Now by Definition 2.6.3, the $\mathscr{M}$-restriction of Δ is:

$$\Delta|_{\mathscr{M}} : E_1\dot{z}_1 = H_1 z_1.$$

Find $Q \in Gl(l, \mathbb{R})$ such that $QE_1 = \begin{bmatrix}\tilde{E}_1\\ 0\end{bmatrix}$, where $\tilde{E}_1$ is of full row rank, then denote $QH_1 = \begin{bmatrix}\tilde{H}_1\\ \bar{H}_1\end{bmatrix}$. By $H\mathscr{M} \subseteq E\mathscr{M}$, we can deduce that $\bar{H}_1 = 0$ (since $QH\mathscr{M} \subseteq QE\mathscr{M} \Rightarrow \operatorname{Im}\begin{bmatrix}\tilde{H}_1\\ \bar{H}_1\end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix}\tilde{E}_1\\ 0\end{bmatrix}$). Thus a reduction of $\Delta|_{\mathscr{M}}$, according to Definition 2.6.4, is $\Delta|^{red}_{\mathscr{M}} = (E|^{red}_{\mathscr{M}}, H|^{red}_{\mathscr{M}}) = (\tilde{E}_1, \tilde{H}_1)$. Clearly $E|^{red}_{\mathscr{M}}$ is of full row rank. □

Define $\Lambda|_{(\mathcal{V}^*,\mathcal{U}^*)}$ as the control system $\Lambda = (A, B, C, D)$ restricted to $\mathcal{V}^*$ (which is well-defined because $\mathcal{V}^*$ can be made invariant by a suitable feedback) and with controls u restricted to $\mathcal{U}^* = (B^{-1}\mathcal{V}^*) \cap \ker D$. The output $y = Cx + Du$ of Λ becomes $y = 0$ and by $\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)}$, we denote$\Lambda|_{(\mathcal{V}^*,\mathcal{U}^*)}$ without the trivial output $y = 0$.

Proposition 2.6.7. *For a DAE $\Delta_{l,n} = (E, H)$, consider its maximal invariant subspace $\mathscr{M}^*$ and its largest (E^{-1}, H)-invariant subspace $\mathscr{V}^*$. Then we have*

(i) $\mathscr{M}^*$=$\mathscr{V}^*$;

*(ii) Let $\Lambda \in Expl(\Delta)$ and $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{M}^*})$. Then $\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)}$ and Λ^* are explicit control systems without outputs i.e., the **MCF** of the two control systems has no MCF^3 and MCF^4 parts, and $\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)}$ is feedback equivalent to Λ^*.*

The proof is given in Section 2.7. Analogously to the ex-equivalence of DAEs, we define the internal equivalence of two DAEs as follows.

Definition 2.6.8. For two DAEs $\Delta_{l,n} = (E, H)$ and $\tilde{\Delta}_{\tilde{l},\tilde{n}} = (\tilde{E}, \tilde{H})$, let $\mathscr{M}^*$ and $\tilde{\mathscr{M}}^*$ be the maximal invariant subspace of Δ and $\tilde{\Delta}$, respectively. Then Δ and $\tilde{\Delta}$ are called internally equivalent, shortly in-equivalent, if $\Delta|^{red}_{\mathscr{M}^*}$ and $\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*}$ are ex-equivalent and we will denote the in-equivalence of two DAEs as $\Delta \overset{in}{\sim} \tilde{\Delta}$.

Remark 2.6.9. A similar definition to the above internal equivalence above is given in [18], called the behavioral equivalence, proposed via the behavioral approach of DAEs. The difference between the internal equivalence and the behavioral equivalence is that, in the definition of internal equivalence, two DAEs are not necessarily of the same dimension, we only require their reductions of $\mathscr{M}^*$-restrictions to be of the same dimension (since they are ex-equivalent), but for the behavioral equivalence, the two DAEs are required to have the same dimension.

Any $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{M}^*})$ is an explicit system without outputs (see Proposition 2.6.7(ii)) and denote the dimensions of its state space and input space by n^* and m^*, respectively, and its corresponding matrices by A^*, B^* and thus $\Lambda^*_{n^*,m^*} = (A^*, B^*)$.

Theorem 2.6.10. *Let $\mathscr{M}^*$ and $\tilde{\mathscr{M}}^*$ be the maximal invariant subspaces of Δ and $\tilde{\Delta}$, respectively. Consider two control systems:*

$$\Lambda^* = (A^*, B^*) \in Expl(\Delta|^{red}_{\mathscr{M}^*}), \quad \tilde{\Lambda}^* = (\tilde{A}^*, \tilde{B}^*) \in Expl(\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*}).$$

Then the following are equivalent:

(i) $\Delta \overset{in}{\sim} \tilde{\Delta}$*;*

(ii) Λ^* *and* $\tilde{\Lambda}^*$ *are feedback equivalent;*

(iii) Δ *and* $\tilde{\Delta}$ *have isomorphic trajectories, i.e, there exists a linear and invertible map* $S : \mathscr{M}^* \to \tilde{\mathscr{M}}^*$ *transforming any trajectory* $x(t, x^0)$*, where* $x^0 \in \mathscr{M}^*$ *of* $\Delta|^{red}_{\mathscr{M}^*}$ *into a trajectory* $\tilde{x}(t, \tilde{x}^0)$*,* $\tilde{x}^0 \in \tilde{\mathscr{M}}^*$ *of* $\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*}$*, where* $\tilde{x}^0 = Sx^0$*, and vice versa.*

The proof is given in Section 2.7. In most of the DAEs literature, regularity of DAEs is frequently studied and various definitions are proposed. From the point of view of the existence and uniqueness of solutions, we propose the following definition of internal regularity of DAEs.

Definition 2.6.11. Δ is internally regular if through any point $x^0 \in \mathscr{M}^*$, there passes only one solution.

Recall that $\operatorname{rank}_{\mathbb{R}[s]}(sE - H)$ denotes the rank of a polynomial matrix $sE - H$ over the ring $\mathbb{R}[s]$.

Proposition 2.6.12. *For a DAE $\Delta_{l,n} = (E, H)$, denote* $\operatorname{rank} E = q$. *The following statements are equivalent:*

(i) Δ *is internally regular;*

(ii) *Any* $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{M}^*})$ *has no inputs;*

(iii) *The* ***MCF*** *of* $\Lambda \in Expl(\Delta)$ *has no* MCF^1 *part.*

(iv) $\operatorname{rank} E = \dim E\mathscr{M}^*$*;*

(v) $\operatorname{rank}_{\mathbb{R}(s)}(sE - H) = q$*;*

(vi) *The* ***MCF*** *of* $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{M}^*})$ *has the* MCF^2 *part only.*

The proof is given in Section 2.7.

Remark 2.6.13. (i) The above definition of internal regularity is actually equivalent to the definition of an autonomous DAE in [11]. Both of them mean that the DAE is not under-determined (there is no $L(s)$ in the **KCF** of $sE - H$).

(ii) Our notion of internal regularity does not imply that the matrices E and H are square, since the presence of the overdetermined part KCF^4 (or $L^p(s)$) is allowed for $\Delta = (E, H)$.

(iii) If E and H are square ($l = n$), then Δ (equivalently, $sE - H$) is internally regular if and only if $|sE - H| \not\equiv 0$, $s \in \mathbb{C}$. It means that for the case of square matrices, the classical notion of regularity and internal regularity coincide.

2.7 Proofs of the results

2.7.1 Proof of Theorem 2.3.4

Proof. (i) This result can be easily deduced from Definition 2.3.1 and 2.3.2.

(ii) Consider two control systems

$$\Lambda = (A, B, C, D) \in Expl(\Delta) \quad \text{and} \quad \tilde{\Lambda} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in Expl(\tilde{\Delta}).$$

Then by (i) of Theorem 2.3.4, there exist invertible matrices $Q, \tilde{Q}, P, \tilde{P}$ of appropriate sizes such that

$$Q\,(sE - H)\,P^{-1} = \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix}, \qquad \tilde{Q}\left(s\tilde{E} - \tilde{H}\right)\tilde{P}^{-1} = \begin{bmatrix} sI - \tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}. \quad (2.26)$$

"If". Suppose $\Lambda \overset{M}{\sim} \tilde{\Lambda}$, then there exist Morse transformation matrices T_s, T_i, T_o, F, K such that

$$\begin{bmatrix} T_s & T_sK \\ 0 & T_o \end{bmatrix} \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} = \begin{bmatrix} sI - \tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}. \quad (2.27)$$

By (2.27), we have

$$\begin{bmatrix} T_s & T_sK \\ 0 & T_o \end{bmatrix} Q \left(Q^{-1} \begin{bmatrix} sI-A & -B \\ -C & -D \end{bmatrix} P \right) P^{-1} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} = \tilde{Q} \left(\tilde{Q}^{-1} \begin{bmatrix} sI-\tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix} \tilde{P} \right) \tilde{P}^{-1}$$

Substitute (2.26) into the above equation, to have

$$\tilde{Q}^{-1} \begin{bmatrix} T_s & T_sK \\ 0 & T_0 \end{bmatrix} Q\left(sE-H\right)P^{-1} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} \tilde{P} = s\tilde{E}-\tilde{H}.$$

Thus $\Delta \overset{ex}{\sim} \tilde{\Delta}$ via $(\bar{Q}, \bar{P})$, where

$$\bar{Q} = \tilde{Q}^{-1} \begin{bmatrix} T_s & T_sK \\ 0 & T_0 \end{bmatrix} Q \text{ and } \bar{P}^{-1} = P^{-1} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} \tilde{P}.$$

"Only if". Suppose $\Delta \overset{ex}{\sim} \tilde{\Delta}$, then there exist invertible matrices $\bar{Q}$ and $\bar{P}$ of appropriate sizes such that $\bar{Q}\left(sE-H\right)\bar{P}^{-1} = \left(s\tilde{E}-\tilde{H}\right)$, which implies that

$$\bar{Q}Q^{-1}\left(Q\left(sE-H\right)P^{-1}\right)P\bar{P}^{-1} = \tilde{Q}^{-1}\left(\tilde{Q}\left(s\tilde{E}-\tilde{H}\right)\tilde{P}^{-1}\right)\tilde{P}$$

$$\overset{(2.26)}{\Rightarrow} \tilde{Q}\bar{Q}Q^{-1} \begin{bmatrix} sI-A & -B \\ -C & -D \end{bmatrix} P\bar{P}^{-1}\tilde{P}^{-1} = \begin{bmatrix} sI-\tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}.$$

Denote $\tilde{Q}\bar{Q}Q^{-1} = \begin{bmatrix} Q^1 & Q^2 \\ Q^3 & Q^4 \end{bmatrix}$ and $P\bar{P}^{-1}\tilde{P}^{-1} = \begin{bmatrix} P^1 & P^2 \\ P^3 & P^4 \end{bmatrix}$, where Q^i and P^i, for $i = 1,2,3,4$, are matrices of suitable sizes. Then we get

$$\begin{bmatrix} Q^1 & Q^2 \\ Q^3 & Q^4 \end{bmatrix} \begin{bmatrix} sI-A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} P^1 & P^2 \\ P^3 & P^4 \end{bmatrix} = \begin{bmatrix} sI-\tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}.$$

Now by the invertibility of $\tilde{Q}\bar{Q}Q^{-1}$ and $P\bar{P}^{-1}\tilde{P}^{-1}$, we get $\begin{bmatrix} Q^1 & Q^2 \\ Q^3 & Q^4 \end{bmatrix}$ and $\begin{bmatrix} P^1 & P^2 \\ P^3 & P^4 \end{bmatrix}$ are invertible. By a direct calculation, we get $Q^3 = 0$, $P^2 = 0$, $Q^1 = (P^1)^{-1}$, thus Q^4 and P^4 are invertible as well. Therefore, $\Lambda \overset{M}{\sim} \tilde{\Lambda}$ via the Morse transformation

$$M_{tran} = \left(Q^1, (P^4)^{-1}, Q^4, P^3Q^1, (Q^1)^{-1}Q^2\right).$$

(iii) Given two control systems $\Lambda = (A,B,C,D)$ and $\tilde{\Lambda} = (\tilde{A},\tilde{B},\tilde{C},\tilde{D})$, the corresponding matrix pencils of $\Delta^{Impl} = Impl(\Lambda)$ and $\tilde{\Delta}^{Impl} = Impl(\tilde{\Lambda})$, by Definition 2.3.1, are $\begin{bmatrix} sI-A & -B \\ -C & -D \end{bmatrix}$ and $\begin{bmatrix} sI-\tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}$, respectively.

"If". Suppose $\Delta^{Impl} \overset{ex}{\sim} \tilde{\Delta}^{Impl}$, that is, there exist invertible matrices Q and P such that

$$Q \begin{bmatrix} sI-A & -B \\ -C & -D \end{bmatrix} P^{-1} = \begin{bmatrix} sI-\tilde{A} & -\tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}. \tag{2.28}$$

Denote $Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$ and $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ with matrices Q^i and P^i, for $i = 1, 2, 3, 4$, of suitable dimensions. Then by (2.28), we get $Q_3 = 0$, $P_2 = 0$, $Q_1 = (P_1)^{-1}$. Since Q and P are invertible, we can conclude that Q_4 and P_4 are invertible as well. Therefore, $\Lambda \overset{M}{\sim} \tilde{\Lambda}$ via the Morse transformation $M_{tran} = (Q_1, (P_4)^{-1}, Q_4, P_3 Q_1, (Q_1)^{-1} Q_2)$.

"Only if". Suppose $\Lambda \overset{M}{\sim} \tilde{\Lambda}$ via a Morse transformation $M_{tran} = (T_s, T_i, T_o, F, K)$ (see equation (2.4)), then we have $\Delta^{Impl} \overset{ex}{\sim} \tilde{\Delta}^{Impl}$ via (Q, P), where $Q = \begin{bmatrix} T_s & T_s K \\ 0 & T_o \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} T_s^{-1} & 0 \\ F T_s^{-1} & T_i^{-1} \end{bmatrix}$. □

2.7.2 Proof of Proposition 2.4.4

Proof. (i) It can be observed from (2.10) that $\mathscr{V}_i$ is non-increasing. By a dimensional argument, the sequence $\mathscr{V}_i$ gets stabilized at $i = k^* \leq n$ and it can be directly seen from $\mathscr{V}_{k^*} = H^{-1} E \mathscr{V}_{k^*}$ that $\mathscr{V}_{k^*}$ is a (H^{-1}, E)-invariant subspace. We now prove by induction that it is the largest. Choose any other (H^{-1}, E)-invariant subspace $\hat{\mathscr{V}}$ and consider (2.10). For $i = 0$, $\hat{\mathscr{V}} \subseteq \mathscr{V}_0$; Suppose $\hat{\mathscr{V}} \subseteq \mathscr{V}_i$, then $H^{-1} E \hat{\mathscr{V}} \subseteq H^{-1} E \mathscr{V}_i$ (since taking the image and preimage preserves inclusion), thus $\hat{\mathscr{V}} = H^{-1} E \hat{\mathscr{V}} \subseteq H^{-1} E \mathscr{V}_i = \mathscr{V}_{i+1}$. Therefore, $\hat{\mathscr{V}} \subseteq \mathscr{V}_i$ for $i \in \mathbb{N}$, i.e., $\hat{\mathscr{V}} \subseteq \mathscr{V}_{k^*}$, it follows $\mathscr{V}_{k^*}$ is the largest u(H^{-1}, E)-invariant subspace.

Now consider (2.11), observe that the sequence $\mathscr{W}_i$ is non-decreasing and by a dimensional argument, $\mathscr{W}_i$ gets stabilized at $i = l^* \leq n$. It can be directly seen from $\mathscr{W}_{l^*} = E^{-1} H \mathscr{W}_{l^*}$ that $\mathscr{W}_{l^*}$ is a (E^{-1}, H)-invariant subspace. We then prove that any other (E^{-1}, H)-invariant subspace $\hat{\mathscr{W}}$ contains $\mathscr{W}^*$, for $i = 0$, $\mathscr{W}_0 \subseteq \hat{\mathscr{W}}$; if $\mathscr{W}_i \subseteq \hat{\mathscr{W}}$, then $E^{-1} H \mathscr{W}_i \subseteq E^{-1} H \hat{\mathscr{W}}$, so $\mathscr{W}_{i+1} = E^{-1} H \mathscr{W}_i \subseteq E^{-1} H \hat{\mathscr{W}} = \hat{\mathscr{W}}$ that is $\mathscr{W}_i \subseteq \hat{\mathscr{W}}$ for $i \in \mathbb{N}$, which gives $\mathscr{W}_{l^*} \subseteq \hat{\mathscr{W}}$ and $\mathscr{W}_{l^*}$ is the smallest (E^{-1}, H)-invariant subspace.

(ii) By Definition 2.4.3, $\mathscr{V}^*$ satisfies $\mathscr{V}^* = H^{-1} E \mathscr{V}^*$, thus it is seen that $H \mathscr{V}^* \subseteq E \mathscr{V}^*$. We then prove, by induction that, $\mathscr{V}^*$ is the largest satisfying that property. Choose any other subspace $\hat{\mathscr{V}}$ which satisfies $H \hat{\mathscr{V}} \subseteq E \hat{\mathscr{V}}$, consider (2.10), for $i = 0$, so $\hat{\mathscr{V}} \subseteq \mathscr{V}_0$. Suppose $\hat{\mathscr{V}} \subseteq \mathscr{V}_i$, then $\hat{\mathscr{V}} \subseteq H^{-1} E \hat{\mathscr{V}} \subseteq H^{-1} E \mathscr{V}_i = \mathscr{V}_{i+1}$, thus $\hat{\mathscr{V}} \subseteq H^{-1} E \mathscr{V}_i = \mathscr{V}_{i+1}$, therefore $\hat{\mathscr{V}} \subseteq \mathscr{V}_i$ for $i \in \mathbb{N}$, i.e., $\hat{\mathscr{V}} \subseteq \mathscr{V}_{k^*}$, which implies $\mathscr{V}^* = \mathscr{V}_{k^*}$ is the largest subspace such that $H \mathscr{V}^* \subseteq E \mathscr{V}^*$

Obviously, $\{0\}$ is the smallest subspace satisfying $H\{0\} \subseteq E\{0\}$, but $\mathscr{W}^*$ is not always $\{0\}$, so we prove that $\mathscr{W}^*$ is not necessarily the smallest subspace such that $E \mathscr{W}^* \subseteq H \mathscr{W}^*$. □

2.7.3 Proof of Proposition 2.4.10

Proof. Observe that, by Definition 2.2.1 and 2.4.1, if two DAEs Δ and $\tilde{\Delta}$ are ex-equivalent via (Q,P), then direct calculations of the Wong sequences of Δ and $\tilde{\Delta}$ give that $\mathscr{V}_i(\tilde{\Delta}) = P\mathscr{V}_i(\Delta)$ and $\mathscr{W}_i(\tilde{\Delta}) = P\mathscr{W}_i(\Delta)$. As Λ is a (Q,P)-explicitation of Δ, by Theorem 2.3.4(i), we have $\Delta \overset{ex}{\sim} \Delta^{Impl}$ via (Q,P), where $\Delta^{Impl} = Impl(\Lambda)$. Thus we have

$$\mathscr{V}_i(\Delta^{Impl}) = P\mathscr{V}_i(\Delta), \quad \mathscr{W}_i(\Delta^{Impl}) = P\mathscr{W}_i(\Delta). \tag{2.29}$$

Notice that

$$\Delta^{Impl}_{l,n} = \left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right), \ \ \mathbf{\Lambda}_{n,m,p} = (\mathbf{A},\mathbf{B},\mathbf{C}) = \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \begin{bmatrix} C & D \end{bmatrix}\right),$$

where $m = n-q$ and $p = l-q$. The proof of (i) will be done in 3 steps :

Step 1: First we show that for $i \in \mathbb{N}$,

$$\mathscr{V}_i(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}_i(\mathbf{\Lambda}). \tag{2.30}$$

Calculate $\boldsymbol{\mathcal{V}}_{i+1}(\mathbf{\Lambda})$ using (2.13), to get

$$\boldsymbol{\mathcal{V}}_{i+1}(\mathbf{\Lambda}) = \ker \begin{bmatrix} C & D \end{bmatrix} \cap \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^{-1} \left(\boldsymbol{\mathcal{V}}_i(\mathbf{\Lambda}) + \operatorname{Im} \begin{bmatrix} 0 \\ I_m \end{bmatrix}\right). \tag{2.31}$$

Equation (2.31) can be written as

$$\boldsymbol{\mathcal{V}}_{i+1}(\mathbf{\Lambda}) = \left\{\tilde{v} \mid \begin{bmatrix} A & B \end{bmatrix}\tilde{v} \in \begin{bmatrix} I_q & 0 \end{bmatrix}\boldsymbol{\mathcal{V}}_i(\mathbf{\Lambda}), \ \begin{bmatrix} C & D \end{bmatrix}\tilde{v} = 0\right\}$$

or, equivalently,

$$\boldsymbol{\mathcal{V}}_{i+1}(\mathbf{\Lambda}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{\mathcal{V}}_i(\mathbf{\Lambda}). \tag{2.32}$$

Now, observe that the inductive formula (2.32) for $\boldsymbol{\mathcal{V}}_{i+1}(\mathbf{\Lambda})$ coincides with the inductive formula (2.10) for the Wong sequence $\mathscr{V}_{i+1}(\Delta^{Impl})$. Since $\mathscr{V}_0(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}_0(\mathbf{\Lambda}) = \mathbb{R}^n$, we conclude that $\mathscr{V}_i(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}_i(\mathbf{\Lambda})$ for all $i \in \mathbb{N}$.

Step 2: We then prove that for $i \in \mathbb{N}$,

$$\mathscr{V}_{i+1}(\Delta^{Impl}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_i(\Lambda) \\ 0 \end{bmatrix}. \tag{2.33}$$

By calculating $\mathcal{V}_{i+1}(\Lambda)$ via (2.15), we get

$$\mathcal{V}_{i+1}(\Lambda) = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\begin{bmatrix} I \\ 0 \end{bmatrix}\mathcal{V}_i(\Lambda) + \operatorname{Im}\begin{bmatrix} B \\ D \end{bmatrix}\right).$$

We can rewrite the above equation as

$$\mathcal{V}_{i+1}(\Lambda) = \begin{bmatrix} I_q & 0_{q\times m} & 0 \end{bmatrix} \ker \begin{bmatrix} A & B & \bar{V}_i \\ C & D & 0 \end{bmatrix}, \tag{2.34}$$

where $\bar{V}_i$ is a matrix with independent columns such that $\operatorname{Im}\bar{V}_i = \mathcal{V}_i(\Lambda)$.

From basic knowledge of linear algebra, for two matrices $M \in \mathbb{R}^{l\times n}$ and $N \in \mathbb{R}^{l\times n}$, the preimage $M^{-1}\operatorname{Im}N = [I_n, 0]\ker[M, N]$. With this formula, calculate $\mathscr{V}_{i+1}(\Delta^{Impl})$ via (2.10), to get

$$\mathscr{V}_{i+1}(\Delta^{Impl}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_q & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} \ker \begin{bmatrix} A & B & \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} V_i \\ C & D & \end{bmatrix}, \quad (2.35)$$

where V_i is a matrix with independent columns such that $\operatorname{Im} V_i = \mathscr{V}_i(\Delta)$.

In order to show that (2.33) holds, we will first prove inductively that for all $i \in \mathbb{N}$,

$$\begin{bmatrix} \mathcal{V}_i(\Lambda) \\ 0 \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \mathscr{V}_i(\Delta^{Impl}). \quad (2.36)$$

For $i = 0$, $\begin{bmatrix} \mathcal{V}_0(\Lambda) \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbb{R}^q \\ 0 \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \mathscr{V}_0(\Delta^{Impl})$. Suppose that for $i = k \in \mathbb{N}$, equation (2.36) holds, or equivalently, $\begin{bmatrix} \bar{V}_k \\ 0 \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} V_k(\Delta^{Impl})$. Then we have

$$\begin{aligned} \begin{bmatrix} \mathcal{V}_{k+1}(\Lambda) \\ 0 \end{bmatrix} &\overset{(2.34)}{=} \begin{bmatrix} I_q & 0_{q\times m} & 0 \\ 0 & 0_{p\times m} & 0 \end{bmatrix} \ker \begin{bmatrix} A & B & \bar{V}_k \\ C & D & 0 \end{bmatrix} \\ &= \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_q & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} \ker \begin{bmatrix} A & B & \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} V_k \\ C & D & \end{bmatrix} \overset{(2.35)}{=} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \mathscr{V}_{k+1}(\Delta^{Impl}). \end{aligned}$$

Therefore, equation (2.36) holds for all $i \in \mathbb{N}$.

Consequently, we have for $i \in \mathbb{N}$,

$$\mathscr{V}_{i+1}(\Delta^{Impl}) \overset{(2.10)}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \mathscr{V}_i(\Delta^{Impl}) \overset{(2.36)}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_i(\Lambda) \\ 0 \end{bmatrix}.$$

Step 3: Finally, since $\mathscr{V}^*$ and $\boldsymbol{\mathcal{V}}^*$ are the limits of the sequences $\mathscr{V}_i$ and $\boldsymbol{\mathcal{V}}_i$, respectively, it follows from (2.30) that $\mathscr{V}^*(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}^*(\boldsymbol{\Lambda})$. Since $\mathscr{V}^*$ and $\mathcal{V}^*$ are the limits of $\mathscr{V}_i$ and $\mathcal{V}_i$, respectively, it follows from (2.33) that $\mathscr{V}^*(\Delta^{Impl}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}^*(\Lambda) \\ 0 \end{bmatrix}$. Thus by (2.29), we have $P\mathscr{V}^*(\Delta) = \mathscr{V}^*(\Delta^{Impl}) = \boldsymbol{\mathcal{V}}^*(\boldsymbol{\Lambda}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}^*(\Lambda) \\ 0 \end{bmatrix}$.

The proof of (ii) will be done in 3 steps :

Step 1: Firstly, we show that for $i \in \mathbb{N}$,

$$\mathscr{W}_i(\Delta^{Impl}) = \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}). \quad (2.37)$$

Calculate $\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda})$ by (2.14), as

$$\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}) = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \left(\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}) \cap \ker \begin{bmatrix} C & D \end{bmatrix} + \operatorname{Im} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \right)$$

$$
\begin{aligned}
&= \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \left(\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}) \cap \ker \begin{bmatrix} C & D \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda})\right) \cap \left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \ker \begin{bmatrix} C & D \end{bmatrix}\right).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \ker \begin{bmatrix} C & D \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \ker \begin{bmatrix} C & D \end{bmatrix} \\ * \end{bmatrix} + \operatorname{Im} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \\
&= \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \operatorname{Im} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\end{aligned}
$$

Then we have

$$
\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}). \tag{2.38}
$$

Observe that the inductive formula (2.38) for $\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda})$ coincides with the inductive formula (2.11) for the Wong sequences $\mathscr{W}_{i+1}(\Delta^{Impl})$. Since $\mathscr{W}_0(\Delta^{Impl}) = \boldsymbol{\mathcal{W}}_0(\boldsymbol{\Lambda}) = \{0\}$, we deduce that $\mathscr{W}_i(\Delta^{Impl}) = \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda})$ for $i \in \mathbb{N}$.

Step 2: Subsequently, we will prove that for $i \in \mathbb{N}$,

$$
\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_i(\Lambda) \\ 0 \end{bmatrix}. \tag{2.39}
$$

Considering (2.17) for Λ, we have

$$
\begin{aligned}
\begin{bmatrix} \mathcal{W}_{i+1}(\Lambda) \\ 0 \end{bmatrix} &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} \mathcal{W}_i(\Lambda) \\ \mathbb{R}^m \end{bmatrix} \cap \ker \begin{bmatrix} C & D \end{bmatrix} \right) \\
&= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \left(\left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_i(\Lambda) \\ 0 \end{bmatrix} \right) \cap \ker \begin{bmatrix} C & D \end{bmatrix} \right),
\end{aligned}
$$

which implies that

$$
\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_{i+1}(\Lambda) \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_i(\Lambda) \\ 0 \end{bmatrix} \cap \ker \begin{bmatrix} C & D \end{bmatrix} \right) + \operatorname{Im} \begin{bmatrix} 0 \\ I_m \end{bmatrix}. \tag{2.40}
$$

Observe that the inductive formula (2.40) for $\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_{i+1}(\Lambda) \\ 0 \end{bmatrix}$ coincides with the inductive formula (2.14) for $\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda})$. Since $\boldsymbol{\mathcal{W}}_1(\Lambda) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_0(\Lambda) \\ 0 \end{bmatrix} = \operatorname{Im} \begin{bmatrix} 0 \\ I_m \end{bmatrix}$, we have $\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_i(\Lambda) \\ 0 \end{bmatrix}$ for all $i \in \mathbb{N}$.

Step 3: Equation (2.37) and the fact that $\mathscr{W}^*$ and $\boldsymbol{\mathcal{W}}^*$ are the limits of $\mathscr{W}_i$ and $\boldsymbol{\mathcal{W}}_i$, respectively, yield $\mathscr{W}^*(\Delta) = \boldsymbol{\mathcal{W}}^*(\boldsymbol{\Lambda})$. Equation (2.39) and the fact that $\mathcal{W}^*$ and $\boldsymbol{\mathcal{W}}^*$ are the limits of $\mathcal{W}_i$ and $\boldsymbol{\mathcal{W}}_i$, respectively, yield $\boldsymbol{\mathcal{W}}^*(\boldsymbol{\Lambda}) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}^*(\Lambda) \\ 0 \end{bmatrix}$. Thus using equation (2.29), we prove (ii) of Proposition 2.4.10. □

2.7.4 Proof of Proposition 2.4.13

In this proof, we will need the following two lemmata. Denote by $\mathbb{F}(\mathcal{V}_i(\Lambda))$ the class of maps $F:\mathbb{R}^q\to\mathbb{R}^m$ satisfying $(A+BF)\mathcal{V}_{i+1}(\Lambda)\subset\mathcal{V}_i(\Lambda)$ and $(C+DF)\mathcal{V}_{i+1}(\Lambda)=0$.

Lemma 2.7.1. *Given $\Delta_{l,n}=(E,H)$, its (Q,P)-explicitation $\Lambda=(A,B,C,D)\in Expl(\Delta)$, and $\Delta^{Impl}=Impl(\Lambda)$, consider the Wong sequences $\mathscr{V}_i$, $\mathscr{W}_i$ of both Δ and Δ^{Impl}, given by Definition 2.4.1 and the subspaces $\mathcal{V}_i$, $\mathcal{W}_i$ of Λ, given by Lemma 2.4.6. Then for $i\in\mathbb{N}$, we have*

$$\mathscr{V}_{i+1}(\Delta^{Impl})=P\mathscr{V}_{i+1}(\Delta)=\begin{bmatrix}\mathcal{V}_{i+1}(\Lambda)\\ F_i\mathcal{V}_{i+1}(\Lambda)\end{bmatrix}+\begin{bmatrix}0\\ \mathcal{U}_i(\Lambda)\end{bmatrix}, \tag{2.41}$$

where $F_i\in\mathbb{F}(\mathcal{V}_i(\Lambda))$ and

$$\mathscr{W}_{i+1}(\Delta^{Impl})=P\mathscr{W}_{i+1}(\Delta)=\begin{bmatrix}\mathcal{W}_i(\Lambda)\\ *\end{bmatrix}+\begin{bmatrix}0\\ \mathscr{U}(\Lambda)\end{bmatrix}. \tag{2.42}$$

Lemma 2.7.2. *Consider the subspace sequences $\boldsymbol{\mathcal{V}}_i$ and $\boldsymbol{\mathcal{W}}_i$ of $\boldsymbol{\Lambda}^d$, given by Lemma 2.4.6. Then for $i\in\mathbb{N}$, the following hold*

$$P^T\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}^d)=H^T(E^T)^{-1}\left(P^T\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}^d)\right), \tag{2.43}$$
$$P^T\boldsymbol{\mathcal{V}}_{i+1}\left(\boldsymbol{\Lambda}^d\right)=E^T(H^T)^{-1}\left(P^T\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}^d)\right). \tag{2.44}$$

Proof of Lemma 2.7.1. We first show that equation (2.41) holds. Let independent vectors $v_1=\begin{bmatrix}v_1^1\\ v_1^2\end{bmatrix},...,v_\alpha=\begin{bmatrix}v_\alpha^1\\ v_\alpha^2\end{bmatrix}\in\mathbb{R}^n$ form a basis of

$$P\mathscr{V}_{i+1}(\Delta)\overset{(2.29)}{=}\mathscr{V}_{i+1}(\Delta^{Impl})\overset{(2.33)}{=}\begin{bmatrix}A&B\\ C&D\end{bmatrix}^{-1}\begin{bmatrix}\mathcal{V}_i(\Lambda)\\ 0\end{bmatrix},$$

where $v_j^1\in\mathbb{R}^q, v_j^2\in\mathbb{R}^m, j=1,2,...,\alpha$ (implying that $\dim(\mathscr{V}_{i+1}(\Delta^{Impl}))=\alpha$). Now without loss of generality, assume $v_j^1\neq 0$ for $j=1,...,\kappa$ and $v_j^1=0$ for $j=\kappa+1,...,\alpha$, where $\kappa<\alpha$ is the number of non-zero vectors v_j^1. Then from equation (2.36), it can be deduced that v_j^1 for $j=1,...,\kappa$ form a basis of $\mathcal{V}_{i+1}(\Lambda)$. Moreover, from (2.33), it is not hard to see that v_j^2 for $j=\kappa+1,...,\alpha$ form a basis of $\mathcal{U}_i(\Lambda)$. Let $F_i\in\mathbb{R}^{m\times\kappa}$ be such that $F_iv_j^1=v_j^2$ for $j=1,...,\kappa$ (such F_i exists), then $v_1,\dots,v_\alpha$ form a basis of $\begin{bmatrix}\mathcal{V}_{i+1}(\Lambda)\\ F_i\mathcal{V}_{i+1}(\Lambda)\end{bmatrix}+\begin{bmatrix}0\\ \mathcal{U}_i(\Lambda)\end{bmatrix}$. Therefore,

$$\begin{bmatrix}\mathcal{V}_{i+1}(\Lambda)\\ F_i\mathcal{V}_{i+1}(\Lambda)\end{bmatrix}+\begin{bmatrix}0\\ \mathcal{U}_i(\Lambda)\end{bmatrix}=\begin{bmatrix}A&B\\ C&D\end{bmatrix}^{-1}\begin{bmatrix}\mathcal{V}_i(\Lambda)\\ 0\end{bmatrix},$$

because both spaces have the same basis $v_1,\dots,v_\alpha$. We now prove that for any choice of F_i, we have $F_i\in\mathbb{F}(\mathcal{V}_i(\Lambda))$. Pre-multiply the above equation by $\begin{bmatrix}A&B\\ C&D\end{bmatrix}$ on the left to obtain

$$\begin{bmatrix}(A+BF_i)\mathcal{V}_{i+1}(\Lambda)\\ (C+DF_i)\mathcal{V}_{i+1}(\Lambda)\end{bmatrix}+\begin{bmatrix}B\mathcal{U}_i(\Lambda)\\ D\mathcal{U}_i(\Lambda)\end{bmatrix}\subseteq\begin{bmatrix}\mathcal{V}_i(\Lambda)\\ 0\end{bmatrix}.$$

Moreover, we get $\begin{bmatrix} B\mathcal{U}_i(\Lambda) \\ D\mathcal{U}_i(\Lambda) \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{V}_i(\Lambda) \\ 0 \end{bmatrix}$ by (2.16). Thus it is easy to see that $(A + BF_i)\mathcal{V}_{i+1}(\Lambda) \subseteq \mathcal{V}_i$ and $(C + DF_i)\mathcal{V}_{i+1}(\Lambda) = 0$.

Subsequently, we show that equation (2.42) holds. By (2.37) and (2.39), it follows that for $i \in \mathbb{N}$,

$$\mathscr{W}_{i+1}(\Delta^{Impl}) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{W}_i(\Lambda) \\ 0 \end{bmatrix}. \tag{2.45}$$

Then by (2.29), we have $\mathscr{W}_{i+1}(\Delta^{Impl}) = P\mathscr{W}_{i+1}(\Delta)$ and we complete the proof of (2.42) by calculating explicitly the right-hand side of (2.45). □

Proof of Lemma 2.7.2. Notice that $\boldsymbol{\Lambda}^d_{n,p,m} = \left(\begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix}, \begin{bmatrix} C^T \\ D^T \end{bmatrix}, \begin{bmatrix} 0 & I_m \end{bmatrix} \right)$. We first prove that the following relations hold,

$$\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}^d) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-T} \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}^d), \quad \boldsymbol{\mathcal{V}}_{i+1}(\boldsymbol{\Lambda}^d) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-T} \boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}^d). \tag{2.46}$$

For $\boldsymbol{\Lambda}^d$, calculate $\boldsymbol{\mathcal{W}}_{i+1}$ via (2.14), to get for $i \in \mathbb{N}$:

$$\boldsymbol{\mathcal{W}}_{i+1}\left(\boldsymbol{\Lambda}^d\right) = \begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix} \left(\boldsymbol{\mathcal{W}}_i\left(\boldsymbol{\Lambda}^d\right) \cap \ker \begin{bmatrix} 0 & I_m \end{bmatrix} \right) + \operatorname{Im} \begin{bmatrix} C^T \\ D^T \end{bmatrix}.$$

Moreover, it is not hard to see that

$$\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-T} \boldsymbol{\mathcal{W}}_i\left(\boldsymbol{\Lambda}^d\right) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \left(\boldsymbol{\mathcal{W}}_i\left(\boldsymbol{\Lambda}^d\right) \cap \ker \begin{bmatrix} 0 & I_m \end{bmatrix} \right) + \operatorname{Im} \begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

Pre-multiply both sides of the above equation by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T$, it follows that

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-T} \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}^d) &= \begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix} \left(\boldsymbol{\mathcal{W}}_i\left(\boldsymbol{\Lambda}^d\right) \cap \ker \begin{bmatrix} 0 & I_m \end{bmatrix} \right) + \operatorname{Im} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \\ &= \boldsymbol{\mathcal{W}}_{i+1}\left(\boldsymbol{\Lambda}^d\right). \end{aligned}$$

Then calculate $\boldsymbol{\mathcal{V}}_{i+1}$ for $\boldsymbol{\Lambda}^d$, via (2.13), to get for $i \in \mathbb{N}$,

$$\boldsymbol{\mathcal{V}}_{i+1}\left(\boldsymbol{\Lambda}^d\right) = \ker \begin{bmatrix} 0 & I_m \end{bmatrix} \cap \begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix}^{-1} \left(\boldsymbol{\mathcal{V}}_i\left(\boldsymbol{\Lambda}^d\right) + \operatorname{Im} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \right). \tag{2.47}$$

Rewrite (2.47) as

$$\boldsymbol{\mathcal{V}}_{i+1}\left(\boldsymbol{\Lambda}^d\right) = \operatorname{Im} \begin{bmatrix} I_q \\ 0 \end{bmatrix} \cap \left(\begin{bmatrix} I_n & 0 \end{bmatrix} \ker \begin{bmatrix} (A)^T & 0 & \tilde{V}_i\left(\boldsymbol{\Lambda}^d\right) & (C)^T \\ (B)^T & 0 & & (D)^T \end{bmatrix} \right)$$

$$= \begin{bmatrix} \begin{bmatrix} I_q & 0 \end{bmatrix} \ker \begin{bmatrix} A^T & C^T & \tilde{V}_i\left(\mathbf{\Lambda}^d\right) \\ B^T & D^T & \\ & 0 & \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}^{-1} \boldsymbol{\mathcal{V}}_i\left(\mathbf{\Lambda}^d\right).$$

Therefore, the proof of (2.46) is complete. Consequently, substitute

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = QHP^{-1}, \quad \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} = QEP^{-1}$$

into (2.46), then it is straightforward to see that (2.43) and (2.44) hold for any $i \in \mathbb{N}$. $\square$

Proof of Proposition 2.4.13. Notice that since $\Lambda \in Expl(\Delta)$, by Proposition 2.4.12, we have $\Lambda^d \in Expl(\Delta^d)$. Moreover, it is easy to see if Λ is the (Q, P)-explicitation of Δ, then Λ^d is the (P^{-T}, Q^{-T})-explicitation of Δ^d. The proof will be done in 3 steps.

Step 1; Step 1a: We show that for $i \in \mathbb{N}$,

$$\mathscr{W}_{i+1}(\Delta^d) = (E\mathscr{V}_i(\Delta))^\perp \Leftrightarrow \mathcal{W}_i(\Lambda^d) = (\mathcal{V}_i(\Lambda))^\perp. \tag{2.48}$$

By $\Lambda^d \in Expl(\Delta^d)$ and (2.42) of Lemma 2.7.1, we get

$$Q^{-T}\mathscr{W}_{i+1}(\Delta^d) = \begin{bmatrix} \mathcal{W}_i(\Lambda^d) \\ * \end{bmatrix} + \mathrm{Im} \begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

Moreover, we have

$$(E\mathscr{V}_i(\Delta))^\perp = (Q^{-1}QEP^{-1}P\mathscr{V}_i(\Delta))^\perp \overset{(2.29)}{=} (Q^{-1} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \mathscr{V}_i(\Delta^{Impl}))^\perp \overset{(2.36)}{=} Q^T \begin{bmatrix} \mathcal{V}_i(\Lambda) \\ 0 \end{bmatrix}^\perp$$
$$= Q^T \left(\begin{bmatrix} (\mathcal{V}_i(\Lambda))^\perp \\ * \end{bmatrix} + \mathrm{Im} \begin{bmatrix} 0 \\ I_p \end{bmatrix} \right).$$

It is seen that $\mathscr{W}_{i+1}(\Delta^d) = (E\mathscr{V}_i(\Delta))^\perp$ if and only if $\mathcal{W}_i(\Lambda^d) = (\mathcal{V}_i(\Lambda))^\perp$.

Step 1b: In this step, we will prove that for $i \in \mathbb{N}$,

$$\mathscr{V}_i(\Delta^d) = (H\mathscr{W}_i(\Delta))^\perp \Leftrightarrow \mathcal{V}_i(\Lambda^d) = (\mathcal{W}_i(\Lambda))^\perp. \tag{2.49}$$

We first prove "$\Rightarrow$" of (2.49): Considering equation (2.29) and (2.36) for Δ^d, we can deduce that

$$E^T\mathscr{V}_i(\Delta^d) = P^T \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^T Q^{-T}\mathscr{V}_i(\Delta^d) = P^T \begin{bmatrix} \mathcal{V}_i(\Lambda^d) \\ 0 \end{bmatrix}.$$

On the other hand, we have

$$E^T(H\mathscr{W}_i(\Delta))^\perp = (E^{-1}H\mathscr{W}_i(\Delta))^\perp \overset{(2.11)}{=} (\mathscr{W}_{i+1}(\Delta))^\perp = (P^{-1}P\mathscr{W}_{i+1}(\Delta))^\perp$$
$$= (P^{-1})^{-T}(P\mathscr{W}_{i+1}(\Delta))^\perp \overset{(2.42)}{=} P^T \left(\begin{bmatrix} \mathcal{W}_i(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} \right)^\perp$$

$$= P^T \begin{bmatrix} \mathcal{W}_i(\Lambda)^\perp \\ 0 \end{bmatrix}.$$

Now we can see that for $i \in \mathbb{N}$, if $\mathscr{V}_i(\Delta^d) = (H\mathscr{W}_i(\Delta))^\perp$, then $\mathcal{V}_i(\Lambda^d) = (\mathcal{W}_i(\Lambda))^\perp$.

We then prove "⇐" of (2.49): By equation (2.29) and (2.33), we can deduce that

$$Q^{-T}\mathscr{V}_{i+1}(\Delta^d) = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_i(\Lambda^d) \\ 0 \end{bmatrix}. \tag{2.50}$$

We have

$$\begin{aligned}(H\mathscr{W}_{i+1}(\Delta))^\perp &= (Q^{-1}QHP^{-1}P\mathscr{W}_{i+1}(\Delta))^\perp = (Q^{-1}\begin{bmatrix} A & B \\ C & D \end{bmatrix} P\mathscr{W}_{i+1}(\Delta))^\perp \\ &= \left(Q^{-1}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)^{-T} (P\mathscr{W}_{i+1}(\Delta))^\perp \\ &\overset{(2.42)}{=} Q^T \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathcal{W}_i(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix}\right)^\perp.\end{aligned}$$

The above equation gives

$$Q^{-T}(H\mathscr{W}_{i+1}(\Delta))^\perp = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}^{-1} \begin{bmatrix} (\mathcal{W}_i(\Lambda))^\perp \\ 0 \end{bmatrix}. \tag{2.51}$$

Now equations (2.50) and (2.51) yield that for $i \in \mathbb{N}$, if $\mathcal{V}_i(\Lambda^d) = (\mathcal{W}_i(\Lambda))^\perp$, then $\mathscr{V}_i(\Delta^d) = (H\mathscr{W}_i(\Delta))^\perp$. Thus the proof of (2.49) is complete.

Step 2; Step 2a: We prove that for $i \in \mathbb{N}$,

$$\mathscr{W}_{i+1}(\Delta^d) = (E\mathscr{V}_i(\Delta))^\perp \Leftrightarrow \boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}^d) = (\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}))^\perp. \tag{2.52}$$

Using equation (2.43) of Lemma 2.7.2, we will prove by induction that for $i \in \mathbb{N}$,

$$H^T\mathscr{W}_i(\Delta^d) = P^T\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}^d). \tag{2.53}$$

For $i = 0$, $H^T\mathscr{W}_0(\Delta^d) = P^T\boldsymbol{\mathcal{W}}_0(\boldsymbol{\Lambda}^d) = 0$; If $H^T\mathscr{W}_i(\Delta^d) = P^T\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}^d)$, then

$$H^T\mathscr{W}_{i+1}(\Delta^d) \overset{(2.11)}{=} H^T(E^T)^{-1}H^T\mathscr{W}_i(\Delta^d) = H^T(E^T)^{-1}P^T\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}^d) \overset{(2.43)}{=} P^T\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}^d).$$

By an induction argument, (2.53) holds for $i \in \mathbb{N}$.

We now prove " ⇒ " of (2.52): Assume for $i \in \mathbb{N}$, $\mathscr{W}_{i+1}(\Delta^d) = (E\mathscr{V}_i(\Delta))^\perp$, it follows that

$$\begin{aligned}\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}^d) &\overset{(2.53)}{=} P^{-T}H^T\mathscr{W}_{i+1}(\Delta^d) = P^{-T}H^T(E\mathscr{V}_i(\Delta))^\perp = (PH^{-1}E\mathscr{V}_i(\Delta))^\perp \\ &\overset{(2.10)}{=}(P\mathscr{V}_{i+1}(\Delta)) = (\mathscr{V}_{i+1}(\Delta^{Impl}))^\perp \overset{(2.30)}{=}(\boldsymbol{\mathcal{V}}_{i+1}(\boldsymbol{\Lambda}))^\perp.\end{aligned}$$

We then prove " ⇐ " of (2.52): Assume for $i \in \mathbb{N}$, $\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}^d) = (\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}))^\perp$, it follows that

$$(E\mathscr{V}_i(\Delta))^\perp = E^{-T}(\mathscr{V}_i(\Delta))^\perp = E^{-T}(P^{-1}\mathscr{V}_i(\Delta^{Impl}))^\perp \overset{(2.30)}{=} E^{-T}(P^{-1}\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}))^\perp$$

$$= E^{-T}P^T\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}^d) \overset{(2.53)}{=} E^{-T}H^T\mathscr{W}_i(\Delta^d) \overset{(2.11)}{=} \mathscr{W}_{i+1}(\Delta^d),$$

and the proof of (2.52) is complete.

Step 2b: In this step, we show that for $i \in \mathbb{N}$,

$$\mathscr{V}_i(\Delta^d) = (H\mathscr{W}_i(\Delta))^\perp \Leftrightarrow \boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}^d) = (\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}))^\perp. \tag{2.54}$$

Using equation (2.44) of Lemma 2.7.2, we will prove by induction that for $i \in \mathbb{N}$,

$$\mathscr{V}_i(\Delta^d) = (H^T)^{-1}\left(P^T\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}^d)\right). \tag{2.55}$$

For $i = 0$, $\mathscr{V}_0(\Delta^d) = \mathbb{R}^n = (H^T)^{-1}P^T\boldsymbol{\mathcal{V}}_0(\boldsymbol{\Lambda}^d)$; If $\mathscr{V}_i(\Delta^d) = (H^T)^{-1}P^T\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}^d)$, then we get

$$\begin{aligned}\mathscr{V}_{i+1}(\Delta^d) &\overset{(2.10)}{=} (H^T)^{-1}E^T\mathscr{V}_i(\Delta^d) = (H^T)^{-1}E^T(H^T)^{-1}P^T\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}^d)\\ &\overset{(2.44)}{=} (H^T)^{-1}P^T\boldsymbol{\mathcal{V}}_{i+1}(\boldsymbol{\Lambda}^d).\end{aligned}$$

By an induction argument, (2.55) holds for $i \in \mathbb{N}$.

We now prove " $\Rightarrow$ " of (2.54). Assume $\mathscr{V}_i(\Delta^d) = (H\mathscr{W}_i(\Delta))^\perp$, then

$$\begin{aligned}P^T\boldsymbol{\mathcal{V}}_{i+1}(\boldsymbol{\Lambda}^d) &\overset{(2.44)}{=} E^TH^{-T}P^T\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}^d) \overset{(2.55)}{=} E^T\mathscr{V}_i(\Delta^d) = E^T(H\mathscr{W}_i(\Delta))^\perp\\ = (E^{-1}H\mathscr{W}_i(\Delta))^\perp &\overset{(2.11)}{=} (\mathscr{W}_{i+1}(\Delta))^\perp = (P^{-1}\mathscr{W}_{i+1}(\Delta^{Impl}))^\perp \overset{(2.37)}{=} P^T\boldsymbol{\mathcal{W}}_{i+1}(\boldsymbol{\Lambda}),\end{aligned}$$

We then prove " $\Leftarrow$ " of (2.54): Assume $\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}^d) = (\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}))^\perp$, then for $i \in \mathbb{N}$,

$$\begin{aligned}(H\mathscr{W}_i(\Delta))^\perp &= (H^T)^{-1}(\mathscr{W}_i(\Delta))^\perp = (H^T)^{-1}(P^{-1}\mathscr{W}_i(\Delta^{Impl}))^\perp\\ &\overset{(2.37)}{=} (H^T)^{-1}(P^{-1}\boldsymbol{\mathcal{W}}_i(\boldsymbol{\Lambda}))^\perp = (H^T)^{-1}P^T\boldsymbol{\mathcal{V}}_i(\boldsymbol{\Lambda}^d) \overset{(2.55)}{=} \mathscr{V}_i(\Delta^d),\end{aligned}$$

which completes the proof of (2.54).

Step 3: Since $\mathscr{V}^*$, $\mathscr{V}^*$, $\mathcal{V}^*$, $\mathcal{W}^*$, $\boldsymbol{\mathcal{V}}^*$, $\boldsymbol{\mathcal{W}}^*$ are the limites of $\mathscr{V}_i$, $\mathscr{V}_i$, $\mathcal{V}_i$, $\mathcal{W}_i$, $\boldsymbol{\mathcal{V}}_i$, $\boldsymbol{\mathcal{W}}_i$, respectively, equations (2.48) and (2.49) prove that $(i) \Leftrightarrow (ii)$ holds, and equations (2.52) and (2.54) prove that $(i) \Leftrightarrow (iii)$ holds. □

2.7.5 Proof of Proposition 2.5.3

Proof. Note that the Kronecker indices are invariant under ex-equivalence. By $\Delta \overset{ex}{\sim} \Delta^{Impl}$, in our proof we can work with the Kronecker indices of Δ^{Impl} instead of those of Δ. In what follows, we will use the results of Lemma 2.7.1 given in Section 2.7.4.

(i) Recall Lemma 2.5.1(i) for Δ^{Impl} and Lemma 2.5.2(i) for Λ. For $i \in \mathbb{N}^+$, it holds that,

$$\mathscr{K}_i(\Delta^{Impl}) = \mathscr{W}_i(\Delta^{Impl}) \cap \mathscr{V}^*(\Delta^{Impl})$$

$$\overset{\text{Lemma 2.7.1}}{=} \left(\begin{bmatrix} \mathcal{W}_{i-1}(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} \right) \cap \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ F^*\mathcal{V}^*(\Lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda) \end{bmatrix} \right)$$
$$= \begin{bmatrix} \mathcal{W}_{i-1}(\Lambda) \cap \mathcal{V}^*(\Lambda) \\ F^* \left(\mathcal{W}_{i-1}(\Lambda) \cap \mathcal{V}^*(\Lambda)\right) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda) \end{bmatrix}, \tag{2.56}$$

for a suitable $F^* \in \mathbb{F}(\mathcal{V}^*(\Lambda))$. Then we have

$$a \overset{\text{Lemma 2.5.1(i)}}{=} \dim \left(\mathscr{K}_1(\Delta^{Impl})\right) \overset{(2.56)}{=} \dim \left(\begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda) \end{bmatrix} \right) = \dim \left(\mathcal{U}^*(\Lambda)\right) \overset{\text{Lemma 2.5.2(i)}}{=} a'.$$

Moreover, it is seen that for $i \in \mathbb{N}$,

$$\begin{aligned} \omega_i &\overset{\text{Lemma 2.5.1(i)}}{=} \dim \left(\mathscr{K}_{i+2}(\Delta^{Impl})\right) - \dim \left(\mathscr{K}_{i+1}(\Delta^{Impl})\right) \\ &\overset{(2.56)}{=} \dim \left(\mathcal{W}_{i+1}(\Lambda) \cap \mathcal{V}^*(\Lambda)\right) - \dim \left(\mathcal{W}_i(\Lambda) \cap \mathcal{V}^*(\Lambda)\right) \\ &= \dim \left(\mathcal{R}_{i+1}(\Lambda)\right) - \dim \left(\mathcal{R}_i(\Lambda)\right) \overset{\text{Lemma 2.5.2(i)}}{=} \omega_i'. \end{aligned}$$

Now consider equations (2.19) and (2.23) and it is sufficient to show

$$\begin{cases} \varepsilon_j = \varepsilon_j' = 0 & \text{for} \qquad 1 \le j \le a - \omega_0 = a' - \omega_0', \\ \varepsilon_j = \varepsilon_j' = i & \text{for} \quad a' - \omega_{i-1}' + 1 = a - \omega_{i-1} + 1 \le j \le a - \omega_i = a' - \omega'. \end{cases}$$

The statement that $d = d'$, $\eta_i = \eta_i'$ can be proved in a similar way using dual objects. It is not hard to see that for $i \in \mathbb{N}^+$,

$$\begin{aligned} \hat{\mathscr{K}_i}(\Delta^{Impl}) &= \left(E\mathscr{V}_{i-1}(\Delta^{Impl})\right)^\perp \cap (H\mathscr{W}^*(\Delta^{Impl}))^\perp \\ &\overset{\text{Prop. 2.4.13(i)}}{=} \mathscr{W}_i((\Delta^{Impl})^d) \cap \mathscr{V}^*((\Delta^{Impl})^d) \\ &\overset{\text{Lemma 2.7.1}}{=} \begin{bmatrix} \mathcal{W}_{i-1}(\Lambda^d) \cap \mathcal{V}^*(\Lambda^d) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda^d) \end{bmatrix}, \end{aligned}$$

where $(\Delta^{Impl})^d$ is the dual system of Δ^{Impl}, which coincides with $Impl(\Lambda^d)$. It follows that

$$d \overset{\text{Lemma 2.5.1(i)}}{=} \dim \left(\hat{\mathscr{K}_1}(\Delta^{Impl})\right) = \dim \left(\begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda^d) \end{bmatrix} \right) = \dim \left(\mathcal{Y}^*(\Lambda)\right) \overset{\text{Lemma 2.5.2(i)}}{=} d'.$$

We can also see that for $i \in \mathbb{N}$,

$$\begin{aligned} \hat{\omega}_i &= \dim \left(\hat{\mathscr{K}_{i+2}}(\Delta^{Impl})\right) - \dim \left(\hat{\mathscr{K}_{i+1}}(\Delta^{Impl})\right) \\ &= \dim \left(\mathcal{W}_{i+1}(\Lambda^d) \cap \mathcal{V}^*(\Lambda^d)\right) - \dim \left(\mathcal{W}_i(\Lambda^d) \cap \mathcal{V}^*(\Lambda^d)\right) \\ &\overset{\text{Prop.2.4.13}}{=} \dim \left((\mathcal{V}_{i+1})^\perp \cap (\mathcal{W}^*)^\perp\right) - \dim \left((\mathcal{V}_i)^\perp \cap (\mathcal{W}^*)^\perp\right) \\ &= \dim (\hat{\mathcal{R}}_{i+1}(\Lambda)) - \dim (\hat{\mathcal{R}}_i(\Lambda)) = \hat{\omega}_i'. \end{aligned}$$

Now it is sufficient to show that

$$\begin{cases} \eta_j = \eta_j' = 0 & \text{for} \qquad 1 \le j \le d - \hat{\omega}_0 = h - \hat{\omega}_0', \\ \eta_j = \eta_j' = i & \text{for} \quad h - \omega_{i-1}' + 1 = d - \hat{\omega}_{i-1} + 1 \le j \le d - \hat{\omega}_i = h - \hat{\omega}'. \end{cases}$$

(ii) Recall Lemma 2.5.1(ii) for Δ^{Impl} and Lemma 2.5.2(ii) for Λ. We have for all $i \in \mathbb{N}^+$,

$$\begin{aligned}\mathscr{V}^*(\Delta^{Impl}) + \mathscr{W}_i(\Delta^{Impl}) &\overset{\text{Lemma2.7.1}}{=} \begin{bmatrix} \mathcal{V}^*(\Lambda) \\ F^* * \mathcal{V}^*(\Lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}_i(\Lambda) \end{bmatrix} + \begin{bmatrix} \mathcal{W}_{i-1}(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{V}^*(\Lambda) + \mathcal{W}_{i-1}(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix}.\end{aligned}$$

If $\nu = 0$, then we have the following result by (2.21):

$$\begin{aligned}&\mathscr{V}^*(\Delta^{Impl}) + \mathscr{W}_0(\Delta^{Impl}) = \mathscr{V}^*(\Delta^{Impl}) + \mathscr{W}_1(\Delta^{Impl}) \Rightarrow \\ &\left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ F^*\mathcal{V}^*(\Lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda) \end{bmatrix} \right) = \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} \right) \Rightarrow \mathscr{U}(\Lambda) = \mathcal{U}^*(\Lambda).\end{aligned}$$

It follows that $c' = \dim\, (\mathscr{U}(\Lambda)) - \dim\, (\mathcal{U}^*(\Lambda)) = 0$. Therefore, in this case, the MCF^3-part of **MCF** is absent. As a consequence, if $N(s)$ of **KCF** is absent, then MCF^3 of **MCF** is absent as well. If $\nu > 0$, from (2.21) we get

$$\begin{aligned}\nu &= \min \left\{ i \in \mathbb{N}^+ \,\middle|\, \begin{bmatrix} \mathcal{V}^*(\Lambda) + \mathcal{W}_{i-1}(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} = \begin{bmatrix} \mathcal{V}^*(\Lambda) + \mathcal{W}_i(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} \right\} \\ &= \min \left\{ i \in \mathbb{N}^+ \,|\mathcal{V}^*(\Lambda) + \mathcal{W}_{i-1}(\Lambda) = \mathcal{V}^*(\Lambda) + \mathcal{W}_i(\Lambda) \right\} = \nu' + 1.\end{aligned}$$

We have

$$\begin{aligned}c = \pi_0 &= \dim\, \left(\mathscr{V}^*(\Delta^{Impl}) + \mathscr{W}_1(\Delta^{Impl})\right) - \dim\, \left(\mathscr{V}^*(\Delta^{Impl}) + \mathscr{W}_0(\Delta^{Impl})\right) \\ &\overset{\text{Lemma 2.7.1}}{=} \dim \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} \right) - \dim \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}(\Lambda) \end{bmatrix} \right) \\ &= \dim\, (\mathscr{U}(\Lambda)) - \dim\, (\mathcal{U}(\Lambda)) = c'.\end{aligned}$$

We also have for $i \in \mathbb{N}^+$,

$$\begin{aligned}\pi_i &= \dim\, \left(\mathscr{V}^*(\Delta^{Impl}) + \mathscr{W}_{i+1}(\Delta^{Impl})\right) - \dim\, \left(\mathscr{V}^*(\Delta^{Impl}) + \mathscr{W}_i(\Delta^{Impl})\right) \\ &= \dim \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) + \mathcal{W}_i(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} \right) - \dim \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) + \mathcal{W}_{i-1}(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix} \right) \\ &= \dim\, (\mathcal{W}_i(\Lambda) + \mathcal{V}^*(\Lambda)) - \dim\, (\mathcal{W}_{i-1}(\Lambda) + \mathcal{V}^*(\Lambda)) = \pi'_{i-1}.\end{aligned}$$

Now substituting $c = c'$, $\pi_i = \pi'_{i-1}$ and $\nu = \nu' + 1$ into (2.22), we can rewrite equation (2.22) as

$$\begin{cases} \sigma_j = 0 \quad \text{for} \quad 1 \le j \le c - \pi_1 = c' - \pi'_0 = \delta, \\ \sigma_j = i \quad \text{for} \quad c' - \pi'_{i-2} + 1 = c - \pi_{i-1} + 1 \le j \le c - \pi_i = c' - \pi'_{i-1}, \;\; i = 2, ..., \nu' + 1. \end{cases}$$

Replacing i by $i - 1$, we get

$$\sigma_j = i - 1 \quad for \quad c' - \pi'_{i-1} + 1 \le j \le c' - \pi'_i, \;\; i = 1, 2, ..., \nu'.$$

Finally, compare the above expression of σ_j with that for σ'_j of (2.25), it is not hard to see that $\sigma_j + 1 = \sigma'_j$ for $j = 1, \ldots, c$.

(iii) We only show that the invariant factors of MCF^2 of Λ coincide with the invariant factors of the real Jordan pencil $J(s)$ of Δ^{Impl}, then the equalities $d = h$, $\eta_1 = \eta_1', \cdots, \eta_d = \eta_h'$ are immediately satisfied. First, let two subspaces $\mathscr{X}_2 \subseteq \mathscr{V}^*(\Delta^{Impl})$ and $\mathscr{Z}_2 \subseteq \mathcal{V}^*(\Lambda)$ be such that

$$\mathscr{X}_2 \oplus \left(\mathscr{V}^*(\Delta^{Impl}) \cap \mathscr{W}^*(\Delta^{Impl})\right) = \mathscr{V}^*(\Delta^{Impl}), \quad \mathscr{Z}_2 \oplus (\mathcal{V}^*(\Lambda) \cap \mathcal{W}^*(\Lambda)) = \mathcal{V}^*(\Lambda).$$

The above construction gives $\Delta^{Impl}|\mathscr{X}_2 = KCF^2$ and $\Lambda|\mathscr{Z}_2 = MCF^2$, where KCF^2 corresponds to the Jordan pencil $J(s)$. Use Lemma 2.7.1 conclude that

$$\mathscr{X}_2 \oplus \left(\mathscr{V}^*(\Delta^{Impl}) \cap \mathscr{W}^*(\Delta^{Impl})\right) = \mathscr{V}^*(\Delta^{Impl})$$

implies

$$\begin{aligned}
&\mathscr{X}_2 \oplus \left(\left(\begin{bmatrix} \mathcal{W}^*(\Lambda) \\ * \end{bmatrix} + \begin{bmatrix} 0 \\ \mathscr{U}(\Lambda) \end{bmatrix}\right) \cap \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ F^*\mathcal{V}^*(\Lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda) \end{bmatrix}\right)\right) \\
&= \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ F^*\mathcal{V}^*(\Lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda) \end{bmatrix}\right) \\
&\Rightarrow \mathscr{X}_2 \oplus \left(\begin{bmatrix} \mathcal{W}^*(\Lambda) \cap \mathcal{V}^*(\Lambda) \\ F'\left(\mathcal{W}^*(\Lambda) \cap \mathcal{V}^*(\Lambda)\right) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda) \end{bmatrix}\right) = \left(\begin{bmatrix} \mathcal{V}^*(\Lambda) \\ F^*\mathcal{V}^*(\Lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U}^*(\Lambda) \end{bmatrix}\right),
\end{aligned}$$

where $F \in \mathbb{F}(\mathcal{V}^*(\Lambda))$, $F' \in \mathbb{F}(\mathcal{W}^*(\Lambda) \cap \mathcal{V}^*(\Lambda))$. Since $\mathscr{Z}_2 \oplus \mathcal{V}^*(\Lambda) \cap \mathcal{W}^*(\Lambda) = \mathcal{V}^*(\Lambda)$, we have $\mathscr{X}_2 = \begin{bmatrix} \mathscr{Z}_2 \\ F''\mathscr{Z}_2 \end{bmatrix}$, where $F'' \in \mathbb{F}(\mathscr{Z}_2)$. Then, it follows that

$$\begin{aligned}
\begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix}\Bigg|_{\mathscr{X}_2} &= \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} \mathscr{Z}_2 \\ F''\mathscr{Z}_2 \end{bmatrix} = \begin{bmatrix} (sI - (A + BF''))\,\mathscr{Z}_2 \\ (C + DF'')\mathscr{Z}_2 \end{bmatrix} \\
&= \begin{bmatrix} (sI - (A + BF''))\,\mathscr{Z}_2 \\ 0 \end{bmatrix}.
\end{aligned}$$

Now it is known from Lemma 4.1 of [146] that $(A + BF'')|\mathscr{Z}_2$ does not dependent on the choice of F''. Thus the invariant factors of $(sI - (A + BF''))\,\mathscr{Z}_2$ coincide with the invariant factors of MCF^2 for Λ. Finally, from the above equation, it is easy to see that the invariant factors of $J(s)$ in **KCF** of Δ coincide with those of MCF^2 of Λ. □

2.7.6 Proof of Proposition 2.6.7

Proof. (i) By Proposition 2.6.6, $\mathscr{M}$ is an invariant subspace if and only if $H\mathscr{M} \subseteq E\mathscr{M}$. Therefore, $\mathscr{M}^*$ is the largest subspace such that $H\mathscr{M}^* \subseteq E\mathscr{M}^*$, then by Proposition 2.4.4(ii), we have $\mathscr{M}^* = \mathscr{V}^*$.

(ii) By Proposition 2.6.6, for $\Delta|^{red}_{\mathscr{M}^*} = (E|^{red}_{\mathscr{M}^*}, H|^{red}_{\mathscr{M}^*})$, the matrix $E|^{red}_{\mathscr{M}^*}$ is of full row rank. Thus from the explicitation procedure, it is straightforward to see that $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{M}^*})$ is a control system without outputs. Note that, by the definitions of reduction and restriction, if two DAEs $\Delta \overset{ex}{\sim} \tilde{\Delta}$, then $\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*}$. In the following, without loss of generality, we assume Δ is in its **KCF** (which is invariant under ex-equivalence). Denote the four parts of the **KCF** of Δ as KCF^k, $k = 1, \ldots, 4$ and the corresponding matrix pencil of each part is:

$$L(s) \text{ for } KCF^1, \quad J(s) \text{ for } KCF^2, \quad N(s) \text{ for } KCF^3, \quad L^p(s) \text{ for } KCF^4.$$

Thus $\Lambda \in Expl(\Delta)$ is in the **MCF**. It is easily seen that

$$\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)} = \left(MCF^1, MCF^2\right),$$

which can be seen as a control system without outputs. From the one-to-one correspondence of the **KCF** and the **MCF** discussed in Section 2.5, it is straightforward to see that $(MCF^1, MCF^2) \in Expl(KCF^1, KCF^2)$, which implies

$$\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)} = \left(MCF^1, MCF^2\right) \in Expl(\Delta|^{red}_{\mathscr{V}^*}).$$

Since $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{V}^*})$, by Theorem 2.3.4(ii), we have $\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)} \overset{M}{\sim} \Lambda^*$. Finally, since Λ^* and $\Lambda|^{red}_{(\mathcal{V}^*,\mathcal{U}^*)}$ are two control systems without outputs, their Morse equivalence reduces to their feedback equivalence (see Remark 2.2.4) □

2.7.7 Proof of Theorem 2.6.10

Proof. $(i) \Leftrightarrow (ii)$: By Definition 2.6.8, we have $\Delta \overset{in}{\sim} \tilde{\Delta}$ if and only if $\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \tilde{\Delta}|^{red}_{\mathscr{M}^*}$. Consider $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{M}^*})$ and $\tilde{\Lambda}^* \in Expl(\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*})$, then by Theorem 2.3.4(ii), it follows that $\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \tilde{\Delta}|^{red}_{\mathscr{M}^*}$ if and only if $\Lambda^* \overset{M}{\sim} \tilde{\Lambda}^*$. Thus by Proposition 2.6.7(ii), Λ^* and $\tilde{\Lambda}^*$ are two control systems without outputs, which implies that their Morse equivalence reduces to their feedback equivalence (see Remark 2.2.4).

$(ii) \Leftrightarrow (iii)$: We first prove that two DAEs $\Delta^* = Impl(\Lambda^*)$ and $\tilde{\Delta}^* = Impl(\tilde{\Lambda}^*)$ have isomorphic trajectories if and only if Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent. Let $(z(t), u(t))$ and $(\tilde{z}(t), \tilde{u}(t))$ denote trajectories of Δ^* and $\tilde{\Delta}^*$, respectively. Suppose Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent, then there exist matrices $T_s \in Gl(n^*, \mathbb{R})$, $T_i \in Gl(m^*, \mathbb{R})$, $F \in \mathbb{R}^{m^* \times n^*}$ such that $\tilde{A}^* = T_s(A^* + B^*F)T_s^{-1}$, $\tilde{B}^* = T_s B T_i^{-1}$. Since Λ^* has no output, its implicitation (see Definition 2.3.1) is

$$\Delta^* : \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A^* & B^* \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}.$$

For $\tilde{\Lambda}^*$, its implicitation is

$$\tilde{\Delta}^* : \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{u}} \end{bmatrix} = \begin{bmatrix} \tilde{A}^* & \tilde{B}^* \end{bmatrix} \begin{bmatrix} \tilde{z} \\ \tilde{u} \end{bmatrix} \Rightarrow \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{u}} \end{bmatrix} = T_s \begin{bmatrix} A^* & B^* \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix} \begin{bmatrix} \tilde{z} \\ \tilde{u} \end{bmatrix}.$$

It can be seen that any trajectory $(z(t), u(t))$ of Δ^* satisfying $z(0) = z^0$ and $u(0) = u^0$, is mapped via $T = \begin{bmatrix} T_s^{-1} & 0 \\ FT_s^{-1} & T_i^{-1} \end{bmatrix}^{-1}$ into a trajectory $(\tilde{z}(t), \tilde{u}(t))$ of $\tilde{\Delta}^*$ passing through $\begin{bmatrix} \tilde{z}^0 \\ \tilde{u}^0 \end{bmatrix} = T \begin{bmatrix} z^0 \\ u^0 \end{bmatrix}$.

Conversely, suppose that there exists an invertible matrix $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$ such that $\begin{bmatrix} \tilde{z}(t) \\ \tilde{u}(t) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}$. It follows that $(\tilde{z}(t), \tilde{u}(t))$, being a solution of $\tilde{\Delta}^*$, satisfies

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{pmatrix} \dot{\tilde{z}}(t) \\ \dot{\tilde{u}}(t) \end{pmatrix} = \begin{bmatrix} \tilde{A}^* & \tilde{B}^* \end{bmatrix} \begin{pmatrix} \tilde{z}(t) \\ \tilde{u}(t) \end{pmatrix},$$

which implies

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{pmatrix} \dot{z}(t) \\ \dot{u}(t) \end{pmatrix} = \begin{bmatrix} \tilde{A}^* & \tilde{B}^* \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}.$$

Since $(z(t), u(t))$ satisfies $\dot{z}(t) = A^* z(t) + B^* u(t)$, it follows that

$$\begin{aligned} T_1 \dot{z}(t) + T_2 \dot{u}(t) &= (\tilde{A}^* T_1 + \tilde{B}^* T_3) z(t) + (\tilde{A}^* T_2 + \tilde{B}^* T_4) u(t) \Rightarrow \\ T_1(A^* z(t) + B^* u(t)) + T_2 \dot{u}(t) &= (\tilde{A}^* T_1 + \tilde{B}^* T_3) z(t) + (\tilde{A}^* T_2 + \tilde{B}^* T_4) u(t). \end{aligned} \tag{2.57}$$

Notice that equation (2.57) is satisfied for any solution $(z(t), u(t))$ of Δ^*. (a). Let $u(t) \equiv 0$ and $(z(t, z^0), 0)$ (where $z^0 \neq 0$) be a solution of Δ^* (obviously, such a solution always exists). By substituting this solution into (2.57) and considering it for $t = 0$, we have $T_1 A^* z^0 = (\tilde{A}^* T_1 + \tilde{B}^* T_3) z^0$, where $z^0 = z(0)$ can be taken arbitrary, which implies $A^* = T_1^{-1}(\tilde{A}^* + \tilde{B}^*(T_3 T_1^{-1})) T_1$. (b). Fix $z(0) = z^0 = 0$ and set $u(t) = u^i(t) = [0, \ldots, t, \ldots, 0]^T$, where t is in the i-th row. Evaluating at $t = 0$, we have $z(0) = 0$, $u(0) = 0$ and $\dot{u}^i(0) = [0, \ldots, 1, \ldots, 0]^T$, and thus by (2.57) we have $T_2 \dot{u}^i(0) = 0$. So taking control $u^1(t), \ldots, u^{m^*}(t)$ of that form, we conclude that $T_2 = 0$. Now it is easy to see from (2.57) that $B^* = T_1^{-1} \tilde{B}^* T_4$. Thus Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent (see Remark 2.2.4) via $T_s = T_1$, $T_i = T_4^{-1}$ and $F = T_3 T_1^{-1}$. Therefore, any trajectory of Δ^* is transformed via T into a trajectory of $\tilde{\Delta}^*$ if and only if Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent.

Then by Theorem 2.3.4(i), we have

$$\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \Delta^* = Impl(\Lambda^*) \text{ and } \tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*} \overset{ex}{\sim} \tilde{\Delta}^* = Impl(\tilde{\Lambda}^*)$$

(since $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{M}^*})$ and $\tilde{\Lambda}^* \in Expl(\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*})$). Moreover, by Remark 2.2.2, there exist matrices $P \in Gl(n^*, \mathbb{R})$ and $\tilde{P} \in Gl(n^*, \mathbb{R})$ such that any trajectory of $\Delta|^{red}_{\mathscr{M}^*}$ is mapped via P into the corresponding trajectory of Δ^* and any trajectory of $\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*}$ is mapped via $\tilde{P}$ into the corresponding trajectory of $\tilde{\Delta}^*$. Now we can conclude that the linear and invertible map $S = PT\tilde{P}^{-1}$ sends any trajectory of $\Delta|^{red}_{\mathscr{M}^*}$ into the corresponding trajectory of $\tilde{\Delta}|^{red}_{\tilde{\mathscr{M}}^*}$ if and only if Λ^* and $\tilde{\Lambda}^*$ are feedback equivalent. □

2.7.8 Proof of Proposition 2.6.12

Proof. $(i) \Leftrightarrow (ii)$: Consider a DAE $\Delta^* = Impl(\Lambda^*)$. We have $\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \Delta^*$ (implied by $\Lambda^* \in Expl(\Delta|^{red}_{\mathscr{M}^*})$ and Theorem 2.3.4(i)), we get $\Delta|^{red}_{\mathscr{M}^*} \overset{ex}{\sim} \Delta^*$. Actually, since Λ^* is defined on $\mathscr{M}^*$, it follows from Definition 2.6.8 that $\Delta|^{red}_{\mathscr{M}^*} \overset{in}{\sim} \Delta^* = Impl(\Lambda^*)$. Thus by the equivalence of item (i) and (iii) of Theorem 2.6.10, the solutions of Δ passing through $x^0 \in \mathscr{M}^*$ are mapped, via a certain linear isomorphism S, into the solutions of Δ^*, which means that Δ is internally regular if and only if Δ^* has only one solution passing through any initial point in $\mathscr{M}^*$. This is true if and only if the input of Λ^* is absent, i.e., Δ^* is an ODE without free variables. Therefore, Δ is internally regular if and only if Λ^* has no inputs.

$(ii) \Leftrightarrow (iii) \Leftrightarrow (vi)$: From the proof of Proposition 2.6.7(ii), we can see that the input is absent in Λ^* if and only if $\Lambda^* = MCF^2$ of Λ, that is, MCF^1 is absent in the **MCF** of Λ.

$(i) \Leftrightarrow (iv) \Leftrightarrow (v)$: Using $\mathscr{V}^* = \mathscr{M}^*$ and the **KCF** of Δ, it is straightforward to see this equivalence. □

2.8 Conclusion

In this chapter, we propose a procedure named explicitation for DAEs. The explicitation of a DAE is, simply speaking, attaching to the DAE a class of linear control systems defined up to a coordinates change, a feedback and an output injection. We prove that the invariant subspaces of the attached control systems have direct relations with the limits of the Wong sequences of the DAE. We show that the Kronecker indices of the DAE have direct relations with the Morse indices of the attached control systems, and as a consequence, the Kronecker canonical form **KCF** of the DAE and the Morse canonical from **MCF** of control systems have a perfect correspondence. We also propose a notion named internal equivalence for DAEs and show that the internal equivalence is useful when analyzing the existence and uniqueness of solutions (internal regularity).

2.9 Appendix

Kronecker Canonical Form (KCF) [117],[75]: For any matrix pencil $sE - H \in \mathbb{R}^{l\times n}[s]$, there exist matrices $Q \in Gl(l,\mathbb{R})$, $P \in Gl(n,\mathbb{R})$ and integers $\varepsilon_1, ..., \varepsilon_a \in \mathbb{N}, \rho_1, ..., \rho_b \in \mathbb{N}, \sigma_1, ..., \sigma_c \in \mathbb{N}, \eta_1, ..., \eta_d \in \mathbb{N}$ with $a, b, c, d \in \mathbb{N}$ such that

$$Q(sE - H)P^{-1} =$$
$$\operatorname{diag}\left(L_{\varepsilon_1}(s), ..., L_{\varepsilon_a}(s), J_{\rho_1}(s), ..., J_{\rho_a}(s), N_{\sigma_1}(s), ..., N_{\sigma_c}(s), L^p_{\eta_1}(s), ..., L^p_{\eta_d}(s)\right),$$

where (omitting, for simplicity, the index i of $\varepsilon_i, \rho_i, \sigma_i, \eta_i$) the bidiagonal pencil $L_\varepsilon(s) \in \mathbb{R}^{\varepsilon\times(\varepsilon+1)}[s]$, the real Jordan pencil $J_\rho(s) \in \mathbb{R}^{\rho\times\rho}[s]$, the nilpotent pencil $N_\sigma(s) \in \mathbb{R}^{\sigma\times\sigma}[s]$

and the "per-transpose" pencil $L^p_\eta(s) \in \mathbb{R}^{\eta\times(\eta+1)}[s]$ have the following form:

$$L_\varepsilon(s) = \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}, \quad N_\sigma(s) = \begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix}, \quad L^p_\eta(s) = \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix},$$

$$J_\rho(s) = \begin{bmatrix} s-\lambda_\rho & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & s-\lambda_\rho \end{bmatrix}, \text{ or } J_\rho(s) = \begin{bmatrix} S-\Lambda_\rho & -I & & \\ & \ddots & \ddots & \\ & & \ddots & -I \\ & & & S-\Lambda_\rho \end{bmatrix}, \; S-\Lambda_\rho = \begin{bmatrix} s-\phi_\rho & -\varphi_\rho \\ \varphi_\rho & s-\phi_\rho \end{bmatrix}.$$

where λ_ρ, φ_ρ, $\phi_\rho \in \mathbb{R}$. The integers ε_i, ρ_i, σ_i, η_i are, respectively, called Kronecker column (minimal) indices, the degrees of the finite elementary divisors, the degrees of the infinite elementary divisors and ,Kronecker row (minimal) indices. In addition, λ_ρ and $\varphi_\rho + i\phi_\rho$ are the corresponding eigenvalues of $J(s)$. These indices are invariant under external equivalence of Definition 2.2.1.

Definition 2.9.1. (Prime system) [145] A control system $\Lambda = (A, B, C, D)$ is called prime if there exists a Morse transformation M_{tran} such that $M_{tran}(\Lambda) = \Lambda^3 = (A^3, B^3, C^3, D^3)$, where the 4-tuple (A^3, B^3, C^3, D^3) is given by (2.58) below.

Lemma 2.9.2. [145] *A control system $\Lambda = (A, B, C, D)$ is prime if and only if*

$$\mathcal{W}^* = \mathscr{X}, \;\; \mathcal{Y}^* = \mathscr{Y}, \;\; \mathcal{V}^* = 0, \;\; \mathcal{U}^* = 0.$$

Morse Canonical Form MCF [146],[145]: Any linear control system $\Lambda = (A, B, C, D)$ is Morse equivalent to the Morse canonical form **MCF** shown below:

$$\textbf{MCF}: \begin{cases} MCF^1: \dot z^1 = A^1 z^1 + B^1 u^1 \\ MCF^2: \dot z^2 = A^2 z^2 \\ MCF^3: \dot z^3 = A^3 z^3 + B^3 u^3, \quad y^3 = C^3 z^3 + D^3 u^3 \\ MCF^4: \dot z^4 = A^4 z^4, \qquad\qquad\quad y^4 = C^4 z^4. \end{cases}$$

If a control system $\Lambda = (A, B, C, D)$ is in the **MCF**, then the matrices A, B, C, D, together with all invariants are thus given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cccc|cc} A^1 & 0 & 0 & 0 & B^1 & 0 \\ 0 & A^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & A^3 & 0 & 0 & B^3 \\ 0 & 0 & 0 & A^4 & 0 & 0 \\ \hline 0 & 0 & C^3 & 0 & 0 & D^3 \\ 0 & 0 & 0 & C^4 & 0 & 0 \end{array}\right],$$

(i) with $A^1 = \mathrm{diag}\{A^1_{\varepsilon'_1}, ..., A^1_{\varepsilon'_{a'}}\}$, $B^1 = \mathrm{diag}\{B^1_{\varepsilon'_1}, ..., B^1_{\varepsilon'_{a'}}\}$, where

$$A^1_{\varepsilon'} = \begin{bmatrix} 0 & I_{\varepsilon'-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\varepsilon'\times\varepsilon'}, \quad B^1_{\varepsilon'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\varepsilon'\times 1},$$

The integers $\varepsilon'_1, ..., \varepsilon'_{a'} \in \mathbb{N}$ are the controllability indices of (A^1, B^1).

(ii) $A^2 = \operatorname{diag}\{A^2_{\rho'_1}, ..., A^1_{\rho'_{b'}}\}$, where A^2_i is given by

$$A^2_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \quad or \quad A^2_i = \begin{bmatrix} \Lambda_i & I & & \\ & \ddots & \ddots & \\ & & \ddots & I \\ & & & \Lambda_i \end{bmatrix}, \quad \Lambda_i = \begin{bmatrix} s-\phi_i & -\varphi_i \\ \varphi_i & s-\phi_i \end{bmatrix},$$

where $\lambda_i, \varphi_i, \phi_i \in \mathbb{R}$.

(iii) The 4-tuple (A^3, B^3, C^3, D^3) is controllable and observable (prime). That is,

$$\begin{bmatrix} A^3 & B^3 \\ C^3 & D^3 \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A}^3 & \hat{B}^3 & 0 \\ \hline \hat{C}^3 & 0 & 0 \\ 0 & 0 & I_\delta \end{array}\right], \tag{2.58}$$

where $\begin{bmatrix} \hat{A}^3 & \hat{B}^3 \\ \hat{C}^3 & 0 \end{bmatrix}$ is square and invertible and $\delta = \operatorname{rank} D^3 \in \mathbb{N}$, and the matrices

$$\hat{A}^3 = \operatorname{diag}\{\hat{A}^3_{\sigma'_{\delta+1}}, ..., \hat{A}^3_{\sigma'_{c'}}\}, \quad \hat{B}^3 = \operatorname{diag}\{\hat{B}^3_{\sigma'_{\delta+1}}, ..., \hat{B}^3_{\sigma'_{c'}}\}, \quad \hat{C}^3 = \operatorname{diag}\{\hat{C}^3_{\sigma'_{\delta+1}}, ..., \hat{C}^3_{\sigma'_{c'}}\},$$

where

$$\hat{A}^3_{\sigma'} = \begin{bmatrix} 0 & I_{\sigma'-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\sigma' \times \sigma'}, \quad \hat{B}^3_{\sigma'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\sigma' \times 1}, \quad \hat{C}^3_{\sigma'} = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1 \times \sigma'}.$$

The integers $\sigma_1 = \cdots = \sigma_\delta = 0$, and $\sigma_{\delta+1}, ..., \sigma_{c'} \in \mathbb{N}^+$ are the controllability indices of the pair $(\hat{A}^3, \hat{B}^3)$ and they are equal to the observability indices of the pair $(\hat{C}^3, \hat{A}^3)$.

(iv) $A^4 = \operatorname{diag}\{A^4_{\eta'_1}, ..., A^4_{\eta'_{d'}}\}$, $C^4 = \operatorname{diag}\{C^4_{\eta'_1}, ..., C^4_{\eta'_{d'}}\}$, where

$$A^4_{\eta'} = \begin{bmatrix} 0 & I_{\eta'-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\eta' \times \eta'}, \quad C^4_{\eta'} = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1 \times \eta'}.$$

The integers $\eta'_1, ..., \eta'_{d'} \in \mathbb{N}$ are the observability indices of the pair (C^4, A^4).

Clearly, the subsystem MCF^2 is in the real Jordan canonical form. Denote $\mu_i = \epsilon'_i$ if $k = 1$, $\mu_i = \sigma'_i$ if $k = 3$, and $\mu_i = \eta'_i$ if $k = 4$. Then for $k = 1, 3, 4$, the subsystem MCF^k consists of a', c', d', subsystems (indexed by i) for which either $\mu_i \geq 1$ and then they are given by

$$\dot{z}^{k,j}_i = \begin{cases} z^{k,j+1}_i, & 1 \leq j \leq \mu_i - 1, \quad \text{for } k = 1, 3, 4, \\ u^k_i, & j = \mu_i, \quad \text{for } k = 1, 3, \\ 0, & j = \mu_i, \quad \text{for } k = 4, \end{cases} \qquad y^k_i = z^{k,1}_i, \text{ for } k = 3, 4,$$

or $\mu_i = 0$ (notice that we allow for the Morse indices to be equal to zero) in which case the input u^1 contains components u^1_i that do not affect the system at all (if $\epsilon'_i = 0$), the output y^4 contains trivial components $y^4_i = 0$ (if $\eta'_i = 0$) and the output y^3 contains $\delta = \operatorname{rank} D^3$ static relations $y^3_i = u^3_i$ (if $\sigma'_i = 0$).

We call the integers ε'_i, ρ'_i, σ'_i, η'_i the Morse indices of control systems, together with a', b', c', d', δ and $\lambda_i \in \mathbb{R}$ or $\lambda_i = \varphi + j\phi \in \mathbb{C}$, where $j = \sqrt{-1}$, they are all invariant under Morse equivalence.

Chapter 3

From Morse Triangular Form of ODE Control Systems to Feedback Canonical Form of DAE Control Systems

Abstract: In this chapter, we study connections between the feedback canonical form **FBCF** of DAE control systems, shortly DAECSs, proposed in [131] and the famous Morse canonical form **MCF** of ODE control systems ODECSs, see [146],[145]. First, in order to connect DAECSs with ODECSs, we propose a procedure named explicitation (with driving variables). This procedure attaches a class of ODECSs with two kinds of inputs (the original control input and a vector of driving variables) to a given DAECS. On the other hand, for classical linear ODECSs (with one type of controls), we propose a Morse triangular form **MTF** to modify the construction of the **MCF** given in [145]. Based on this **MTF**, we propose an extended **MTF** and an extended **MCF** for ODECSs with two kinds of inputs. Finally, an algorithm is proposed to transform a given DAECS to its **FBCF**. This algorithm is based on the extended **MCF** of an ODECS given by the explicitation procedure. At last, a numerical example is given to show the efficiency of the proposed algorithm.

3.1 Introduction

Consider a linear control system described by a differential-algebraic equation DAE of the following form:

$$\Delta^u : E\dot{x} = Hx + Lu, \tag{3.1}$$

where $x \in \mathscr{X} \cong \mathbb{R}^n$ is called the "generalized" state, $u \in \mathbb{R}^m$ is the vector of control inputs, and where $E \in \mathbb{R}^{l\times n}$, $H \in \mathbb{R}^{l\times n}$ and $L \in \mathbb{R}^{l\times m}$. A linear DAE control system DAECS of form (3.1) will be denoted by $\Delta^u_{l,n,m} = (E, H, L)$ or, simply, Δ^u. The motivation of studying DAECSs comes from the mathematical models of such constrained dynamical systems as electrical circuits [63],[176], mechanical systems [159],[143], chemical processes [60],[121], etc.

In order to connect DAEs of the form $E\dot{x} = Hx$ with ODE control systems, shortly ODECSs, and analyze DAEs using classical control theory, we proposed a procedure named explicitation in Chapter 2, see also [47]. In the present chapter, we will propose a more general explicitation procedure called *explicitation with driving variables* (see Definition 3.2.2) for linear DAECSs. Since the vector of driving variables enters statically into the system (similarly as the control input u), we can regard it as another kind of input. More specifically, the *explicitation with driving variables* of a DAECS is a class of ODECSs with two kinds of inputs of the form:

$$\Lambda^{uv} : \begin{cases} \dot{x} = Ax + B^u u + B^v v \\ y = Cx + D^u u, \end{cases} \tag{3.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B^u \in \mathbb{R}^{n\times m}$, $B^v \in \mathbb{R}^{n\times s}$, $C \in \mathbb{R}^{p\times n}$ and $D^u \in \mathbb{R}^{p\times m}$, where $u \in \mathbb{R}^m$ is the vector of control variables and $v \in \mathbb{R}^s$ is the vector of driving variables. An ODECS of form (3.2) will be denoted by $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$ or, simply, Λ^{uv}. Note that although both u and v may be considered as inputs of system (3.2), we distinguish them because they play different roles for the system and, as a consequence, their feedback transformation rules are different (see Remark 3.2.7). A classical ODECS (with a control input only) is of the form

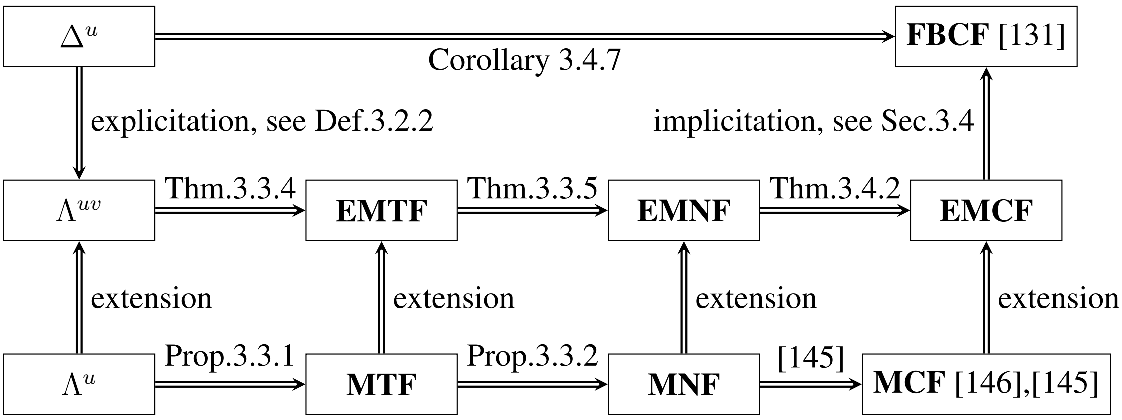
$$\Lambda^{u} : \begin{cases} \dot{x} = Ax + B^u u \\ y = Cx + D^u u, \end{cases} \tag{3.3}$$

where $A \in \mathbb{R}^{n\times n}$, $B^u \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D^u \in \mathbb{R}^{p\times m}$. An ODECS of form (3.3) will be denoted by $\Lambda^{u}_{n,m,p} = (A, B^u, C, D^u)$ or, simply, Λ^u.

Remark 3.1.1. Observe that we can express an ODECS Λ^{uv} of form (3.2), as a classical ODECS $\Lambda^w = (A, B^w, C, D^w)$ of form (3.3) by denoting $w = [u^T, v^T]^T$, $B^w = \begin{bmatrix} B^u & B^v \end{bmatrix}$ and $D^w = \begin{bmatrix} D^u & 0 \end{bmatrix}$. Throughout the chapter, depending on the context, we will use either Λ^{uv} or Λ^w to denote an ODECS with two kinds of inputs.

The feedback canonical form **FBCF** obtained in [131] (we restate it as Corollary 3.4.7 of the present chapter) for linear DAECSs plays an important role in DAECS theory, e.g. controllability analysis [17], regularization [32],[18], pole assignment [132],[27] are discussed based on this **FBCF**. The purpose of the present chapter is to find an efficient geometric way to transform a DAECS Δ^u into its **FBCF** via the explicitation procedure. More specifically, instead of using transformations directly on a DAECS, we will first transform an ODECS, given by the explicitation of our DAECS, into its canonical form (called the extended Morse canonical form **EMCF**, see Theorem 3.4.2). Then by the relation between DAECSs and ODECSs given in Section 3.2, we can easily get the **FBCF** from the **EMCF**.

The **FBCF** of DAECSs is actually an extension of the Kronecker canonical form (see [117],[75]) of singular matrix pencils $sE - H$. Some methods (most are numerical) of transforming a matrix pencil into its Kronecker canonical form can be found in [62],[184],[10]. The authors of [20] proposed recently a geometric method to get a quasi-Kronecker triangular form for singular matrix pencils based on the Wong sequences and there the quasi-Kronecker triangular form is transformed into the quasi-Kronecker form by solving some generalized Sylvester equations. Inspired by the quasi-Kronecker triangular form of [20], we will propose a Morse triangular form **MTF** (see Proposition 3.3.1) to transform an ODECS (with one type of controls) into its Morse normal form **MNF** (see Proposition 3.3.2). Then we show that the **MTF** can be easily generalized to an extended Morse triangular form **EMTF** for ODECS with two kinds of inputs. After solving some constrained Sylvester equations, the transformations from the **EMTF** to an extended Morse canonical form **EMCF** are also easy to construct. We use the following diagram to show the relations of the results in the present chapter:



Note that a procedure of transforming an ODECS Λ^u into its **MCF** was given by Morse [146] for $D^u = 0$ and by Molinari [145] for the general case $D^u \neq 0$. We propose to do it via two intermediate normal forms **MTF** and **MNF**.

This chapter is organized as follows. In Section 3.2, we introduce the explicitation with driving variables procedure and build geometric connections between DAECSs and ODECSs. In Section 3.3, we show a method of constructing the **MTF** and the **MNF** for classical ODECSs of form (3.3), then we extend them to the **EMTF** and the **EMNF** for ODECSs (of form (3.2)) with two kinds of inputs. In Section 3.4, we propose an **EMCF** for ODECSs of form (3.2) and show a way of calculating its indices via invariant subspaces. These results allow to construct the **FBCF** of DAECSs and to calculate the **FBCF** indices as corollaries. Finally, a simple algorithm is proposed to construct the **FBCF** for a given DAECS. In Section 3.5, we give a numerical example to show the effectiveness of the algorithm. Section 3.6 and 3.7 contain conclusions and proofs of this chapter, respectively. Notations and definitions of geometric invariant subspaces for ODCSs and DAECSs are given in Appendix.

3.2 Explicitation with driving variables for linear DAE control systems

Throughout, we will use the notations given in Appendix. Consider a DAECS $\Delta^u_{l,n,m} = (E, H, L)$, given by (3.1). The solution of Δ^u is a map $(x(t), u(t)) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m$ with

$x(t) \in \mathcal{C}^1$ and $u(t) \in \mathcal{C}^0$ satisfying $E\dot{x}(t) = Hx(t) + Lu(t)$.

Definition 3.2.1. Two DAECSs $\Delta^u_{l,n,m} = (E, H, L)$ and $\tilde{\Delta}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{H}, \tilde{L})$ are called externally feedback equivalent, shortly ex-fb-equivalent, if there exist matrices $Q \in Gl(l, \mathbb{R})$, $P \in Gl(n, \mathbb{R})$, $F \in \mathbb{R}^{m\times n}$ and $G \in Gl(m, \mathbb{R})$ such that

$$\tilde{E} = QEP^{-1}, \quad \tilde{H} = Q(H + LF)P^{-1}, \quad \tilde{L} = QLG. \tag{3.4}$$

We denote the ex-fb-equivalence of two DAECSs as $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$.

The notion of ex-fb-equivalence is the classical equivalence of DAECSs via left multiplication by Q and right multiplication by P^{-1}, completed by feedback transformations of the controls via $u = Fx + G\tilde{u}$. Now we introduce the *explicitation with driving variables* procedure for Δ^u as follows.

- Denote the rank of E by $q \in \mathbb{N}$, define $s = n - q$ and $p = l - q$. Then there exists a matrix $Q \in Gl(l, \mathbb{R})$ such that $QE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}$, where $E_1 \in \mathbb{R}^{q\times n}$ and $\operatorname{rank} E_1 = q$. Via Q, DAECS Δ^u is ex-fb-equivalent to

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} x + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} u, \tag{3.5}$$

  where $QH = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$, $QL = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, and where $H_1 \in \mathbb{R}^{q\times n}$, $H_2 \in \mathbb{R}^{(l-q)\times n}$, $L_1 \in \mathbb{R}^{q\times m}$, $L_2 \in \mathbb{R}^{(l-q)\times m}$.

- The matrix E_1 is of full row rank q, so let $E_1^\dagger \in \mathbb{R}^{n\times q}$ denote its right inverse. Set $A = E_1^\dagger H_1$ and $B^u = E_1^\dagger L_1$. Consider the differential part of (3.5):

$$E_1\dot{x} = H_1x + L_1u. \tag{3.5a}$$

  The collection of all $\dot{x}$ satisfying (3.5a) is given by the differential inclusion:

$$\dot{x} \in Ax + B^u u + \ker E_1. \tag{3.6}$$

- Choose a full column rank matrix $B^v \in \mathbb{R}^{n\times s}$ such that $\operatorname{Im} B^v = \ker E_1 = \ker E$ (note that the kernels of E_1 and E coincide since any invertible Q preserves the kernel). Thus, by (3.6), there exists a vector of *driving variables* $v \in \mathbb{R}^s$ parameterizing the affine subspace $Ax + B^u u + \ker E_1$ and all solutions of the differential inclusion (3.6) correspond to all solutions of

$$\dot{x} = Ax + B^u u + B^v v. \tag{3.7}$$

  Observe that the columns of B^v span the subspace $\ker E$ with the help of driving variables v. Now all solutions of DAE (3.5) can be expressed as all solutions (corresponding to all controls $v(t)$) of

$$\begin{cases} \dot{x} = Ax + B^u u + B^v v \\ 0 = Cx + D^u u, \end{cases} \tag{3.8}$$

where $C = H_2 \in \mathbb{R}^{p\times n}$ and $D^u = L_2 \in \mathbb{R}^{p\times m}$. Recall that a control system of form (3.2) is denoted by $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$. It is immediate see that equation (3.8) can be obtained from the ODECS Λ^{uv} by setting the output $y = 0$. In the above way, we attach an ODECS Λ^{uv} to a DAECS Δ^u.

The above procedure of attaching a control system $\Lambda^{u,v}$ to a DAECS Δ^u will be called *explicitation with driving variables* and is formalized as follows.

Definition 3.2.2. Given a DAECS $\Delta^u_{l,n,m} = (E, H, L)$, by a (Q, v)-explicitation, we will call a control system $\Lambda^{uv} = (A, B^u, B^v, C, D^u)$, with

$$A = E_1^\dagger H_1,\ \ B^u = E_1^\dagger L_1,\ \ \operatorname{Im} B^v = \ker E_1 = \ker E,\ \ C = H_2,\ \ D^u = L_2.$$

where

$$QE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix},\ \ QH = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},\ \ QL = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.$$

The class of all (Q, v)-explicitations will be called the explicitation with driving variables class or, shortly explicitation class, of Δ^u, denoted by $\mathbf{Expl}(\Delta^u)$. If a particular ODECS Λ^{uv} belongs to the explicitation class $\mathbf{Expl}(\Delta^u)$, we will write $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$.

The definition of the explicitation class $\mathbf{Expl}(\Delta^u)$ suggests that a given Δ^u has many (Q, v)-explicitations. Indeed, the construction of $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$ is not unique at three stages: there is a freedom in choosing Q, $E_1^\dagger$, and B^v. Notice that the choices of B^v and $E_1^\dagger$ can be seen together as a choice of a driving variable v to express (3.5a) explicitly.

Now we will analyze these three choices. We start with B^v. Choosing B^v and $\tilde{B}^{\tilde{v}}$ such that $\operatorname{Im} B^v = \operatorname{Im} \tilde{B}^v = \ker E$ means that there exists $T_v \in Gl(s, \mathbb{R})$ such that $\tilde{B}^{\tilde{v}} = T_v^{-1} B_v$ or, equivalently, $\tilde{v} = T_v v$. To analyze the role of the choice of $E_1^\dagger$, fix B^v, consider the differential part (3.5a) of the semi-explicit system (3.5). Any (Q, v)-explicitation of (3.5a) is a control system without outputs, so we will denote it by $\Lambda^{uv}_{n,m,s,0} = (A, B^u, B^v)$.

Proposition 3.2.3. *Assume that a control system $\Lambda^{uv}_{n,m,s,0} = (A, B^u, B^v)$ is a (Q, v)-explicitation of (3.5a) corresponding to a choice of right inverse $E_1^\dagger$ of E_1. Then a control system $\tilde{\Lambda}^{u\tilde{v}}_{n,m,s,0} = (\tilde{A}, \tilde{B}^u, \tilde{B}^{\tilde{v}})$ is a $(Q, \tilde{v})$-explicitation of (3.5a) corresponding to another choice of right inverse $\tilde{E}_1^\dagger$ of E_1 with the same choice $B^v = \tilde{B}^{\tilde{v}}$ yielding* $\operatorname{Im} B^v = \operatorname{Im} \tilde{B}^{\tilde{v}} = \ker E_1$ *if and only if Λ^{uv} and $\tilde{\Lambda}^{u\tilde{v}}$ are equivalent via a v-feedback transformation of the form $v = F_v x + Ru + \tilde{v}$, which maps*

$$A \mapsto \tilde{A} = A + B^v F_v,\ \ B^u \mapsto \tilde{B}^u = B^u + B^v R.$$

To analyze the role of choosing Q, go back to the start of the explicitation procedure to find an invertible Q such that Δ^u is transformed to a DAE of form (3.5). Notice that Q is an invertible matrix such that E_1 of $QE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}$ is of full row rank. Any other $\tilde{Q}$ such that $\tilde{E}_1$ of $\tilde{Q}E = \begin{bmatrix} \tilde{E}_1 \\ 0 \end{bmatrix}$ full row rank is of the form $\tilde{Q} = Q'Q$, where $Q' = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix}$,

$\tilde{Q} \in Gl(l, \mathbb{R})$ and $Q_1 \in \mathbb{R}^{q \times q}$, thus Q_1 and Q_4 are invertible matrices as well. Then via $\tilde{Q}$, Δ^u is ex-fb-equivalent to

$$\begin{bmatrix} Q_1 E_1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} Q_1 H_1 + Q_2 H_2 \\ Q_4 H_2 \end{bmatrix} x + \begin{bmatrix} Q_1 L_1 + Q_2 L_2 \\ Q_4 L_2 \end{bmatrix} u. \tag{3.9}$$

Proposition 3.2.4. *Assume that $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$ is a (Q, v)-explicitation of Δ^u corresponding to a choice of right inverse $E_1^\dagger$ of E_1. Then $\tilde{\Lambda}^{uv}_{n,m,s,p} = (\tilde{A}, \tilde{B}^u, \tilde{B}^v, \tilde{C}, \tilde{D}^u)$ is a $(\tilde{Q}, v)$-explicitation of Δ^u, where $\tilde{Q} = Q'Q$ and $Q' = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix}$, corresponding to a choice of right inverse $\tilde{E}_1^\dagger Q_1^{-1}$ of $Q_1 E_1$ with $\tilde{E}_1^\dagger = E_1^\dagger$ and $\tilde{B}^v = B^v$, if and only if Λ^{uv} and $\tilde{\Lambda}^{uv}$ are equivalent via an output injection $Ky = K(Cx + D^u u)$ and an output multiplication $\tilde{y} = T_y y$, which map*

$$A \mapsto \tilde{A} = A + KC, \quad B^u \mapsto \tilde{B}^u = B^u + KD^u, \quad B^v \mapsto \tilde{B}^v = B^v,$$
$$C \mapsto \tilde{C} = T_y C, \quad D^u \mapsto \tilde{D}^u = T_y D.$$

The proofs of Proposition 3.2.3 and 3.2.4 will be given in Section 3.7.1. In view of the above analysis, it is seen that $\mathbf{Expl}(\Delta^u)$ a class of ODECSs of the following form, given by all choices of K, F_v, R, and invertible T_v, T_y:

$$\begin{cases} \dot{x} = Ax + B^u u + Ky + B^v(F_v x + Ru + T_v^{-1}\tilde{v}) \\ y = T_y(Cx + Du). \end{cases}$$

Notice that the definition of (Q, v)-explicitation in the present chapter is different in two aspects from the (Q, P)-explicitation of Chapter 2: in this chapter we consider the explicitation of a DAECS but in Chapter 2, we only consider DAEs. The other difference is shown in the remark below. Nevertheless, in this chapter we use the same name by calling *the explicitation with driving variables* as *explicitation* for simplicity.

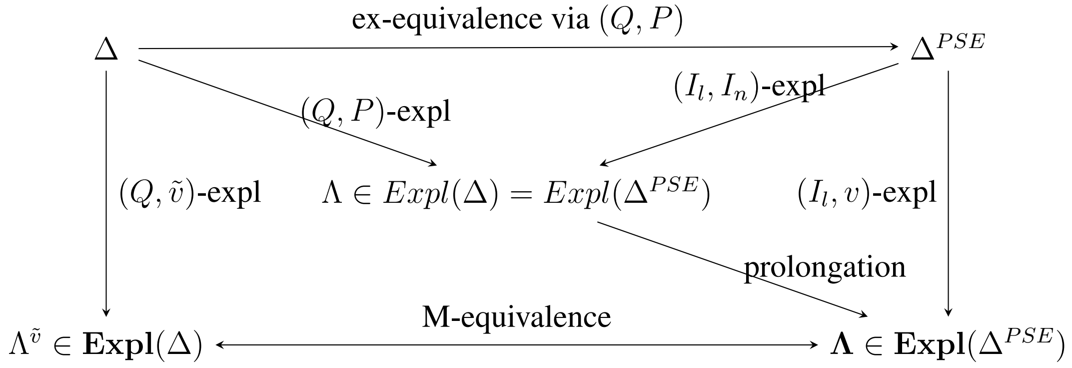
Remark 3.2.5. (i) Consider a DAE $E\dot{x} = Hx$, denoted by $\Delta = (E, H, 0)$. Via two invertible matrices Q and P, Δ is ex-fb-equivalent (actually externally equivalent) to a pure semi-explicit PSE DAE Δ^{PSE} below. Then the (Q, P)-explicitation of Δ defined in Chapter 2 is a control system Λ below (and the class of all (Q, P)-explicitations is denoted by $Expl(\Delta)$) and by adding $v = \dot{u}$, we get the prolongation $\mathbf{\Lambda}$ (which is actually an (I_l, v)-explicitation of Δ^{PSE}) of Λ :

$$QEP^{-1}P\dot{x} = QHP^{-1}Px \Rightarrow \Delta^{PSE} : \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} \Rightarrow$$
$$\Lambda : \begin{cases} \dot{z} = H_1 z + H_2 u \\ y = H_3 z + H_4 u, \end{cases} \Rightarrow \mathbf{\Lambda} : \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} v, \; y = H_3 z + H_4 u.$$

where $Px = [z^T \quad u^T]^T$.

(ii) The state z and control u of the (Q, P)-explicitation $Expl(\Delta)$ are linear combination $z = P_1 x$ and $u = P_2 x$, respectively, of the original "generalized" state x of Δ. On the other hand, the state x of a (Q, v)-explicitation $\mathbf{Expl}(\Delta)$ is the "generalized" state x of Δ and the driving variables v are extra variables not present in Δ.

(iii) The differences and relations between (Q, P)-explicitations in Chapter 2 and (Q, v)-explicitations in this chapter can be illustrated by the following diagram



Note that the implication that the $(Q, \tilde{v})$-explicitation $\Lambda^{\tilde{v}}$ of Δ is Morse equivalent (see Remark 3.2.7(ii)) to the prolo. system $\mathbf{\Lambda}$ is a corollary of Theorem 3.2.8 below applied to DAEs (without the original control u) since $\mathbf{\Lambda} \in \mathbf{Expl}(\Delta^{PSE})$, $\Lambda^{\tilde{v}} \in \mathbf{Expl}(\Delta)$ and Δ^{PSE} is ex-equivalent to Δ.

Since the *explicitation* (with driving variables) of Δ^u is a class of ODECSs of form (3.2), we give the following definition of equivalence for ODECSs of form (3.2). This definition is a natural extension of the Morse equivalence (see Chapter 2 and [146],[145]) of classical ODECSs of form (3.3).

Definition 3.2.6. (Extended Morse equivalence and extended Morse transformation) Two ODECSs

$$\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u), \quad \tilde{\Lambda}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^u)$$

are called extended Morse equivalent, shortly EM-equivalent, denoted by $\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$, if there exist matrices $T_x \in Gl(n, \mathbb{R})$, $T_u \in Gl(m, \mathbb{R})$, $T_v \in Gl(s, \mathbb{R})$, $T_y \in Gl(p, \mathbb{R})$, $F_u \in \mathbb{R}^{m\times n}$, $F_v \in \mathbb{R}^{s\times n}$, $R \in \mathbb{R}^{s\times m}$, $K \in \mathbb{R}^{n\times p}$ such that the system matrices of Λ^{uv} and $\tilde{\Lambda}^{\tilde{u}\tilde{v}}$ satisfy:

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{bmatrix} = \begin{bmatrix} T_x & T_x K \\ 0 & T_y \end{bmatrix} \begin{bmatrix} A & B^u & B^v \\ C & D^u & 0 \end{bmatrix} \begin{bmatrix} T_x^{-1} & 0 & 0 \\ F_u T_x^{-1} & T_u^{-1} & 0 \\ (F_v + RF_u)T_x^{-1} & RT_u^{-1} & T_v^{-1} \end{bmatrix}. \quad (3.10)$$

An 8-tuple $(T_x, T_u, T_v, T_y, F_u, F_v, R, K)$, acting on the system according to (3.10), will be called an extended Morse transformation and denoted by EM_{tran}.

The matrices T_x, T_u, T_v and T_y are coordinates transformations in the, respectively, state space $\mathscr{X} = \mathbb{R}^n$, input subspace $\mathscr{U}_u = \mathbb{R}^m$, input subspace $\mathscr{U}_v = \mathbb{R}^s$ and, output space $\mathscr{Y} = \mathbb{R}^p$, where F_u defines a state feedback of u, F_v and R define a feedback of v, K defines an output injection.

Remark 3.2.7. (i) An extended Morse transformation, whose action is given by equation (3.10), includes two kinds of feedback transformations:

$$v = F_v x + Ru + T_v^{-1}\tilde{v} \quad \text{and} \quad u = F_u x + T_u^{-1}\tilde{u}. \tag{3.11}$$

The vector of driving variables v is "stronger" than the original control vector u since when transforming v we can use both u and x as feedback, but when transforming u we are not allowed to use v. This is expressed by the triangular form of the matrix multiplying on the right of (3.10).

(ii) Recall the definition of the Morse equivalence and the Morse transformation [146] (and their generalization by Molinari [145] for $D^u \neq 0$, see also Chapter 2): for two ODECSs $\Lambda^u = (A, B^u, C, D^u)$ and $\tilde{\Lambda}^{\tilde{u}} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{C}, \tilde{D}^{\tilde{u}})$ of form (3.3), if

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} \end{bmatrix} = \begin{bmatrix} T_x & T_x K \\ 0 & T_y \end{bmatrix} \begin{bmatrix} A & B^u \\ C & D^u \end{bmatrix} \begin{bmatrix} T_x^{-1} & 0 \\ F_u T_x^{-1} & T_u^{-1} \end{bmatrix},$$

then Λ^u and $\tilde{\Lambda}^{\tilde{u}}$ are called Morse equivalent (shortly M-equivalent) and the Morse transformation (T_x, T_u, T_y, F_u, K) is denoted by M_{tran}. Clearly, M-equivalence is an equivalence relation for ODECSs of form (3.3), defined by a 4-tuple (A, B^u, C, D^u) and EM-equivalence is for ODECSs of form (3.2), defined by a 5-tuple (A, B^u, B^v, C, D^u). Observe that if the vector of driving variables v is of dimension zero (B^v is absent), then the EM-equivalence reduces to the M-equivalence.

(iii) Recall that we can express an ODECS of the form $\Lambda^{uv} = (A, B^u, B^v, C, D^u)$ as a standard ODECS $\Lambda^w = (A, B^w, C, D^w)$ of form (3.3) with one type of controls w, where $w = [u^T, v^T]^T$. Now let

$$F_w = \begin{bmatrix} F_u \\ F_v + RF_u \end{bmatrix}, \quad T_w^{-1} = \begin{bmatrix} T_u^{-1} & 0 \\ RT_u^{-1} & T_v^{-1} \end{bmatrix},$$

then we conclude the following equation from (3.10) (notice that T_w has a block-triangular structure):

$$\begin{bmatrix} \tilde{A} & \tilde{B}^w \\ \tilde{C} & \tilde{D}^w \end{bmatrix} = \begin{bmatrix} T_x & T_x K \\ 0 & T_y \end{bmatrix} \begin{bmatrix} A & B^w \\ C & D^w \end{bmatrix} \begin{bmatrix} T_x^{-1} & 0 \\ F_w T_x^{-1} & T_w^{-1} \end{bmatrix}, \tag{3.12}$$

which is exactly the expression of the M-equivalence for system Λ^w (compare Remark 3.2.7(ii) above). It implies that the EM-equivalence can be expressed as a form of the M-equivalence with a triangular matrix T_w (input coordinates transformation matrix). This triangular form is a consequence of two kinds of feedback transformation shown in equation (3.11).

Now we give the main result of this subsection:

Theorem 3.2.8. *Consider two DAECSs $\Delta^u_{l,n,m} = (E, H, L)$ and $\tilde{\Delta}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{H}, \tilde{L})$ as well as two ODECSs $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$ and $\tilde{\Lambda}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^{\tilde{u}})$ satisfying $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$ and $\tilde{\Lambda}^{\tilde{u}\tilde{v}} \in \mathbf{Expl}(\tilde{\Delta}^{\tilde{u}})$. Then, $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$ if and only if $\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$.*

The proof will be given in Section 3.7.1. In Section 3.8, we recall the definitions of geometric subspaces for DAECSs and ODECSs. More specifically, for a DAECS Δ^u, we recall the augmented Wong sequences $\mathscr{V}_i$ and $\mathscr{W}_i$, together with $\hat{\mathscr{W}_i}$ (see [17],[128]); For an ODECS Λ^w, we recall the subspaces sequences $\mathcal{V}_i$ and $\mathcal{W}_i$ (see [194],[193],[9]), whose limits are controlled and conditioned invariant subspaces, respectively, and we introduce the subspaces sequence $\hat{\mathcal{W}}_i$.

Proposition 3.2.9. *Given $\Delta^u_{l,n,m} = (E, H, L)$ and $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$ (or equivalently, $\Lambda^w_{n,m+s,p} = (A, B^w, C, D^w)$), consider the subspaces $\mathscr{V}_i$, $\mathscr{W}_i$, $\hat{\mathscr{W}_i}$ of Δ^u, given by Definition 3.8.2 and the subspaces $\mathcal{V}_i$, $\mathcal{W}_i$, $\hat{\mathcal{W}}_i$ of Λ^w, given by Lemma 3.8.5 in the Appendix of Section 3.8. Assume that $\Lambda^{uv} \in$ **Expl**(Δ^u). Then we have for $i \in \mathbb{N}$,*

$$\mathscr{V}_i(\Delta^u) = \mathcal{V}_i(\Lambda^w), \quad \mathscr{W}_i(\Delta^u) = \mathcal{W}_i(\Lambda^w),$$

and for $i \in \mathbb{N}^+$,

$$\hat{\mathscr{W}_i}(\Delta^u) = \hat{\mathcal{W}}_i(\Lambda^w).$$

The proof will be given in Section 3.7.2. Note that Theorem 3.2.8 and Proposition 3.2.9 are fundamental results for the remaining part of the chapter. Our purpose is to find the **FBCF** of DAECSs via explicitation. We have proven in Theorem 3.2.8 that the EM-equivalence for explicitation systems corresponds to the ex-fb-equivalence for DAECSs. Thus rather than transforming a DAECS Δ^u directly into its **FBCF** under ex-fb-equivalence, we will look for the canonical form for $\Lambda^{uv} \in$ **Expl**(Δ^u) under EM-equivalence.

3.3 The Morse triangular form and its extension

In the beginning of this section, we show that the normal form given in [145] (called Morse normal form **MNF** in the present chapter) for 4-tuple ODECS Λ^u, given by equation (3.3), can be constructed through a Morse triangular form **MTF** that we propose. Although the constructed normal form is the same as the one in [145], we will give explicit transformations with the help of the invariant subspaces given in Lemma 3.8.5 of Appendix in Section 3.8, which makes the normalizing procedure simple and transparent. Similar results can be found in [42], whose authors consider the general 4-tuple (A, B, C, D) and transform the system matrices into a normal form by choosing a special basis and corresponding coordinates. Their procedure is illustrated in [44] by examples.

Proposition 3.3.1. (Morse triangular form **MTF**) *For an ODECS $\Lambda^u_{n,m,p} = (A, B^u, C, D^u)$, consider the subspaces $\mathcal{V}^*$, $\mathcal{U}^*_u$, $\mathcal{W}^*$, $\mathcal{Y}^*$ given by Definition 3.8.4 of Appendix. Choose full rank matrices $T^1_s \in \mathbb{R}^{n\times n_1}$, $T^2_s \in \mathbb{R}^{n\times n_2}$, $T^3_s \in \mathbb{R}^{n\times n_3}$, $T^4_s \in \mathbb{R}^{n\times n_4}$, $T^1_i \in \mathbb{R}^{m\times m_1}$, $T^2_i \in \mathbb{R}^{m\times m_2}$, $T^1_o \in \mathbb{R}^{p\times p_1}$, $T^2_o \in \mathbb{R}^{p\times p_2}$ such that*

$$\begin{array}{ll} \operatorname{Im} T^1_s = \mathcal{V}^* \cap \mathcal{W}^*, & \mathcal{V}^* \cap \mathcal{W}^* \oplus \operatorname{Im} T^2_s = \mathcal{V}^*, \\ \mathcal{V}^* \cap \mathcal{W}^* \oplus \operatorname{Im} T^3_s = \mathcal{W}^*, & (\mathcal{V}^* + \mathcal{W}^*) \oplus \operatorname{Im} T^4_s = \mathscr{X} = \mathbb{R}^n, \\ \operatorname{Im} T^1_i = \mathcal{U}^*_u, & \operatorname{Im} T^2_i \oplus \operatorname{Im} T^1_i = \mathscr{U}_u = \mathbb{R}^m, \\ \operatorname{Im} T^1_o = \mathcal{Y}^*, & \operatorname{Im} T^2_o \oplus \operatorname{Im} T^1_o = \mathscr{Y} = \mathbb{R}^p, \end{array}$$

where $n = n_1 + n_2 + n_3 + n_4$, $m = m_1 + m_2$, $p = p_1 + p_2$. Then

$$T_s = [T_s^1 \;\; T_s^2 \;\; T_s^3 \;\; T_s^4]^{-1} \in Gl(n, \mathbb{R}),$$
$$T_i = [T_i^1 \;\; T_i^2]^{-1} \in Gl(m, \mathbb{R}), \quad T_o = [T_o^1 \;\; T_o^2]^{-1} \in Gl(p, \mathbb{R}),$$

*and there exist matrices $F_{MT} \in \mathbb{R}^{m\times n}$ and $K_{MT} \in \mathbb{R}^{n\times p}$ such that the Morse transformation $M_{tran} = (T_s, T_i, T_o, F_{MT}, K_{MT})$ brings Λ^u into $\tilde{\Lambda}^{\tilde{u}} = M_{tran}(\Lambda^u)$, represented in the Morse triangular form **MTF**, that is given by $\tilde{\Lambda}^{\tilde{u}} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{C}, \tilde{D}^{\tilde{u}})$, where*

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} \end{bmatrix} = \left[\begin{array}{cccc|cc} \tilde{A}_1 & \tilde{A}_1^2 & \tilde{A}_1^3 & \tilde{A}_1^4 & \tilde{B}_1 & \tilde{B}_1^2 \\ 0 & \tilde{A}_2 & 0 & \tilde{A}_2^4 & 0 & 0 \\ 0 & 0 & \tilde{A}_3 & \tilde{A}_3^4 & 0 & \tilde{B}_3 \\ 0 & 0 & 0 & \tilde{A}_4 & 0 & 0 \\ \hline 0 & 0 & \tilde{C}_3 & \tilde{C}_3^4 & 0 & \tilde{D}_3 \\ 0 & 0 & 0 & \tilde{C}_4 & 0 & 0 \end{array}\right]. \tag{3.13}$$

*In the above **MTF**, the pair $(\tilde{A}_1, \tilde{B}_1)$ is controllable, the pair $(\tilde{C}_4, \tilde{A}_4)$ is observable and the 4-tuple $(\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$ is prime (see Definition 2.9.1 in the Appendix of Chapter 2).*

The proof is given in Section 3.7.3. In the next proposition, we describe a way to transform the above **MTF** into a Morse normal form **MNF**, which is a further simplification of the **MTF**. We will use the same notations as in Proposition 3.3.1.

Proposition 3.3.2. (Morse normal form **MNF**) *There exists a feedback transformation matrix $F_{MN} \in \mathbb{R}^{m\times n}$, an output injection matrix $K_{MN} \in \mathbb{R}^{n\times p}$ and a state space coordinate transformation matrix $T_{MN} \in Gl(n, \mathbb{R})$, which can be chosen by Algorithm 3.3.3 below, such that the Morse transformation $M_{tran} = (T_{MN}, I_u, I_y, F_{MN}, K_{MN})$ brings $\tilde{\Lambda}^{\tilde{u}}$ of Proposition 3.3.1 into $\bar{\Lambda}^{\bar{u}} = M_{tran}(\tilde{\Lambda}^{\tilde{u}})$, represented in the Morse normal form **MNF**, that is given by $\bar{\Lambda}^{\bar{u}} = (\bar{A}, \bar{B}^{\bar{u}}, \bar{C}, \bar{D}^{\bar{u}})$, where*

$$\begin{bmatrix} \bar{A} & \bar{B}^{\bar{u}} \\ \bar{C} & \bar{D}^{\bar{u}} \end{bmatrix} = \left[\begin{array}{cccc|cc} \bar{A}_1 & 0 & 0 & 0 & \bar{B}_1 & 0 \\ 0 & \bar{A}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{A}_3 & 0 & 0 & \bar{B}_3 \\ 0 & 0 & 0 & \bar{A}_4 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_3 & 0 & 0 & \bar{D}_3 \\ 0 & 0 & 0 & \bar{C}_4 & 0 & 0 \end{array}\right]. \tag{3.14}$$

*In the above **MNF**, the pair $(\bar{A}_1, \bar{B}_1)$ is controllable, the pair $(\bar{C}_4, \bar{A}_4)$ is observable, and the 4-tuple $(\bar{A}_3, \bar{B}_3, \bar{C}_3, \bar{D}_3)$ is prime (see Definition 2.9.1 in the Appendix of Chapter 2).*

Notice that in the **MNF**, the system is decoupled into four independent subsystems of exactly the same dimension as in the Morse canonical form **MNF** (see Appendix of Chapter 2). In the latter, correspond to the **MNF**, we additionally normalize the controllable pair $(\bar{A}_1, \bar{B}_1)$ into its Brunovský canonical form [31], the observable pair $(\bar{A}_4, \bar{C}_4)$ into its dual Brunovský canonical form and the controllable and observable 4-tuple $(\bar{A}_3, \bar{B}_3, \bar{C}_3, \bar{D}_3)$ into its prime form, and the matrix $\bar{A}_2$ into its (real) Jordan canonical form. The proof of

Proposition 3.3.2 will be given in Section 3.7.4 and in that proof, we will use the construction of transformation matrices F_{MN}, K_{MN} and T_{MN}, which is formulated in the following algorithm.

Algorithm 3.3.3. *Step 1: Choose F_{MN} and K_{MN}:*

$$F_{MN} = \begin{bmatrix} F^1_{MN} & 0 & 0 & 0 \\ 0 & 0 & F^2_{MN} & F^3_{MN} \end{bmatrix}, \quad K_{MN} = \begin{bmatrix} K^1_{MN} & 0 \\ 0 & 0 \\ K^2_{MN} & 0 \\ 0 & K^3_{MN} \end{bmatrix},$$

such that the eigenvalues of $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ and $\bar{A}_4$ of the equation below are disjoint (notice that F_{MN} and K_{MN} preserve the zero blocks of $\tilde{\Lambda}^{\tilde{u}} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{C}, \tilde{D}^{\tilde{u}})$):

$$\begin{bmatrix} I_n & K_{MN} \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ F_{MN} & I_m \end{bmatrix} = \left[\begin{array}{cccc|cc} \bar{A}_1 & \bar{A}^2_1 & \bar{A}^3_1 & \bar{A}^4_1 & \bar{B}_1 & \bar{B}^2_1 \\ 0 & \bar{A}_2 & 0 & \bar{A}^4_2 & 0 & 0 \\ 0 & 0 & \bar{A}_3 & \bar{A}^4_3 & 0 & \bar{B}_3 \\ 0 & 0 & 0 & \bar{A}_4 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_3 & \bar{C}^4_3 & 0 & \bar{D}_3 \\ 0 & 0 & 0 & \bar{C}_4 & 0 & 0 \end{array}\right].$$

Step 2: Find matrices T^1_{MN}, T^2_{MN}, T^3_{MN}, T^4_{MN}, T^5_{MN} via the following (constrained) Sylvester equations:

$$\begin{array}{c} \bar{A}_1 T^1_{MN} - T^1_{MN}\bar{A}_2 = -\bar{A}^2_1, \quad \bar{A}_2 T^4_{MN} - T^4_{MN}\bar{A}_4 = -\bar{A}^4_2, \\ \bar{A}_1 T^3_{MN} - T^3_{MN}\bar{A}_4 = -\bar{A}^4_1 - \bar{A}^2_1 T^4_{MN} - \bar{A}^3_1 T^5_{MN}; \end{array} \tag{3.15}$$

$$\begin{array}{c} \bar{A}_1 T^2_{MN} - T^2_{MN}\bar{A}_3 = -\bar{A}^3_1, \quad T^2_{MN}\bar{B}_3 = -\bar{B}^2_1, \\ \bar{A}_3 T^5_{MN} - T^5_{MN}\bar{A}_4 = -\bar{A}^4_3, \quad \bar{C}_3 T^5_{MN} = -\bar{C}_4. \end{array} \tag{3.16}$$

Step 3: Set

$$T_{MN} = \begin{bmatrix} I & T^1_{MN} & T^2_{MN} & T^3_{MN} \\ 0 & I & 0 & T^4_{MN} \\ 0 & 0 & I & T^5_{MN} \\ 0 & 0 & 0 & I \end{bmatrix}^{-1}.$$

It is not surprising that Proposition 3.3.1 and 3.3.2 describe results similar to those of Theorem 2.3 and Theorem 2.6 of [20], as we have shown in Chapter 2 that there are direct connections between the geometric subspaces (the Wong sequences) of a DAE $\Delta : E\dot{x} = Hx$ and invariant subspaces of a control system $\Lambda = (A, B, C, D) \in Expl(\Delta)$. There are, however, differences between Proposition 3.3.1 and 3.3.2 and results of [20]. In particular, in Theorem 2.6 of [20], one has to solve generalized Sylvester equations, while in Proposition 3.3.2 we use (constrained) Sylvester equations. In addition, our transformations differ from those proposed in [145].

Recall that the explicitation of a DAECS Δ^u is a class of ODECSs with two kinds of inputs of form (3.2). In the following theorems, we will extend the results in Proposition 3.3.1 and 3.3.2 to ODECSs with two kinds of inputs.

Theorem 3.3.4. (Extended Morse triangular form **EMTF**) *For a DAECS*

$$\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u),$$

*there exists an extended Morse transformation EM_{tran} bringing Λ^{uv} into $EM_{tran}(\Lambda^{uv}) = \tilde{\Lambda}^{\tilde{u}\tilde{v}}$ represented in the extended Morse triangular form **EMTF**, that is given by $\tilde{\Lambda}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^{\tilde{u}})$, where*

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{bmatrix} = \left[\begin{array}{cccc|cc|cc} \tilde{A}_1 & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} & \tilde{B}^{\tilde{u}}_1 & \tilde{B}^{\tilde{u}}_{12} & \tilde{B}^{\tilde{v}}_1 & \tilde{B}^{\tilde{v}}_{12} \\ 0 & \tilde{A}_2 & 0 & \tilde{A}_{24} & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{A}_3 & \tilde{A}_{34} & 0 & \tilde{B}^{\tilde{u}}_3 & 0 & \tilde{B}^{\tilde{v}}_3 \\ 0 & 0 & 0 & \tilde{A}_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{C}_3 & \tilde{C}_{34} & 0 & \tilde{D}^{\tilde{u}}_3 & 0 & 0 \\ 0 & 0 & 0 & \tilde{C}_4 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{3.17}$$

Moreover,

(i) The pair $(\tilde{A}_1, \tilde{B}^{\tilde{w}}_1)$ is controllable, where $\tilde{B}^{\tilde{w}}_1 = [\tilde{B}^{\tilde{u}}_1, \tilde{B}^{\tilde{v}}_1]$;

(ii) The pair $(\tilde{C}_4, \tilde{A}_4)$ is observable ;

(iii) The 4-tuple $(\tilde{A}_3, \tilde{B}^{\tilde{w}}_3, \tilde{C}_3, \tilde{D}^{\tilde{w}}_3)$ is prime, where $\tilde{B}^{\tilde{w}}_3 = [\tilde{B}^{\tilde{u}}_3, \tilde{B}^{\tilde{v}}_3]$, $\tilde{D}^{\tilde{w}}_3 = [\tilde{D}^{\tilde{u}}_3, 0]$.

Proof. Recall Remark 3.2.7(iii) that there exists an extended Morse transformation EM_{tran} such that $\tilde{\Lambda}^{\tilde{u}\tilde{v}} = EM_{tran}(\Lambda^{uv})$ is of the **EMTF** if and only if there exists a Morse transformation M_{tran} with a triangular (and not just any) input coordinates transformation bringing $\Lambda^{w}_{n,m+s,p} = (A, B^w, C, D^w)$ into the **MTF**. Now we use the result of Proposition 3.3.1 for Λ^w with a more subtle way to construct the input coordinates transformation matrix T_w. More specifically, choose full rank matrices $T^1_u \in \mathbb{R}^{(m+s)\times m_1}$, $T^2_u \in \mathbb{R}^{(m+s)\times m_2}$, $T^1_v \in \mathbb{R}^{(m+s)\times s_1}$, $T^2_v \in \mathbb{R}^{(m+s)\times s_2}$ with $m_1 + m_2 = m$, $s_1 + s_2 = s$, such that

$$\begin{aligned} &\operatorname{Im} T^1_v = \mathfrak{U}^*_v, && \operatorname{Im} T^1_v \oplus \operatorname{Im} T^2_v = \mathscr{U}_v, \\ &\operatorname{Im} T^1_u \oplus \operatorname{Im} T^1_v = \mathfrak{U}^*_{uv} = \mathfrak{U}^*_w, && \operatorname{Im} T^1_u \oplus \operatorname{Im} T^2_u \oplus \operatorname{Im} T^1_v \oplus \operatorname{Im} T^2_v = \mathscr{U}_{uv} = \mathscr{U}_w, \end{aligned}$$

where $\mathfrak{U}^*_v$ is $\mathfrak{U}^*_{uv}$ when input is restricted to v (i.e., we put $u = 0$). Choose $T_w = [T^1_u \ \ T^1_v \ \ T^2_u \ \ T^2_v]^{-1}$ and set $T_x = T_s$, $T_y = T_o$, $F_w = F_{MT}$, $K_w = K_{MT}$ as in Proposition 3.3.1. Then the Morse transformation $M_{trans} = (T_x, T_w, T_y, F_w, K_w)$ brings Λ^w into $\tilde{\Lambda}^{\tilde{w}} = (\tilde{A}, \tilde{B}^{\tilde{w}}, \tilde{C}, \tilde{D}^{\tilde{w}}) = M_{trans}(\Lambda^w)$, for which

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{w}} \\ \tilde{C} & \tilde{D}^{\tilde{w}} \end{bmatrix} = \left[\begin{array}{cccc|cc} \tilde{A}_1 & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} & \tilde{B}^{\tilde{w}}_1 & \tilde{B}^{\tilde{w}}_{12} \\ 0 & \tilde{A}_2 & 0 & \tilde{A}_{24} & 0 & 0 \\ 0 & 0 & \tilde{A}_3 & \tilde{A}_{34} & 0 & \tilde{B}^{\tilde{w}}_3 \\ 0 & 0 & 0 & \tilde{A}_4 & 0 & 0 \\ \hline 0 & 0 & \tilde{C}_3 & \tilde{C}_{34} & 0 & \tilde{D}^{\tilde{w}}_3 \\ 0 & 0 & 0 & \tilde{C}_4 & 0 & 0 \end{array}\right],$$

where $\tilde{B}_1^{\tilde{w}} = [\tilde{B}_1^{\tilde{u}}, \tilde{B}_1^{\tilde{v}}]$, $\tilde{B}_{12}^{\tilde{w}} = [\tilde{B}_{12}^{\tilde{u}}, \tilde{B}_{12}^{\tilde{v}}]$, $\tilde{B}_3^{\tilde{w}} = [\tilde{B}_3^{\tilde{u}}, \tilde{B}_3^{\tilde{v}}]$, $\tilde{D}_3^{\tilde{w}} = [\tilde{D}_3^{\tilde{u}}, 0]$. Since T_w need not be triangular, we will replace it by

$$T_w^{per} = [T_u^1 \quad T_u^2 \quad T_v^1 \quad T_v^2]^{-1} \in \mathbb{R}^{(m+s)\times(m+s)},$$

which is invertible and has a triangular form (since $\operatorname{Im} T_v^1 \oplus \operatorname{Im} T_v^2 = \mathscr{U}_v$). Now the Morse transformation $M_{trans} = (T_x, T_w^{per}, T_y, F_w, K_w)$ brings Λ^w into the desired form of (3.17). Hence, it proves that there exists an EM_{tran} transforming Λ^{uv} into the **EMTF**. Claims (i), (ii), (iii) of Theorem 3.3.4 are inherited from the corresponding results of Proposition 3.3.1. □

Theorem 3.3.5. (Extended Morse normal form **EMNF**) *For $\tilde{\Lambda}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^{\tilde{u}})$ in the **EMTF**, as given by Theorem 3.3.4, there exists an extended Morse transformation EM_{tran} bringing $\tilde{\Lambda}^{\tilde{u}\tilde{v}}$ into $\bar{\Lambda}^{\bar{u}\bar{v}} = EM_{tran}(\tilde{\Lambda}^{\tilde{u}\tilde{v}})$ represented in the extended Morse normal form **EMNF**, that is given by $\bar{\Lambda}^{\bar{u}\bar{v}}_{n,m,s,p} = (\bar{A}, \bar{B}^{\bar{u}}, \bar{B}^{\bar{v}}, \bar{C}, \bar{D}^{\bar{u}})$, where*

$$\begin{bmatrix} \bar{A} & \bar{B}^{\bar{u}} & \bar{B}^{\bar{v}} \\ \bar{C} & \bar{D}^{\bar{u}} & 0 \end{bmatrix} = \left[\begin{array}{cccc|cc|cc} \bar{A}_1 & 0 & 0 & 0 & \bar{B}_1^{\bar{u}} & 0 & \bar{B}_1^{\bar{v}} & 0 \\ 0 & \bar{A}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{A}_3 & 0 & 0 & \bar{B}_3^{\bar{u}} & 0 & \bar{B}_3^{\bar{v}} \\ 0 & 0 & 0 & \bar{A}_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_3 & 0 & 0 & \bar{D}_3^{\bar{u}} & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_4 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{3.18}$$

Moreover,

(i) The pair $(\bar{A}_1, \bar{B}_1^{\bar{w}})$ is controllable, where $\bar{B}_1^{\bar{w}} = [\bar{B}_1^{\bar{u}}, \bar{B}_1^{\bar{v}}]$;

(ii) The pair $(\bar{C}_4, \bar{A}_4)$ is observable;

(iii) The 4-tuple $(\bar{A}_3, \bar{B}_3^{\bar{w}}, \bar{C}_3, \bar{D}_3^{\bar{w}})$ is prime, where $\bar{B}_3^{\bar{w}} = [\bar{B}_3^{\bar{u}}, \bar{B}_3^{\bar{v}}]$, $\tilde{D}_3^{\tilde{w}} = [\tilde{D}_3^{\tilde{u}}, 0]$.

Proof. As explained in the proof of Theorem 3.3.4, there exists an EM_{tran} such that $\bar{\Lambda}^{\bar{u}\bar{v}} = EM_{tran}(\tilde{\Lambda}^{\tilde{u}\tilde{v}})$ is in the **EMNF** if and only if there exists a Morse transformation M_{tran} with a triangular form input transformation matrix T^w bringing system $\tilde{\Lambda}^{\tilde{w}}$ into the **MNF**. Then as shown in Proposition 3.3.2, the input coordinates transformation matrix of the Morse transformation which brings the **MTF** into the **MNF** is an identity matrix. Thus it is always triangular as we need. Therefore, with the transformation matrices shown in Proposition 3.3.2, we can always bring $\tilde{\Lambda}^{\tilde{w}}$ into the **EMNF**. Moreover, the claims (i) (ii) (iii) of Theorem 3.3.5 follow from the corresponding results of Proposition 3.3.2. □

3.4 From the extended Morse normal form to the feedback canonical form of DAECSs

We show that, with a suitable choice of an extended Morse transformation for each subsystems in the **EMNF** of Theorem 3.3.5, we can bring the **EMNF** into an extended Morse canonical form **EMCF**. Below the upper indices refer to: c to controllable, nn to non-controllable and non-observable, p to prime, o to observable.

Definition 3.4.1. By the extended Morse canonical form, we will mean the system

$$\textbf{EMCF}: \begin{cases} \dot z^{cu} = A^{cu}z^{cu} + B^{cu}u \\ \dot z^{cv} = A^{cv}z^{cv} + B^{cv}v \\ \dot z^{nn} = A^{nn}z^{nn} \\ \dot z^{pu} = A^{pu}z^{pu} + B^{pu}u, & y^{pu} = C^{pu}z^{pu} + D^{pu}u \\ \dot z^{pv} = A^{pv}z^{pv} + B^{pv}v, & y^{pv} = C^{pv}z^{pv} \\ \dot z^{o} = A^{o}z^{o} & y^{o} = C^{o}z^{o}, \end{cases}$$

where both the pairs (A^{cu}, B^{cu}) and (A^{cv}, B^{cv}) are controllable and in the Brunovský canonical forms [31], A^{nn} is arbitrary and given up to a similarity transformation, the 4-tuple $(A^{pu}, B^{pu}, C^{pu}, D^{pu})$ and the triple (A^{pv}, B^{pv}, C^{pv}) are prime and the pair (C^o, A^o) is observable and in the dual Brunovský canonical form.

If an ODECS $\Lambda^{uv}_{EM} = (A_{EM}, B^u_{EM}, B^v_{EM}, C_{EM}, D^u_{EM})$ is in the **EMCF**, then the matrices $A_{EM}, B^u_{EM}, B^v_{EM}, C_{EM}, D^u_{EM}$, together with all invariants are thus given by

$$\begin{bmatrix} A_{EM} & B^u_{EM} & B^v_{EM} \\ C_{EM} & D^u_{EM} & 0 \end{bmatrix} = \left[\begin{array}{cccccc|cc|cc} A^{cu} & 0 & 0 & 0 & 0 & 0 & B^{cu} & 0 & 0 & 0 \\ 0 & A^{cv} & 0 & 0 & 0 & 0 & 0 & 0 & B^{cv} & 0 \\ 0 & 0 & A^{nn} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A^{pu} & 0 & 0 & 0 & B^{pu} & 0 & 0 \\ 0 & 0 & 0 & 0 & A^{pv} & 0 & 0 & 0 & 0 & B^{pv} \\ 0 & 0 & 0 & 0 & 0 & A^{o} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C^{pu} & 0 & 0 & 0 & D^{pu} & 0 & 0 \\ 0 & 0 & 0 & 0 & C^{pv} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C^{o} & 0 & 0 & 0 & 0 \end{array}\right], \tag{3.19}$$

with

(i) $A^{cu} = \text{diag}\{A^{cu}_{\epsilon_1}, ..., A^{cu}_{\epsilon_a}\}$, $B^{cu} = \text{diag}\{B^{cu}_{\epsilon_1}, ..., B^{cu}_{\epsilon_a}\}$, $A^{cv} = \text{diag}\{A^{cv}_{\bar\epsilon_b}, ..., A^{cv}_{\bar\epsilon_b}\}$, $B^{cv} = \text{diag}\{B^{cv}_{\bar\epsilon_1}, ..., B^{cv}_{\bar\epsilon_b}\}$ where

$$A^{cu}_{\epsilon} = \begin{bmatrix} 0 & I_{\epsilon-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\epsilon\times\epsilon}, \quad B^{cu}_{\epsilon} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\epsilon\times 1},$$
$$A^{cv}_{\bar\epsilon} = \begin{bmatrix} 0 & I_{\bar\epsilon-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\bar\epsilon\times\bar\epsilon}, \quad B^{cv}_{\bar\epsilon} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\bar\epsilon\times 1}.$$

The integers $\epsilon_1, ..., \epsilon_a \in \mathbb{N}$ are the controllability indices of (A^{cu}, B^{cu}), the integers $\bar\epsilon_1, ..., \bar\epsilon_b \in \mathbb{N}$ are the controllability indices of (A^{cv}, B^{cv}).

(ii) $A^{nn} \in \mathbb{R}^{n_2\times n_2}$ is unique up to similarity.

(iii) Both the 4-tuple $(A^{pu}, B^{pu}, C^{pu}, D^{pu})$ and the triple (A^{pv}, B^{pv}, C^{pv}) are controllable and observable (prime). That is,

$$\begin{bmatrix} A^{pu} & B^{pu} \\ C^{pu} & D^{pu} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A}^{pu} & \hat{B}^{pu} & 0 \\ \hline \hat{C}^{pu} & 0 & 0 \\ 0 & 0 & I_\delta \end{array}\right],$$

where $\begin{bmatrix} \hat{A}^{pu} & \hat{B}^{pu} \\ \hat{C}^{pu} & 0 \end{bmatrix}$ is square and invertible and $\delta = \operatorname{rank} \hat{D}^{pu} \in \mathbb{N}$, and the matrices

$$\begin{aligned} \hat{A}^{pu} &= \operatorname{diag}\{\hat{A}^{pu}_{\sigma_1}, ..., \hat{A}^{pu}_{\sigma_c}\}, \quad \hat{B}^{pu} = \operatorname{diag}\{\hat{B}^{pu}_{\sigma_1}, ..., \hat{B}^{pu}_{\sigma_c}\}, \\ \hat{C}^{pu} &= \operatorname{diag}\{\hat{C}^{pu}_{\sigma_1}, ..., \hat{C}^{pu}_{\sigma_c}\}, \quad A^{pv} = \operatorname{diag}\{A^{pv}_{\bar{\sigma}_1}, ..., A^{pv}_{\bar{\sigma}_d}\}, \\ B^{pv} &= \operatorname{diag}\{B^{pv}_{\bar{\sigma}_1}, ..., B^{pv}_{\bar{\sigma}_d}\}, \quad C^{pv} = \operatorname{diag}\{C^{pv}_{\bar{\sigma}_1}, ..., C^{pv}_{\bar{\sigma}_d}\}, \end{aligned}$$

where

$$\begin{aligned} \hat{A}^{pu}_\sigma &= \begin{bmatrix} 0 & I_{\sigma-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\sigma\times\sigma}, \quad \hat{B}^{pu}_\sigma = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\sigma\times 1}, \quad \hat{C}^{pu}_\sigma = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times\sigma}, \\ A^{pv}_{\bar{\sigma}} &= \begin{bmatrix} 0 & I_{\bar{\sigma}-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\bar{\sigma}\times\bar{\sigma}}, \quad B^{pv}_{\bar{\sigma}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\bar{\sigma}\times 1}, \quad C^{pv}_{\bar{\sigma}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times\bar{\sigma}}. \end{aligned}$$

The integers $\sigma_1, ..., \sigma_c \in \mathbb{N}^+$ are the controllability indices of the pair $(\hat{A}^{pu}, \hat{B}^{pu})$ and they are equal to the observability indices of the pair $(\hat{C}^{pu}, \hat{A}^{pu})$. The integers $\bar{\sigma}_1, ..., \bar{\sigma}_d \in \mathbb{N}^+$ are the controllability indices of the pair (A^{pv}, B^{pv}) and they are equal to the observability indices of the pair (C^{pv}, A^{pv}).

(iv) $A^o = \operatorname{diag}\{A^o_{\eta_1}, ..., A^o_{\eta_e}\}$, $C^o = \operatorname{diag}\{C^o_{\eta_1}, ..., C^o_{\eta_e}\}$, where

$$A^o_\eta = \begin{bmatrix} 0 & I_{\eta-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\eta\times\eta}, \quad C^o_\eta = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times\eta}.$$

The integers $\eta_1, ..., \eta_e \in \mathbb{N}$ are the observability indices of the pair (C^o, A^o).

Theorem 3.4.2. (Extended Morse canonical form **EMCF**) *For any*

$$\Lambda^{uv} = \Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u),$$

there exists an extended Morse transformation EM_{tran} bringing Λ^{uv} into

$$\Lambda^{uv}_{EM} = (A_{EM}, B^u_{EM}, B^v_{EM}, C_{EM}, D^u_{EM}) = EM_{tran}(\Lambda^{uv}),$$

*represented in the extended Morse canonical form **EMCF**.*

The proof will be given in Section 3.7.5. Throughout if we only consider the differential equation of (3.2) (meaning (3.2) without output y), we denote it as $\Lambda^{uv}_{n,m,s} = (A, B^u, B^v)$. Then by Theorem 3.4.2, we have

Corollary 3.4.3. (Brunovský canonical form for ODE control systems with two kinds of inputs) *For* $\Lambda^{uv}_{n,m,s} = (A, B^u, B^v)$*, assume that* $\operatorname{rank} B^w = m + s$*, where* $B^w = [B^u, B^v]$*. If the pair* (A, B^w) *is controllable, then*

$$(A, B^u, B^v) \overset{EM}{\sim} \left(\begin{bmatrix} A^{cu} & 0 \\ 0 & A^{cv} \end{bmatrix}, \begin{bmatrix} B^{cu} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B^{cv} \end{bmatrix} \right),$$

where (A^{cu}, B^{cu}) *and* (A^{cv}, B^{cv}) *are in the Brunovský canonical form.*

Remark 3.4.4. The difference between the **EMCF** above and the **MCF** of [146],[145], or see Chapter 2, comes from their controllable parts only that correspond to two kinds of inputs u and v. More specifically, the **MCF** has only one prime subsystem and only one controllable but non-observable subsystem. On the other hand, the **EMCF** has two prime subsystems: $(A^{pu}, B^{pu}, C^{pu}, D^{pu})$ and (A^{pv}, B^{pv}, C^{pv}), and two controllable but non-observable subsystems (A^{cu}, B^{cu}) and (A^{cv}, B^{cv}).

All the indices in the above **EMCF** can be calculated with the help of the invariant subspaces defined in Section 3.8 as shown in the following proposition. We will use the following definitions for a multi-index β: define the length of a multi-index $\beta = (\beta_1, \ldots, \beta_k)$ as $\ell(\beta) = k$, and define $|\beta| = \sum_{i=1}^{\ell(\beta)} \beta_i$. Given a index $\hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_{\hat{k}})$, we will define the dual index $\beta = (\beta_1, \ldots, \beta_k)$ by

$$\beta_i = \mathfrak{d}_i(\hat{\beta}) = \left\{ \text{number of } \hat{\beta}_j \text{ such that } \hat{\beta}_j \geq i \right\}, \ 1 \leq i \leq k,$$

and define $\mathfrak{d} = (\mathfrak{d}_1(\hat{\beta}), \ldots, \mathfrak{d}_k(\hat{\beta}))$.

Proposition 3.4.5. (The **EMCF** indices) *For an ODECS* $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$ *(or equivalently,* $\Lambda^{w}_{n,m+s,p} = (A, B^w, C, D^w)$*), consider the subspaces* $\mathcal{V}_i$*,* $\mathcal{W}_i$*,* $\hat{\mathcal{W}}_i$ *of Lemma 3.8.5 in the Appendix of Section 3.8. Assume that* $\begin{bmatrix} B^w \\ D^w \end{bmatrix}$ *is of full column rank and* $[C, D^w]$ *is of full row rank. Then the* ***EMCF*** *indices* ϵ_i*,* $\bar{\epsilon}_i$*,* σ_i*,* $\bar{\sigma}_i$*,* η_i*, together with a, b, c, d, e,* δ *in Theorem 3.4.2 can be calculated as follows, and thus are invariant under EM-transformations.*

(i) Set

$$\begin{aligned} \hat{\epsilon}_i &= \dim(\mathcal{V}^* \cap \mathcal{W}_i) - \dim(\mathcal{V}^* \cap \hat{\mathcal{W}}_i), \ i \geq 1, \\ \hat{\bar{\epsilon}}_i &= \dim(\mathcal{V}^* \cap \hat{\mathcal{W}}_i) - \dim(\mathcal{V}^* \cap \mathcal{W}_{i-1}), \ i \geq 1, \\ \hat{\bar{\sigma}}_i &= \dim \hat{\mathcal{W}}_i - \dim \mathcal{W}_{i-1} - \hat{\bar{\epsilon}}_i, \ i \geq 1, \\ \hat{\eta}_i &= \dim(\mathcal{W}^* + \mathcal{V}_{i-1}) - \dim(\mathcal{W}^* + \mathcal{V}_i), \ i \geq 1. \end{aligned}$$

Then $a = \hat{\epsilon}_1$*,* $b = \hat{\bar{\epsilon}}_1$*,* $d = \hat{\bar{\sigma}}_1$*,* $e = \hat{\eta}_1$*. The indices* $(\epsilon_1, \ldots, \epsilon_a) = \mathfrak{d}(\hat{\epsilon})$*,* $(\bar{\epsilon}_1, \ldots, \bar{\epsilon}_b) = \mathfrak{d}(\hat{\bar{\epsilon}})$*,* $(\bar{\sigma}_1, \ldots, \bar{\sigma}_d) = \mathfrak{d}(\hat{\bar{\sigma}})$ *and* $(\eta_1, \ldots, \eta_e) = \mathfrak{d}(\hat{\eta})$*.*

(ii) Set

$$\hat{\sigma}_1 = m - \hat{\epsilon}_1, \ \ \hat{\sigma}_i = \dim \mathcal{W}_{i-1} - \dim \hat{\mathcal{W}}_{i-1} - \hat{\epsilon}_{i-1}, \ i \geq 2.$$

Then $c = \hat{\sigma}_2$ *and* $\delta = \hat{\sigma}_1 - c$*. The indices* $(\sigma_1, \ldots, \sigma_c) = \mathfrak{d}(\hat{\sigma}) - 1$*.*

Proof. In order to prove the invariance of all indices, notice first that for ODECSs, the subspace sequences $\mathcal{V}_i$ and $\mathcal{W}_i$ are invariant under M-equivalence (see [146],[145], or Chapter 2). Recall that the M-equivalence (with triangular input coordinates transformation matrix T_i^w) for Λ^w coincide with the EM-equivalence for Λ^{uv}. Moreover, since the subspace sequences $\mathcal{W}_i$ and $\hat{\mathcal{W}}_i$ only differ from their initial conditions, it is not hard to see that $\hat{\mathcal{W}}_i$ is also invariant under the EM-equivalence. Thus without loss of generality, we assume that Λ^{uv} is in the **EMCF** of (3.19). A direct calculation by the formula of Lemma 3.8.5 of Appendix gives that

$$\mathcal{V}_i = \mathbb{R}^{|\epsilon|} \times \mathbb{R}^{|\bar{\epsilon}|} \times \mathbb{R}^{n_2} \times \operatorname{Im} N_\sigma^i \times \operatorname{Im} N_{\bar{\sigma}}^i \times \operatorname{Im}(N_\eta^T)^i, \quad i \geq 0, \tag{3.20}$$

$$\mathcal{W}_i = \ker N_\epsilon^i \times \ker N_{\bar{\epsilon}}^i \times \{0\}^{n_2} \times \ker N_\sigma^i \times \ker N_{\bar{\sigma}}^i \times \{0\}^{|\eta|}, \quad i \geq 0, \tag{3.21}$$

$$\hat{\mathcal{W}}_i = \ker N_\epsilon^{i-1} \times \ker N_{\bar{\epsilon}}^i \times \{0\}^{n_2} \times \ker N_\sigma^{i-1} \times \ker N_{\bar{\sigma}}^i \times \{0\}^{|\eta|}, \quad i \geq 1. \tag{3.22}$$

Note that by equations (3.20), (3.21) and, (3.22), we have

$$\mathcal{V}^* = \mathbb{R}^{|\epsilon|} \times \mathbb{R}^{|\bar{\epsilon}|} \times \mathbb{R}^{n_2} \times \{0\}^{|\sigma|} \times \{0\}^{|\bar{\sigma}|} \times \{0\}^{|\eta|},$$
$$\mathcal{W}^* = \hat{\mathcal{W}}^* = \mathbb{R}^{|\epsilon|} \times \mathbb{R}^{|\bar{\epsilon}|} \times \{0\}^{n_2} \times \mathbb{R}^{|\sigma|} \times \mathbb{R}^{|\bar{\sigma}|} \times \{0\}^{|\eta|}.$$

Thus, the following hold:

$$\begin{aligned}
&\hat{\epsilon}_1 = \operatorname{rank} B^{cu}, \quad \hat{\bar{\epsilon}}_1 = \operatorname{rank} B^{cv}, \quad \hat{\bar{\sigma}}_1 = \operatorname{rank} B^{pv}, \quad \hat{\eta}_1 = \operatorname{rank} C^o,\\
&\hat{\epsilon}_{i+1} = \operatorname{rank}\left[B^{cu}, A^{cu}B^{cu}, \ldots, (A^{cu})^i B^{cu}\right] - \operatorname{rank}\left[B^{cu}, A^{cu}B^{cu}, \ldots, (A^{cu})^{i-1}B^{cu}\right],\ i \geq 1,\\
&\hat{\bar{\epsilon}}_{i+1} = \operatorname{rank}\left[B^{cv}, A^{cv}B^{cv}, \ldots, (A^{cv})^i B^{cv}\right] - \operatorname{rank}\left[B^{cv}, A^{cv}B^{cv}, \ldots, (A^{cv})^{i-1}B^{cv}\right],\ i \geq 1,\\
&\hat{\bar{\sigma}}_{i+1} = \operatorname{rank}\left[B^{pv}, A^{pv}B^{pv}, \ldots, (A^{pv})^i B^{pv}\right] - \operatorname{rank}\left[B^{pv}, A^{pv}B^{pv}, \ldots, (A^{pv})^{i-1}B^{pv}\right],\ i \geq 1,\\
&\hat{\eta}_{i+1} = \operatorname{rank}\ \operatorname{col}\left[C^o, C^oA^o, \ldots, (C^o)^i A^o\right] - \operatorname{rank}\ \operatorname{col}\left[C^o, C^oA^o, \ldots, (C^o)^{i-1}A^o\right],\ i \geq 1.
\end{aligned}$$

Moreover, we have $\hat{\sigma}_1 = m - \operatorname{rank} B^{pv} = \operatorname{rank} B^{pu} = \operatorname{rank} \hat{B}^{pu} + \delta$ and $\hat{\sigma}_1 = \operatorname{rank} \hat{B}^{pu}$, and for $i \geq 2$,

$$\hat{\sigma}_{i+1} = \operatorname{rank}\left[\hat{B}^{pu}, \hat{A}^{pu}\hat{B}^{pu}, \ldots, (\hat{A}^{pu})^{i-1}\hat{B}^{pu}\right] - \operatorname{rank}\left[\hat{B}^{pu}, \hat{A}^{pu}B^{pu}, \ldots, (\hat{A}^{pu})^{i-2}\hat{B}^{pu}\right].$$

Finally, by the classical controllability and observability indices calculation (see e.g. [31]), we can calculate ϵ_i, $\bar{\epsilon}_i$, σ_i, $\bar{\sigma}_i$, η_i from $\hat{\epsilon}_i$, $\hat{\bar{\epsilon}}_i$, $\hat{\sigma}_i$, $\hat{\bar{\sigma}}_i$, $\hat{\eta}_i$. Finally, from the relations of indices and the invariant subspaces, it is seen that the integers ϵ_i, $\bar{\epsilon}_i$, σ_i, $\bar{\sigma}_i$, η_i, together with cardinalities a, b, c, d, e, δ are also invariant under the EM-equivalence. □

Remark 3.4.6. In general, $\begin{bmatrix} B^w \\ D^w \end{bmatrix}$ may not be monic, i.e., injective, and $[C, D^w]$ may not be epic, i.e., surjective, which implies that some of these indices are allowed to be zero, e.g. for certain i, $\epsilon_i = 0$ meaning that B^{cu} has one zero column, and for certain i, $\eta_i = 0$ implying that C^o has one zero row.

Now we introduce the *driving variable reduction* and *implicitation* procedure (compare Chapter 2 for the case of controls of one kind) to reduce the driving variable v and implicit the **EMCF** to a DAECS. The procedure is that, for each sub-system in **EMCF**, which is affected by v, i.e., (A^{cv}, B^{cv}) and (A^{pv}, B^{pv}, C^{pv}), we reduce the last rows in the dynamics of the subsystems and set the output $y = 0$. Take, for example, a prime subsystem $(A^{pv}_{\bar{\sigma}}, B^{pv}_{\bar{\sigma}}, C^{pv}_{\bar{\sigma}})$ of (A^{pv}, B^{pv}, C^{pv}) for which we get:

$$(A^{pv}_{\bar{\sigma}}, B^{pv}_{\bar{\sigma}}, C^{pv}_{\bar{\sigma}}) : \begin{cases} y &= x^1, \\ \dot{x}^1 &= x^2 \\ & \cdots \\ \dot{x}^{\bar{\sigma}-1} &= x^{\bar{\sigma}} \\ \dot{x}^{\bar{\sigma}} &= v, \end{cases} \to (N_{\bar{\sigma}}, I_{\bar{\sigma}}, 0) : \begin{cases} 0 &= x^1 \\ \dot{x}^1 &= x^2 \\ & \cdots \\ \dot{x}^{\bar{\sigma}-1} &= x^{\bar{\sigma}}. \end{cases}$$

In the above case, the ODECS $\Lambda^{pv} = (A^{pv}_{\bar{\sigma}}, B^{pv}_{\bar{\sigma}}, C^{pv}_{\bar{\sigma}})$ on the left is mapped via an implicitation into a DAECS $\Delta = (N_{\bar{\sigma}}, I_{\bar{\sigma}}, 0)$ on the right. Notice that Λ^{pv} and Δ in the above procedure satisfy $\Lambda^{pv} \in \mathbf{Expl}(\Delta)$.

Then with the help of the above *reduction and implicitation* procedure, we can regard the feedback canonical form **FBCF** for DAECSs of the form $\Delta^u_{l,n,m} = (E, H, L)$ given in [131] as a corollary of Theorem 3.4.2. In the following, in order to save space and simplify notations, we denote

$$K_i = \begin{bmatrix} 0 & I_{i-1} \end{bmatrix} \in \mathbb{R}^{(i-1)\times i}, \quad L_i = \begin{bmatrix} I_{i-1} & 0 \end{bmatrix} \in \mathbb{R}^{(i-1)\times i},$$
$$N_i = \begin{bmatrix} 0 & 0 \\ I_{i-1} & 0 \end{bmatrix} \in \mathbb{R}^{i\times i}, \qquad e_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{i},$$

and

$$N_\beta = \operatorname{diag}\{N_{\beta_1}, \ldots, N_{\beta_k}\} \in \mathbb{R}^{|\beta|\times|\beta|} \qquad K_\beta = \operatorname{diag}\{K_{\beta_1}, \ldots, K_{\beta_k}\} \in \mathbb{R}^{(|\beta|-k)\times|\beta|},$$
$$L_\beta = \operatorname{diag}\{L_{\beta_1}, \ldots, L_{\beta_k}\} \in \mathbb{R}^{(|\beta|-k)\times|\beta|}, \quad \mathcal{E}_\beta = \operatorname{diag}\{e_{\beta_1}, \ldots, e_{\beta_k}\} \in \mathbb{R}^{|\beta|\times k},$$

Corollary 3.4.7. (Feedback canonical form of DAE control systems [131]) *Any DAE control system $\Delta^u_{l,n,m} = (E, H, L)$ is ex-fb-equivalent to the following feedback canonical form **FBCF**:*

$$\left(\begin{bmatrix} I_{|\epsilon'|} & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{\bar{\epsilon}'} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & K^T_{\sigma'} & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{\bar{\sigma}'} & 0 \\ 0 & 0 & 0 & 0 & 0 & L^T_{\eta'} \end{bmatrix}, \begin{bmatrix} N^T_{\epsilon'} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{\bar{\epsilon}'} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & L^T_{\sigma'} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\bar{\sigma}'|} & 0 \\ 0 & 0 & 0 & 0 & 0 & K^T_{\eta'} \end{bmatrix}, \begin{bmatrix} \mathcal{E}_{\epsilon'} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathcal{E}_{\sigma'} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right),$$

where $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_{a'}) \in (\mathbb{N}^+)^{a'}$, $\bar{\epsilon}' = (\bar{\epsilon}'_1, \ldots, \bar{\epsilon}'_{b'}) \in (\mathbb{N}^+)^{b'}$, $\sigma' = (\sigma'_1, \ldots, \sigma'_{c'}) \in (\mathbb{N}^+)^{c'}$, $\bar{\sigma}' = (\bar{\sigma}'_1, \ldots, \bar{\sigma}'_{d'}) \in (\mathbb{N}^+)^{d'}$, $\eta' = (\eta'_1, \ldots, \eta'_{e'}) \in (\mathbb{N}^+)^{e'}$ are multi-indices and the matrix A_ρ is given up to similarity.

Remark 3.4.8. (i) The above **FBCF** of DAECSs is a corollary of Theorem 3.4.2. Indeed, for any DAECS $\Delta^u = (E, H, L)$, we can construct an ODECS $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$. Then, by Theorem 3.4.2, we have $\Lambda^{uv} \overset{EMM}{\sim}$ **EMCF**. From the driving variable reduction and implicitation procedure, it can be observed that the **FBCF** is the implicitation of the **EMCF** of Δ^u. A crucial observation is that **EMCF** $\in$ **Expl**(**FBCF**). Thus, by Theorem 3.2.8, we conclude $\Delta^u \overset{ex-fb}{\sim}$ **FBCF** (since $\Lambda^{uv} \overset{EMM}{\sim}$ **EMCF**).

(ii) There exists a perfect correspondence between the six subsystems of the **EMCF** and their counterparts of the **FBCF**. Morse specifically,

$$\begin{array}{ll} (A^{cu}, B^{cu}) \leftrightarrow (I_{|\epsilon'|}, N^T_{\epsilon'}, E_{\epsilon'}), & (A^{cu}, B^{cu}) \leftrightarrow (L_{\bar{\epsilon}'}, K_{\bar{\epsilon}'}, 0), \\ A_{n_2} \leftrightarrow (I_{n_\rho}, A_\rho), & (A^{pu}, B^{pu}, C^{pu}, D^{pu}) \leftrightarrow (K^T_{\sigma'}, L^T_{\sigma'}, \mathcal{E}_{\sigma'}), \\ (A^{pv}, B^{pv}, C^{pv}) \leftrightarrow (N_{\bar{\sigma}'}, I_{|\bar{\sigma}'|}, 0), & (C^o, A^o) \leftrightarrow (L^T_{\eta'}, K^T_{\eta'}, 0). \end{array}$$

(iii) Since the **FBCF** is the implicitation of the **EMCF**, it is easy to observe that the indices in the **FBCF** and **EMCF** have the following relations:

- $a = a'$ and $\epsilon_k = \epsilon'_k$ for $k = 1, \ldots, a$, $b = b'$ and $\bar{\epsilon}_k = \bar{\epsilon}'_k$ for $k = 1, \ldots, b$;
- $n_2 = n_\rho$ and $A^{nn} \approx A_\rho$;
- $c + \delta = c'$ and $\sigma'_1 = \sigma'_2 =, \cdots, = \sigma'_\delta = 1$, $\sigma'_{\delta+1} = \sigma_1 + 1$, $\sigma'_{\delta+2} = \sigma_2 + 1$, ..., $\sigma'_{\delta+c} = \sigma_c + 1$; Moreover, $d = d'$ and $\bar{\sigma}_k = \bar{\sigma}'_k$ for $k = 1, \ldots, d$;
- $e = e'$ and $\eta_k + 1 = \eta'_k$ for $k = 1, \ldots, e$.

From the indices relations in Remark 3.4.8(iii) and the subspaces relation of Proposition 3.2.9, we can deduce the following calculation of the **FBCF** invariants as a corollary of Proposition 3.4.5.

Corollary 3.4.9. (Invariants of **FBCF** in [131] and [18]) *For $\Delta^u_{l,n,m} = (E, H, L)$, consider the subspaces $\mathscr{V}_i$, $\mathscr{W}_i$, $\hat{\mathscr{W}}_i$ of Definition 3.8.2. Assume that* $\operatorname{rank} L = m$ *and* $\operatorname{Im} E \cap \operatorname{Im}[H, L] = 0$. *Then the* ***FBCF*** *indices $\epsilon', \bar{\epsilon}', \sigma', \bar{\sigma}', \eta'$, together with a', b', c', d', e' of Corollary 3.4.7 can be calculated as follows.*

(i) Set

$$\begin{aligned} \hat{\epsilon}'_i &= \dim(\mathscr{V}^* \cap \mathscr{W}_i) - \dim(\mathscr{V}^* \cap \hat{\mathscr{W}}_i), \ i \geq 1, \\ \hat{\bar{\epsilon}}'_i &= \dim(\mathscr{V}^* \cap \hat{\mathscr{W}}_i) - \dim(\mathscr{V}^* \cap \mathscr{W}_{i-1}), \ i \geq 1, \\ \hat{\bar{\sigma}}'_i &= \dim \hat{\mathscr{W}}_i - \dim \mathscr{W}_{i-1} - \hat{\bar{\epsilon}}'_i, \ i \geq 1. \end{aligned}$$

Then $a' = \hat{\epsilon}'_1$, $b' = \hat{\bar{\epsilon}}'_1$, $d' = \hat{\bar{\sigma}}'_1$. The indices $(\epsilon'_1, \ldots, \epsilon'_{a'}) = \mathfrak{d}(\hat{\epsilon}')$, $(\bar{\epsilon}'_1, \ldots, \bar{\epsilon}'_{b'}) = \mathfrak{d}(\hat{\bar{\epsilon}}')$, $(\bar{\sigma}'_1, \ldots, \bar{\sigma}'_d) = \mathfrak{d}(\hat{\bar{\sigma}}')$.

(ii) Set

$$\begin{aligned} &\hat{\sigma}'_1 = m - \hat{\epsilon}'_1, \ \ \hat{\sigma}'_i = \dim \mathscr{W}_{i-1} - \dim \hat{\mathscr{W}}_{i-1} - \hat{\epsilon}'_{i-1}, \ i \geq 2, \\ &\hat{\eta}'_i = \dim(\mathscr{W}^* + \mathscr{V}_{i-1}) - \dim(\mathscr{W}^* + \mathscr{V}_i), \ i \geq 1. \end{aligned}$$

Then $c' = \hat{\sigma}'_1$, $e' = \hat{\eta}'_1$. The indices $(\sigma'_1, \ldots, \sigma'_{c'}) = \mathfrak{d}(\hat{\sigma}') - 1$ and $(\eta'_1, \ldots, \eta'_{e'}) = \mathfrak{d}(\hat{\eta}') - 1$.

Remark 3.4.10. Note that the assumptions that $\operatorname{rank} L = m$ and $\operatorname{Im} E \cap \operatorname{Im}[H, L] = 0$ correspond to the assumptions of Proposition 3.4.5 that $\begin{bmatrix} B^w \\ D^w \end{bmatrix}$ is of full column rank and

$[C, D^w]$ is of full row rank, respectively. Although more general results about the **FBCF** indices (without these assumptions) are given in [131] and [18], our purpose is to show the connections between **EMCF** indices and **FBCF** indices rather than to calculate them in the case of non full rank. The generalizations of the results of Corollary 3.4.9 to that of the **FBCF** indices in [131] and [18] are straightforward.

Below a simple algorithm is proposed to calculate the **FBCF** for a given DAECS $\Delta^u_{l,n,m} = (E, H, L)$ based on the explicitation procedure.

Algorithm 3.4.11. *Step 1: For Δ^u, construct an ODECS Λ^{uv} such that $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$, via the explicitation procedure described in Section 3.2.*

*Step 2: By Theorem 3.3.4, find an extended Morse transformation EM_{tran} such that the transformed system $\tilde{\Lambda}^{\tilde{u}\tilde{v}} = EM_{tran}(\Lambda^{uv})$ is in the **EMTF**.*

*Step 3: By Theorem 3.3.5, find an extended Morse transformation EM_{tran} such that the transformed system $\bar{\Lambda}^{\bar{u}\bar{v}} = EM_{tran}(\tilde{\Lambda}^{\tilde{u}\tilde{v}})$ is in the **EMNF**. Then by the procedure shown in the proof of Theorem 3.4.2, bring $\bar{\Lambda}^{\bar{u}\bar{v}}$ into the **EMCF**.*

*Step 4: Find the implicitation of **EMCF**, denoted by $\bar{\Delta}^{\bar{u}}$, via the driving variable reduction and implicitation procedure described in Section 3.4. Then $\bar{\Delta}^{\bar{u}}$ is in the **FBCF** and $\Delta^u \overset{ex-fb}{\sim} \bar{\Delta}^{\bar{u}}$.*

3.5 Example

In this section, we illustrate the construction of Algorithm 3.4.11 by an example taken from [20]. Consider the following mathematical model of an electrical circuit (see Fig. 1.1 of [20]), which is a DAECS of the form $E\dot{x} = Hx + Lu$:

$$\begin{bmatrix} 0&0&0&0&L&0&0&0&0&0&0&0&0&0\\ 0&0&-Ca&Ca&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&-1&0&0&0&0&0&R_G&0&0&0&0&0&0\\ 0&1&-1&0&0&0&0&0&R_F&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&R&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&-1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1&0&1&-1&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&1&1\\ 0&0&-1&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 1&0\\ 0&0\\ 0&0\\ 0&0\\ 0&1 \end{bmatrix} \begin{bmatrix} I\\ V \end{bmatrix},$$

where $u = [I^T, V^T]^T$ is the control vector, L, Ca, R, R_G, R_F are real scalars (all assumed to be nonzero). In [20], only the matrix pencil $sE - H$ is transformed into a quasi-Kronecker form. We will transform the whole DAECS into its **FBCF** via Algorithm 3.4.11.

Step 1: Find an ODECS $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$, which we take as

$$\Lambda^{uv}:\begin{cases}\dot{x} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1/Ca&0\\ 1/L&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0 \end{bmatrix} u + \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} v \\[2ex] y = \begin{bmatrix} 0&-1&0&0&0&0&0&R_G&0&0&0&0&0&0\\ 0&1&-1&0&0&0&0&0&R_F&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&R&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&-1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1&0&1&-1&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&1&1\\ 0&0&-1&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0&0\\0&0\\0&0\\0&0\\0&0\\1&0\\0&0\\0&0\\0&0\\0&0\\0&1 \end{bmatrix} u. \end{cases}$$

Step 2: Calculate the subspaces $\mathcal{V}^*$, $\mathcal{U}_w^*$, $\mathcal{U}_v^*$, $\mathcal{W}^*$, $\mathcal{Y}^*$ of $\Lambda^w = (A, B^w, C, D^w)$ by Lemma 3.8.5 of Appendix, where $B^w = \begin{bmatrix} B^u & B^v \end{bmatrix}$, $D^w = \begin{bmatrix} D^u & 0 \end{bmatrix}$ and $w = [u^T, v^T]^T$. They are

$$\mathcal{V}^* = \operatorname{Im} \begin{bmatrix} R_G&0&0&0&0\\ R_G&0&0&0&0\\ R_F+R_G&0&0&0&0\\ 0&R&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&0&0&-1&1\\ -1&-1&0&1&-1 \end{bmatrix}, \quad \mathcal{U}_w^* = \operatorname{Im} \begin{bmatrix} 0&0&0\\ 0&0&0\\ R{*}R_G&0&0\\ R{*}R_G&0&0\\ R{*}(R_F+R_G)&0&0\\ 0&0&0\\ 0&0&0\\ R&0&0\\ R&0&0\\ R_F+R_G&0&0\\ 0&1&0\\ 0&0&1\\ R&-1&1\\ -(R+R_F+R_G)&1&-1 \end{bmatrix},$$

$$\mathcal{U}_v^* = \operatorname{Im} \begin{bmatrix} R{*}R_G&0&0\\ R{*}R_G&0&0\\ R{*}(R_F+R_G)&0&0\\ 0&0&0\\ 0&0&0\\ R&0&0\\ R&0&0\\ R_F+R_G&0&0\\ 0&1&0\\ 0&0&1\\ R&-1&1\\ -(R+R_F+R_G)&1&-1 \end{bmatrix}, \quad \mathcal{W}^* = \mathscr{X} = \mathbb{R}^{14}, \quad \mathcal{Y}^* = \mathscr{Y} = \mathbb{R}^{11}.$$

By the proof of Theorem 3.3.4 and Proposition 3.3.1, we can choose the following transformation matrices: $T_y = I_{11}$, $K_{MT} = 0_{14\times 11}$,

$$T_s = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ -1&1&0&0&0&0&0&0&0&0&0&0&0&0\\ -(RF+RG)/RG&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ -1/RG&0&0&0&0&0&0&1&0&0&0&0&0&0\\ -1/RG&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&-1/R&0&0&0&0&0&1&0&0&0&0\\ -1/RG&0&0&0&0&0&0&0&0&0&1&-1&1&0\\ 1/RG&0&0&1/R&0&0&0&0&0&0&-1&1&0&1 \end{bmatrix},$$

$$T_w = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -(RF+RG)/RG & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/RG & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/RG & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -(RF+RG)/(R*RG) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/RG & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & (R+RF+RG)/(R*RG) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1
\end{bmatrix},$$

$$F_{MT} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(RF+RG)/RG & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/(Ca*R*RG) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/(Ca*R) & 1/(Ca*R) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/(Ca*R*RG) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/(Ca*R) & -1/(Ca*R) & 0 & 0
\end{bmatrix}.$$

Then the Morse transformation $M_{trans}(T_s, T_w, T_y, F_{MT}, K_{MT})$ brings Λ^w into $\tilde{\Lambda}^{\tilde{w}}$, given by $(\tilde{A}, \tilde{B}^{\tilde{w}}, \tilde{C}, \tilde{D}^{\tilde{w}})$, which is in the **EMTF**, where

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B}^{\tilde{w}} \\ \hline \tilde{C} & \tilde{D}^{\tilde{w}} \end{array}\right] = \left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \hline \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{array}\right] = \left[\begin{array}{cc|c|cc} \tilde{A}_1 & \tilde{A}_{13} & 0 & \tilde{B}_1^{\tilde{v}} & \tilde{B}_{12}^{\tilde{v}} \\ 0 & \tilde{A}_3 & 0 & 0 & \tilde{B}_3^{\tilde{v}} \\ \hline 0 & \tilde{C}_3 & \tilde{D}_3^{\tilde{u}} & 0 & 0 \end{array}\right] =$$

$$\left[\begin{array}{cccccccccccccc|cc|cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/(Ca*RG) & 0 & 0 & -1/Ca & 1/Ca & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/Ca & 0 & 0 & 0 & (RF+RG)/RG & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/(Ca*R) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/R & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/(Ca*R) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/R & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & RG & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & RF & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

Step 3: By Algorithm 3.3.3, set

$$K_{MN} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/(Ca*RG) & 0 & 0 & 0 & -1/Ca & 1/(Ca*RG) & 0 & 1/Ca & -1/Ca & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad F_{MN} = 0_{14\times 14}.$$

Then solve T_{MN}^2 via the following constrained Sylvester equation,

$$\bar{A}_1 T_{MN}^2 - T_{MN}^2 \bar{A}_3 = -\bar{A}_1, \quad T_{MN}^2 \bar{B}_3^{\bar{w}} = -\bar{B}_{12}^{\bar{w}},$$

where $\bar{A} = \tilde{A} + K_{MN}\tilde{C}$, $\bar{B}^{\bar{w}} = \tilde{B}^{\tilde{w}} + K_{MN}\tilde{D}^{\tilde{w}}$. The above equation is solvable and the solution is

$$T^2_{MN} = \begin{bmatrix} 0 & 0 & 0&0&0&0&0&0&0&0 \\ 0 & -1 & 0&0&0&0&0&0&0&0 \\ 0 & 0 & 0&0&0&0&0&0&0&0 \\ 0 & 0 & 0&0&0&0&0&0&0&0 \\ 0 & 0 & 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Thus the Morse transformation $M_{tran} = (T_{MN}, I_{14}, I_{11}, F_{MN}, K_{MN})$, where $T_{MN} = \begin{bmatrix} I & T^2_{MN} \\ 0 & I \end{bmatrix}$, brings $\tilde{\Lambda}^{\tilde{w}}$ into $\bar{\Lambda}^{\bar{w}} = (\bar{A}, \bar{B}^{\bar{w}}, \bar{C}, \bar{D}^{\bar{w}})$, which is in the **EMNF**, where

$$\left[\begin{array}{c|c} \bar{A} & \bar{B}^{\bar{w}} \\ \hline \bar{C} & \bar{D}^{\bar{w}} \end{array}\right] = \left[\begin{array}{c|c|c} \bar{A} & \bar{B}^{\bar{u}} & \bar{B}^{\bar{v}} \\ \hline \bar{C} & \bar{D}^{\bar{u}} & 0 \end{array}\right] = \left[\begin{array}{cc|c|cc} \bar{A}_{11} & 0 & 0 & \bar{B}^{\bar{v}}_{11} & 0 \\ 0 & \bar{A}_{33} & 0 & 0 & \bar{B}^{\bar{v}}_{32} \\ \hline 0 & \bar{C}_{13} & \bar{D}^{\bar{u}}_{12} & 0 & 0 \end{array}\right] =$$

$$\left[\begin{array}{cccccccccccccc|cc|cccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
1/(Ca*RG)&0&0&-1/Ca&1/Ca&0&0&0&0&0&0&0&0&0&0&0&(RF+RG)/RG&0&0&0&0&0&0&0&0&0&0&0 \\
1/L&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&-1/(Ca*R)&0&0&0&0&0&0&0&-1/R&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1/(Ca*R)&0&0&0&0&0&0&0&1/R&0&0&0&0&0&0&1 \\
\hline
0&0&0&0&0&-1&0&0&0&RG&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&-1&0&0&0&RF&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&R&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&-1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right].$$

Step 4: Transform each subsystem of $\bar{\Lambda}^{\bar{w}}$ into its canonical form as in Theorem 3.4.2 to obtain the **EMCF**:

$$\left[\begin{array}{cc|c|cc} A^{cv} & 0 & 0 & B^{cv} & 0 \\ 0 & A^{pv} & 0 & 0 & B^{pv} \\ \hline 0 & 0 & D^{pu} & 0 & 0 \\ 0 & \bar{C}^{pv} & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccccccccccccc|cc|cccccccccccc}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right].$$

The **EMCF** indices are $\bar{\epsilon}_1 = 2, \bar{\epsilon}_2 = 2, \bar{\epsilon}_3 = 1, \delta = 2, \bar{\sigma}_1 = \bar{\sigma}_2 =, \ldots, = \bar{\sigma}_9 = 1$. Note that n_2, a, c, e are all zeros and we have 3 subsystems only.

Step 5: Using the driving variable reduction and implicitation procedure, we get the following DAECS from the above **EMCF**:

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \dot{z} = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} z + \begin{bmatrix} 0&0\\ 0&0\\ 1&0\\ 0&1\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0 \end{bmatrix} \tilde{u},$$

where z and $\tilde{u}$ is the new "generalized" state and the input, respectively. Obviously, the above DAECS is in the **FBCF** with indices $\bar{\epsilon}'_1 = 2, \bar{\epsilon}'_2 = 2, \bar{\epsilon}'_3 = 1$, $\sigma_1 = \sigma_2 = 1$, $\bar{\sigma}_1 = \bar{\sigma}_2 =, \ldots, = \bar{\sigma}_9 = 1$. Moreover, $a' = n_\rho = e' = 0$, $c' = \delta = 2$.

Remark 3.5.1. From the above **FBCF**, it is seen that the variables z_1, z_3 are actually the states (they enter the system dynamiclaly) of the system; the variables z_2, z_4, z_5 are the free variables of the "generalized" state; the new control $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ are actually constrained by the algebraic constraints $0 = \tilde{u}$; the variables $z_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}, z_{14}$, are also free "generalized" states (they enter the system statically) but they are constrained by the algebraic constraints.

3.6 Conclusion

In this chapter, on one hand, for linear ODECSs, we modify and simplify the construction of the **MCF** given in [145] by proposing the Morse triangular form **MTF**. On the other hand, the bridge from the **MTF** of ODECSs to the **FBCF** of DAECSs is constructed via the explicitation with driving variables procedure. It is shown that, after attaching a class of ODECSs with two kinds of inputs to a DAECS, we can find connections between their geometric subspaces and canonical forms. Finally, a simple algorithm of constructing transformations from the **MTF** to the **FBCF** is proposed via the explicitation procedure and an example is given to show how our results and algorithms can be applied to concrete physical systems.

3.7 Proofs of the results

3.7.1 Proofs of Proposition 3.2.3, Proposition 3.2.4 and Theorem 3.2.8

Proof of Proposition 3.2.3. If. Suppose that $\Lambda^{uv}_{n,m,s,0}$ and $\tilde{\Lambda}^{u\tilde{v}}_{n,m,s,0}$ are equivalent via $v = F_v x + Ru + \tilde{v}$. Then $\tilde{A} = A + B^v F_v = E_1^\dagger H_1 + B^v F_v$ and $\tilde{B}^u = B^u + B^v R = E_1^\dagger L_1 + B^v R$, which imply that $E_1 \tilde{A} = H_1$ and $E_1 \tilde{B} = L_1$ (since $\operatorname{Im} B^v = \ker E$). Thus $\tilde{A} = \tilde{E}_1^\dagger H_1$ and $\tilde{B} = \tilde{E}_1^\dagger L_1$ for another choice of right inverse $\tilde{E}_1^\dagger$ of E_1. Therefore, $\tilde{\Lambda}^{u\tilde{v}}_{n,m,s,0}$ is also an explicitation of (3.5a).

Only if. Suppose that $\tilde{\Lambda}^{u\tilde{v}}_{n,m,s,0}$ is an explicitation of (3.5a) corresponding the choice of right inverse $\tilde{E}_1^\dagger$ of E_1. Since $E_1^\dagger$ is a right inverse of E_1 if and only if any solution $\dot{x}$ of

$E_1\dot{x} = w$ is given by $E_1^\dagger w$, we have $E_1E_1^\dagger(H_1x + L_1u) = H_1x + L_1u$ and $E_1\tilde{E}_1^\dagger(H_1x + L_1u) = H_1x + L_1u$. It follows that $E_1(\tilde{E}_1^\dagger - E_1^\dagger)(H_1x + L_1u) = 0$ and $(\tilde{E}_1^\dagger - E_1^\dagger)(H_1x + L_1u) \in \ker E_1$, for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, so $(\tilde{E}_1^\dagger - E_1^\dagger)H_1x \in \ker E_1$ and $(\tilde{E}_1^\dagger - E_1^\dagger)L_1u \in \ker E_1$. Since $\ker E_1 = \operatorname{Im} B^v$, it follows that $(\tilde{E}_1^\dagger - E_1^\dagger)L_1 = B^vR$ and $(\tilde{E}_1^\dagger - E_1^\dagger)H_1 = B^vF_v$ (note that we fix B^v) for suitable matrices $R \in \mathbb{R}^{s\times m}$ and $F_v \in \mathbb{R}^{s\times n}$. Therefore $\tilde{A} = \tilde{E}_1^\dagger H_1 = E_1^\dagger H_1 + B^vF_v = A + B^vF_c$ and $\tilde{B}^u = \tilde{E}_1^\dagger L_1 = E_1^\dagger L_1 + B^vR = B^u + B^vR$. □

Proof of Proposition 3.2.4. If. Suppose that Λ^{uv} and $\tilde{\Lambda}^{uv}$ are equivalent via an output injection $Ky = K(Cx + D^uu)$ and an output multiplication $\tilde{y} = T_yy$. Then

$$\tilde{\Lambda}^{uv}: \begin{cases} \dot{x} = \tilde{A}x + \tilde{B}^uu + \tilde{B}^vv = (A + KC)x + (B^u + KD^u)u + B^vv \\ \quad = (E_1^\dagger H_1 + KH_2)x + (E_1^\dagger L_1 + KL_2)u + B^vv \\ \tilde{y} = \tilde{C}x + \tilde{D}u = T_y(Cx + Du) = T_y(H_2x + L_2u), \end{cases}$$

Pre-multiply the differential part of $\tilde{\Lambda}^{uv}$ by E_1, we get (note that $\operatorname{Im} B^v = \ker E_1$)

$$\begin{cases} E_1\dot{x} = (H_1 + E_1KH_2)x + (L_1 + E_1KL_2)u \\ \tilde{y} \ = T_y(H_2x + L_2u), \end{cases}$$

Thus $\tilde{\Lambda}^{uv}$ is an (I, v)-explicitation of the following DAECS:

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} H_1 + E_1KH_2 \\ T_yH_2 \end{bmatrix} x + \begin{bmatrix} L_1 + E_1KL_2 \\ T_yL_2 \end{bmatrix} u.$$

The above DAECS can be transformed from Δ^u via $\tilde{Q} = Q'Q$, where $Q' = \begin{bmatrix} I_q & E_1K \\ 0 & T_y \end{bmatrix}$, it proves that $\tilde{\Lambda}^{uv}$ is a $(\tilde{Q}, v)$-explicitation of Δ^u.

Only if. Suppose that $\tilde{\Lambda}^{uv}_{n,m,s,p} \in \textbf{Expl}(\Delta^u)$ via $\tilde{Q} = Q'Q$, where $Q' = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix}$. Thus via $\tilde{Q}$, Δ^u is ex-fb-equivalent to (3.9) and using $\tilde{E}_1^\dagger Q_1^{-1} = E_1^\dagger Q_1^{-1}$ (note that Q_1 is invertible) and $\tilde{B}^v = B^v$, we can express (3.9) as

$$\begin{cases} \dot{x} = \tilde{E}_1^\dagger Q_1^{-1}(Q_1H_1x + Q_1L_1u) + \tilde{E}_1^\dagger Q_1^{-1}Q_2(H_2x + L_2u) + \tilde{B}^vv \\ \quad = E_1^\dagger(H_1x + L_1u) + E_1^\dagger Q_1^{-1}Q_2(H_2x + L_2u) + B^vv \\ 0 = Q_4(H_2x + L_2u). \end{cases}$$

Thus a $(\tilde{Q}, v)$-explicitation of Δ^u is

$$\tilde{\Lambda}^{uv}: \begin{cases} \dot{x} = Ax + B^uu + K(Cx + D^uu) + B^vv = \tilde{A}x + \tilde{B}^uu + \tilde{B}^vv \\ \tilde{y} = T_y(Cx + Du) = \tilde{C}x + \tilde{D}u, \end{cases}$$

where $T_y = Q_4$, $K = E_1^\dagger Q_1^{-1}Q_2$, which implies that Λ^{uv} and $\tilde{\Lambda}^{uv}$ are equivalent via the output injection $Ky = K(Cx + D^uu)$ and the output multiplication $\tilde{y} = T_yy$. □

Proof of Theorem 3.2.8. Without loss of generality, we assume that the system matrices of $\Delta^u = (E, H, L)$ and $\tilde{\Delta}^{\tilde{u}} = (\tilde{E}, \tilde{H}, \tilde{L})$ are of the following form:

$$E = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} I_{\tilde{q}} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} \tilde{H}_1 \\ \tilde{H}_2 \end{bmatrix}, \quad L = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix},$$

where $H_1 \in \mathbb{R}^{q\times n}$, $L_1 \in \mathbb{R}^{q\times m}$, $\tilde{H}_1 \in \mathbb{R}^{\tilde{q}\times n}$, $\tilde{L}_1 \in \mathbb{R}^{\tilde{q}\times m}$, $q = \operatorname{rank} E$, $\tilde{q} = \operatorname{rank} \tilde{E}$. Since if not, we can always find $Q, \tilde{Q} \in Gl(l,\mathbb{R})$, $P, \tilde{P} \in Gl(n,\mathbb{R})$ such that

$$(QEP^{-1}, QHP^{-1}, QL) \text{ and } (\tilde{Q}\tilde{E}\tilde{P}^{-1}, \tilde{Q}\tilde{H}\tilde{P}^{-1}, \tilde{Q}\tilde{L})$$

are of the above desired form. It is easily seen that the ex-fb-equivalence of (E, H, L) and $(\tilde{E}, \tilde{H}, \tilde{L})$ is equivalent to (implied by and implying) that of (QEP^{-1}, QHP^{-1}, QL) and $(\tilde{Q}\tilde{E}\tilde{P}^{-1}, \tilde{Q}\tilde{H}\tilde{P}^{-1}, \tilde{Q}\tilde{L})$. Thus we can use the above system matrices to represent Δ^u and $\tilde{\Delta}^{\tilde{u}}$ in the remaining part of the proof.

By the assumptions that $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$ and $\tilde{\Lambda}^{\tilde{u}\tilde{v}} \in \mathbf{Expl}(\tilde{\Delta}^{\tilde{u}})$, we have

$$\begin{bmatrix} A & B^u & B^v \\ C & D^u & 0 \end{bmatrix} = \left[\begin{array}{c|c|c} H_1 & L_1 & 0 \\ 0 & 0 & I_{n-q} \\ \hline H_2 & L_2 & 0 \end{array}\right], \quad \begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{bmatrix} = \left[\begin{array}{c|c|c} \tilde{H}_1 & \tilde{L}_1 & 0 \\ 0 & 0 & I_{n-\tilde{q}} \\ \hline \tilde{H}_2 & \tilde{L}_2 & 0 \end{array}\right]. \tag{3.23}$$

We have chosen Λ^{uv} and $\tilde{\Lambda}^{\tilde{u}\tilde{v}}$ as above for convenience, any other choice based on the explicitation procedure could have been made. Since any two ODECSs in an explicitation class are EM-equivalent, the choice of a (Q, v)-explicitation makes no difference when proving EM-equivalence. Therefore, we will use the system matrices in (3.23) for the following proof.

If. Suppose $\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$. Then there exist transformation matrices T_x, T_u, T_v, T_y, F_u, F_v, R, K such that equation (3.10) holds. Substituting the system matrices of (3.23) into (3.10), we have

$$\left[\begin{array}{c|c|c} \tilde{H}_1 & \tilde{L}_1 & 0 \\ 0 & 0 & I_{n-q} \\ \hline \tilde{H}_2 & \tilde{L}_2 & 0 \end{array}\right] = \begin{bmatrix} T_x & T_xK \\ 0 & T_y \end{bmatrix} \left[\begin{array}{c|c|c} H_1 & L_1 & 0 \\ 0 & 0 & I_{n-\tilde{q}} \\ \hline H_2 & L_2 & 0 \end{array}\right] \begin{bmatrix} T_x^{-1} & 0 & 0 \\ F_uT_x^{-1} & T_u^{-1} & 0 \\ (F_v + RF_u)T_x^{-1} & RT_u^{-1} & T_v^{-1} \end{bmatrix}. \tag{3.24}$$

Now represent $T_x = \begin{bmatrix} T_x^1 & T_x^2 \\ T_x^3 & T_x^4 \end{bmatrix}$, where $T_x^1 \in \mathbb{R}^{q\times q}$. By $\tilde{B}^{\tilde{v}} = T_xB^vT_v^{-1}$, we get $\begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} T_x^1 & T_x^2 \\ T_x^3 & T_x^4 \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} T_v^{-1}$, hence it can be deduced that $q = \tilde{q}$ and $T_x^2 = 0$. Moreover, $T_x^4T_v^{-1} = I$ implies that T_x^4 is invertible. Thus by the invertibility of T_x, we have T_x^1 is invertible as well.

Subsequently, premultiply equation (3.24) by $\begin{bmatrix} (T_x^1)^{-1} & 0 & 0 \\ 0 & 0 & I_{l-q} \end{bmatrix}$ and we get

$$\begin{bmatrix} (T_x^1)^{-1} & 0 \\ 0 & I_{l-q} \end{bmatrix} \left[\begin{array}{c|c|c} \tilde{H}_1 & \tilde{L}_1 & 0 \\ \tilde{H}_2 & \tilde{L}_2 & 0 \end{array}\right] = \begin{bmatrix} I_q & K_1 \\ 0 & T_y \end{bmatrix} \left[\begin{array}{c|c|c} H_1 & L_1 & 0 \\ H_2 & L_2 & 0 \end{array}\right] \begin{bmatrix} T_x^{-1} & 0 & 0 \\ F_uT_x^{-1} & T_u^{-1} & 0 \\ (F_v + RF_u)T_x^{-1} & RT_u^{-1} & T_v^{-1} \end{bmatrix},$$

where $K_1 = \begin{bmatrix} I_q & (T_x^1)^{-1}T_x^2 \end{bmatrix} K$. It follows that

$$\left[\begin{array}{c|c} \tilde{H}_1 & \tilde{L}_1 \\ \tilde{H}_2 & \tilde{L}_2 \end{array}\right] = \begin{bmatrix} T_x^1 & T_x^1K_1 \\ 0 & T_y \end{bmatrix} \left[\begin{array}{c|c} H_1 & L_1 \\ H_2 & L_2 \end{array}\right] \begin{bmatrix} T_x^{-1} & 0 \\ F_uT_x^{-1} & T_u^{-1} \end{bmatrix}.$$

Thus $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$ via

$$Q = \begin{bmatrix} T_x^1 & T_x^1 K_1 \\ 0 & T_y \end{bmatrix}, \quad P = T_x, \quad F = F_u, \quad G = T_u^{-1}.$$

Only if: Suppose $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$. Then there exist invertible matrices Q, P, and matrices F, G of appropriate sizes such that equation (3.4) holds. Represent $Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$, where $Q_1 \in \mathbb{R}^{q\times q}$, and $P^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$, where $P_1 \in \mathbb{R}^{q\times q}$. Then by

$$\tilde{E} = QEP^{-1} \Rightarrow \begin{bmatrix} I_{\tilde{q}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix},$$

we immediately get $q = \tilde{q}$ and $Q_1P_1 = I$, $Q_1P_2 = 0$, $Q_3P_1 = 0$, which implies that Q_1, P_1 are invertible matrices, $P_2 = 0$, and $Q_3 = 0$. Thus by the invertibility of Q and P, we have Q_4 and P_4 are invertible matrices as well.

Then by equation (3.4), we get

$$\left[\begin{array}{c|c} \tilde{H}_1 & \tilde{L}_1 \\ \tilde{H}_2 & \tilde{L}_2 \end{array}\right] = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \left[\begin{array}{c|c} H_1 & L_1 \\ H_2 & L_2 \end{array}\right] \begin{bmatrix} P^{-1} & 0 \\ FP^{-1} & G \end{bmatrix}.$$

It implies that the following equation holds:

$$\left[\begin{array}{c|c|c} \tilde{H}_1 & \tilde{L}_1 & 0 \\ 0 & 0 & I_{n-q} \\ \hline \tilde{H}_2 & \tilde{L}_2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} Q_1 & 0 & Q_2 \\ X & P_4^{-1} & 0 \\ \hline 0 & 0 & Q_4 \end{array}\right] \left[\begin{array}{c|c|c} H_1 & L_1 & 0 \\ 0 & 0 & I_{n-\tilde{q}} \\ \hline H_2 & L_2 & 0 \end{array}\right] \begin{bmatrix} P^{-1} & 0 & 0 \\ FP^{-1} & G & 0 \\ Y & Z & P_4 \end{bmatrix},$$

where $X = -P_4^{-1}P_3P_1^{-1}$, $Y = (P_3P_1^{-1}H_1 + P_3P_1^{-1}L_1F)P^{-1}$, $Z = P_3P_1^{-1}L_1G$. Therefore, $\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$ via

$$\begin{aligned} &T_x = P, \quad T_u = G^{-1}, && T_v = P_4^{-1}, && T_y = Q_4, \\ &F_u = F, \quad F_v = P_3P_1^{-1}H_1, && R = P_3P_1^{-1}L_1, && K = \begin{bmatrix} P_1Q_2 \\ P_3Q_2 \end{bmatrix}. \end{aligned}$$

□

3.7.2 Proof of Proposition 3.2.9

Proof. Without loss of generality, we may assume that $\Delta^u_{l,n,m} = (E, H, L)$ is of the following form:

$$\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} u,$$

where $q = \operatorname{rank} E$ and $H_1 \in \mathbb{R}^{q\times q}$, $H_2 \in \mathbb{R}^{q\times(n-q)}$, $H_3 \in \mathbb{R}^{p\times q}$, $H_4 \in \mathbb{R}^{p\times(n-q)}$, $L_1 \in \mathbb{R}^{q\times m}$, $L_2 \in \mathbb{R}^{p\times m}$, where $p = l - q$. Since if not, we can always find $Q \in Gl(l, \mathbb{R})$,

$P \in Gl(n,\mathbb{R})$ such that $\tilde{\Delta}^{\tilde{u}} = (QEP^{-1}, QH, QL)$ is of the above form. Then, it is not hard to verify that $\mathscr{V}_i(\tilde{\Delta}^{\tilde{u}}) = P\mathscr{V}_i(\Delta^u)$, $\mathscr{W}_i(\tilde{\Delta}^{\tilde{u}}) = P\mathscr{W}_i(\Delta^u)$, $\hat{\mathscr{W}}_i(\tilde{\Delta}^{\tilde{u}}) = P\hat{\mathscr{W}}_i(\Delta^u)$. Moreover, for two ODECSs $\Lambda^w = \Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$, $\tilde{\Lambda}^{\tilde{w}} = \tilde{\Lambda}^{\tilde{u}\tilde{v}} \in \mathbf{Expl}(\tilde{\Delta}^{\tilde{u}})$, we can verify that $\mathcal{V}_i(\tilde{\Lambda}^{\tilde{w}}) = P\mathcal{V}_i(\Lambda^w)$, $\mathcal{W}_i(\tilde{\Lambda}^{\tilde{w}}) = P\mathcal{W}_i(\Lambda^w)$, $\hat{\mathcal{W}}_i(\tilde{\Lambda}^{\tilde{w}}) = P\hat{\mathcal{W}}_i(\Lambda^w)$. Therefore, in order to show that the relations of the subspaces (as claimed in Proposition 3.2.9) hold, replacing Δ^u by $\tilde{\Delta}^{\tilde{u}}$ makes no difference and thus we will assume that Δ^u is of the above form in what follows.

Then, the following system, denoted $\Lambda^w = \Lambda^{uv}$, is a (Q,v)-explicitation of Δ^u,

$$\Lambda^w = \Lambda^{uv} : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} L_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I_{n-q} \end{bmatrix} v \\ \quad y \;= H_3 x_1 + H_4 x_2 + L_2 u. \end{cases} \tag{3.25}$$

Firstly, we calculate $\mathcal{V}_i(\Lambda^w)$ through equation (3.44) of Appendix in Section 3.8:

$$\begin{aligned} \mathcal{V}_{i+1}(\Lambda^w) &= \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{V}_i(\Lambda^w) + \operatorname{Im} \begin{bmatrix} B^w \\ D^w \end{bmatrix} \right) \\ &= \begin{bmatrix} H_1 & H_2 \\ 0 & 0 \\ H_3 & H_4 \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathcal{V}_i(\Lambda^w) \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} L_1 & 0 \\ 0 & I_{n-q} \\ L_2 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}^{-1} \left(\begin{bmatrix} [I_q, 0]\mathcal{V}_i(\Lambda^w) \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} L_1 & 0 \\ L_2 & 0 \end{bmatrix} \right) = H^{-1}(E\mathcal{V}_i(\Lambda^w) + \operatorname{Im} L). \end{aligned}$$

Comparing the above expression with equation (3.41) of Appendix, it is easily seen that the subspace sequences $\mathcal{V}_{i+1}(\Lambda^w)$ and $\mathscr{V}_{i+1}(\Delta^u)$ are calculated in the same form. Since $\mathscr{V}_0(\Delta^u) = \mathcal{V}_0(\Lambda^w) = \mathbb{R}^n$, we conclude that $\mathscr{V}_i(\Delta^u) = \mathcal{V}_i(\Lambda^w)$ for $i \in \mathbb{N}$.

Then calculate $\mathscr{W}_{i+1}(\Delta^u)$ via equation (3.42) of Appendix:

$$\begin{aligned} \mathscr{W}_{i+1}(\Delta^u) = E^{-1}(H\mathscr{W}_i(\Delta^u) + \operatorname{Im} L) &= \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \mathscr{W}_i(\Delta^u) + \operatorname{Im} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} H_1 & H_2 & L_1 & 0 \\ H_3 & H_4 & L_2 & 0 \end{bmatrix} \begin{bmatrix} \mathscr{W}_i(\Delta^u) \\ \mathscr{U}_w \end{bmatrix} \right) \\ &= \begin{bmatrix} H_1 & H_2 & L_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} \mathscr{W}_i(\Delta^u) \\ \mathscr{U}_w \end{bmatrix} \cap \ker \begin{bmatrix} H_3 & H_4 & L_2 & 0 \end{bmatrix} \right) + \operatorname{Im} \begin{bmatrix} 0 \\ I_{n-q} \end{bmatrix}. \end{aligned}$$

In the above equation, according to the special form of E, we directly calculate the preimage. Moreover, we can express

$$\begin{bmatrix} 0 \\ I_{n-q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-q} \end{bmatrix} \left(\begin{bmatrix} \mathscr{W}_i(\Delta^u) \\ \mathscr{U}_w \end{bmatrix} \cap \ker \begin{bmatrix} H_3 & H_4 & L_2 & 0 \end{bmatrix} \right).$$

It follows that

$$\mathscr{W}_{i+1}(\Delta^u) = \begin{bmatrix} H_1 & H_2 & L_1 & 0 \\ 0 & 0 & 0 & I_{n-q} \end{bmatrix} \left(\begin{bmatrix} \mathscr{W}_i(\Delta^u) \\ \mathscr{U}_w \end{bmatrix} \cap \ker \begin{bmatrix} H_3 & H_4 & L_2 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} A & B^w \end{bmatrix} \left(\begin{bmatrix} \mathscr{W}_i(\Delta^u) \\ \mathscr{U}_w \end{bmatrix} \cap \ker \begin{bmatrix} C & D^w \end{bmatrix} \right).$$

It is seen from the above equation and (3.46) of Appendix that the subspace sequences $\mathcal{W}_{i+1}(\Lambda^w)$ and $\mathscr{W}_{i+1}(\Delta^u)$ are calculated in the same form. Since the initial conditions $\mathcal{W}_0(\Lambda^w) = \mathscr{W}_0(\Delta^u) = \{0\}$, we conclude that $\mathcal{W}_{i+1}(\Lambda^w) = \mathscr{W}_{i+1}(\Delta^u)$ for all $i \in \mathbb{N}$.

Then from (3.42) and (3.43), it is seen that the subspaces sequences $\mathscr{W}_i$ and $\hat{\mathscr{W}}_i$ are calculated in the same form, their difference comes from their initial conditions only. Similarly, from (3.46) and (3.48), it is seen that $\mathcal{W}_i$ and $\hat{\mathcal{W}}_i$ have different initial conditions but evolve in the same way. Thus, by $\hat{\mathcal{W}}_1(\Lambda^w) = \hat{\mathscr{W}}_1(\Delta^u) = \ker E = \operatorname{Im} B^v$, we get $\hat{\mathcal{W}}_i(\Lambda^w) = \hat{\mathscr{W}}_i(\Delta^u)$ for all $i \in \mathbb{N}^+$. $\square$

3.7.3 Proof of Proposition 3.3.1

Proof. Observe that the transformation matrix T_s decomposes the state space $\mathscr{X}$ of Λ^u into $\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \mathscr{X}_3 \oplus \mathscr{X}_4$, where $\mathscr{X}_1 = \mathcal{V}^* \cap \mathcal{W}^*$, $\mathscr{X}_1 \oplus \mathscr{X}_2 = \mathcal{V}^*$, $\mathscr{X}_1 \oplus \mathscr{X}_3 = \mathcal{W}^*$, $(\mathcal{V}^* + \mathcal{W}^*) \oplus \mathscr{X}_4 = \mathscr{X}$. The transformation matrix T_i decomposes the input space $\mathscr{U}_u$ into $\mathscr{U}_u = \mathscr{U}_1 \oplus \mathscr{U}_2$, where $\mathscr{U}_1 = \mathcal{U}_u^*$, $\mathscr{U}_1 \oplus \mathscr{U}_2 = \mathscr{U}_u$. The transformation matrix T_o decomposes the output space $\mathscr{Y}$ into $\mathscr{Y} = \mathscr{Y}_1 \oplus \mathscr{Y}_2$, where $\mathscr{Y}_1 = \mathcal{Y}^*$, $\mathscr{Y}_1 \oplus \mathscr{Y}_2 = \mathscr{Y}$. Let $\Lambda' = (A', B', C', D') = M_{tran}(\Lambda^u)$, where M_{tran} is the Morse transformation $M_{tran} = (T_s, T_i, T_o, 0, 0)$. Then consider the following equation and subspaces:

$$\begin{aligned} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} &= \begin{bmatrix} T_s & 0 \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B^u \\ C & D^u \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ 0 & T_i^{-1} \end{bmatrix} \\ &= \left[\begin{array}{cccc|cc} A_1^1 & A_1^2 & A_1^3 & A_1^4 & B_1^1 & B_1^2 \\ A_2^1 & A_2^2 & A_2^3 & A_2^4 & B_2^1 & B_2^2 \\ A_3^1 & A_3^2 & A_3^3 & A_3^4 & B_3^1 & B_3^2 \\ A_4^1 & A_4^2 & A_4^3 & A_4^4 & B_4^1 & B_4^2 \\ \hline C_3^1 & C_3^2 & C_3^3 & C_3^4 & D_3^1 & D_3^2 \\ C_4^1 & C_4^2 & C_4^3 & C_4^4 & D_4^1 & D_4^2 \end{array}\right], \quad \begin{array}{cc} \mathcal{V}^*(\Lambda') : \begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix}, & \mathcal{W}^*(\Lambda') : \begin{bmatrix} * \\ 0 \\ * \\ 0 \end{bmatrix}, \\ \mathcal{U}_u{}^*(\Lambda') : \begin{bmatrix} * \\ 0 \end{bmatrix}, & \mathcal{Y}^*(\Lambda') : \begin{bmatrix} * \\ 0 \end{bmatrix}. \end{array} \end{aligned}$$

Now, applying (3.45), for $i = n$, to both Λ' and the dual system of Λ', we have

$$\begin{bmatrix} B' \\ D' \end{bmatrix} \mathcal{U}_u{}^* \subseteq \begin{bmatrix} \mathcal{V}^* \\ 0 \end{bmatrix}, \quad \begin{bmatrix} (C')^T \\ (D')^T \end{bmatrix} (\mathcal{Y}^*)^\perp \subseteq \begin{bmatrix} (\mathcal{W}^*)^\perp \\ 0 \end{bmatrix}.$$

It follows that B_3^1, B_4^1, C_4^1, C_4^3, D_3^1, D_4^1, D_2^4 are all zero.

Then applying (3.44) for $i = n$, to both Λ' and its dual system, we have

$$\begin{bmatrix} A'\mathcal{V}^* \\ C'\mathcal{V}^* \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{V}^* \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} B' \\ D' \end{bmatrix}, \tag{3.26}$$

$$\begin{bmatrix} (A')^T(\mathcal{W}^*)^\perp \\ (B')^T(\mathcal{W}^*)^\perp \end{bmatrix} \subseteq \begin{bmatrix} (\mathcal{W}^*)^\perp \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} (C')^T \\ (D')^T \end{bmatrix}. \tag{3.27}$$

The lower parts of equations (3.26) and (3.27) give $C'\mathcal{V}^* \subseteq \operatorname{Im} D'$ and $(B')^T(\mathcal{W}^*)^\perp \subseteq \operatorname{Im}(D')^T$, which implies that B_2^1 and C_2^4 are zero. On the other hand, equation (3.26) gives that

$$\operatorname{Im}\begin{bmatrix} A_3^1 \\ A_4^1 \\ C_3^1 \end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix} B_3^2 \\ B_4^2 \\ D_3^2 \end{bmatrix} \quad \text{and} \quad \operatorname{Im}\begin{bmatrix} A_3^2 \\ A_4^2 \\ C_3^2 \end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix} B_3^2 \\ B_4^2 \\ D_3^2 \end{bmatrix},$$

implying that there exist matrices $F_1 \in \mathbb{R}^{m_2 \times n_1}$ and $F_2 \in \mathbb{R}^{m_2 \times n_2}$ such that

$$\begin{bmatrix} A_3^1 \\ A_4^1 \\ C_3^1 \end{bmatrix} = -\begin{bmatrix} B_3^2 \\ B_4^2 \\ D_3^2 \end{bmatrix} F_1 \quad \text{and} \quad \begin{bmatrix} A_3^2 \\ A_4^2 \\ C_3^2 \end{bmatrix} = -\begin{bmatrix} B_3^2 \\ B_4^2 \\ D_3^2 \end{bmatrix} F_2.$$

Then setting $F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_1 & F_2 & 0 & 0 \end{bmatrix}$, we have

$$\begin{bmatrix} T_s & 0 \\ 0 & T_o \end{bmatrix}\begin{bmatrix} A & B^u \\ C & D^u \end{bmatrix}\begin{bmatrix} T_s^{-1} & 0 \\ T_i^{-1}F & T_i^{-1} \end{bmatrix} = \left[\begin{array}{cccc|cc} A_1^1 + B_1^2 F_1 & A_1^2 + B_1^2 F_2 & A_1^3 & A_1^4 & B_1^1 & B_1^2 \\ A_2^1 + B_2^2 F_1 & A_2^2 + B_2^2 F_1 & A_2^3 & A_2^4 & 0 & B_2^2 \\ 0 & 0 & A_3^3 & A_3^4 & 0 & B_3^2 \\ 0 & 0 & A_4^3 & A_4^4 & 0 & B_4^2 \\ \hline 0 & 0 & C_3^3 & C_3^4 & 0 & D_3^2 \\ 0 & 0 & 0 & C_4^4 & 0 & 0 \end{array}\right].$$

Since $\mathcal{W}^*$ is feedback invariant, equation (3.27) also holds for the above transformed system. Thus the upper part of (3.27) becomes

$$(A' + B'F)^T(\mathcal{W}^*(\Lambda'))^\perp \subseteq (\mathcal{W}^*(\Lambda'))^\perp + \operatorname{Im}(C')^T,$$

which gives that $(A_2^1 + B_1^2 F_1)^T = 0$,

$$\operatorname{Im}\begin{bmatrix} (A_2^3)^T \\ (B_2^2)^T \end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix} (C_1^3)^T \\ (D_1^2)^T \end{bmatrix} \quad \text{and} \quad \operatorname{Im}\begin{bmatrix} (A_4^3)^T \\ (B_4^2)^T \end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix} (C_3^3)^T \\ (D_3^2)^T \end{bmatrix}.$$

It follows that there exist $K_1 \in \mathbb{R}^{n_2 \times p_1}$ and $K_2 \in \mathbb{R}^{n_4 \times p_1}$ such that

$$\begin{bmatrix} (A_2^3)^T \\ (B_2^2)^T \end{bmatrix} = -\begin{bmatrix} (C_3^3)^T \\ (D_3^2)^T \end{bmatrix} K_1^T \quad \text{and} \quad \begin{bmatrix} (A_4^3)^T \\ (B_4^2)^T \end{bmatrix} = -\begin{bmatrix} (C_3^3)^T \\ (D_3^2)^T \end{bmatrix} K_2^T.$$

Let $K = \begin{bmatrix} 0 & K_1^T & 0 & K_2^T \\ 0 & 0 & 0 & 0 \end{bmatrix}^T$, which implies that

$$\begin{bmatrix} T_s & KT_o \\ 0 & T_o \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} T_s^{-1} & 0 \\ T_i^{-1}F & T_i^{-1} \end{bmatrix} =$$

$$\left[\begin{array}{cccc|cc} A_1^1 + B_1^2 F_1 & A_1^2 + B_1^2 F_2 & A_1^3 & A_1^4 & B_1^1 & B_1^2 \\ 0 & A_2^2 + B_2^2 F_1 & 0 & A_2^4 + K_1 C_1^4 & 0 & 0 \\ 0 & 0 & A_3^3 & A_3^4 & 0 & B_3^2 \\ 0 & 0 & 0 & A_4^4 + K_2 C_3^4 & 0 & 0 \\ \hline 0 & 0 & C_3^3 & C_3^4 & 0 & D_3^2 \\ 0 & 0 & 0 & C_4^4 & 0 & 0 \end{array}\right].$$

Now it is seen that there exist $K_{MT} = T_s^{-1}KT_o$ and $F_{MT} = T_i^{-1}FT_s$ such that $\tilde{\Lambda}^{\tilde{u}} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{C}, \tilde{D}^{\tilde{u}})$ has the form of (3.13), where

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} \end{bmatrix} = \begin{bmatrix} T_s & T_sK_{MT} \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B^u \\ C & D^u \end{bmatrix} \begin{bmatrix} T_s{}^{-1} & 0 \\ F_{MT}T_s{}^{-1} & T_i{}^{-1} \end{bmatrix}.$$

The system matrices of $\tilde{\Lambda}^u$, see (3.13), are $\tilde{A}_1 = A_1^1 + B_1^2F_1$, $\tilde{A}_1^2 = A_1^2$, $\tilde{A}_1^3 = A_1^3$, $\tilde{A}_1^4 = A_1^4$, $\tilde{B}_1 = B_1^1$, $\tilde{B}_1^2 = B_1^2$, $\tilde{A}_2 = A_2^2 + B_2^2F_1$, $\tilde{A}_2^4 = A_2^4 + K_1C_1^4$, $\tilde{A}_3 = A_3^3$, $\tilde{A}_3^4 = A_3^4$, $\tilde{B}_3 = B_3^2$, $\tilde{A}_4 = A_4^4 + K_2C_1^4$, $\tilde{C}_3 = C_3^3$, $\tilde{C}_3^4 = C_3^4$, $\tilde{D}_3 = D_3^2$, $\tilde{C}_4 = C_4^4$.

Now we will show that $(\tilde{A}_1, \tilde{B}_1)$ is controllable. By Lemma 4 of [145], for $\tilde{\Lambda}^{\tilde{u}}$,

$$\mathcal{W}_i|_{\mathcal{U}_u^*}(\tilde{\Lambda}^{\tilde{u}}) = \mathcal{W}_i(\tilde{\Lambda}^{\tilde{u}}) \cap \mathcal{V}^*(\tilde{\Lambda}^{\tilde{u}}), \tag{3.28}$$

where $\mathcal{W}_i|_{\mathcal{U}_u^*}$ is the subspace $\mathcal{W}_i$ when the input is restricted to $\mathcal{U}_u^*$. Use system matrices (3.13) to calculate $\mathcal{W}_i(\tilde{\Lambda})|_{\mathcal{U}_w^*}$ and $\mathcal{W}_i(\tilde{\Lambda}^{\tilde{u}}) \cap \mathcal{V}^*(\tilde{\Lambda}^{\tilde{u}})$, which gives

$$\mathcal{W}_n(\tilde{\Lambda}^{\tilde{u}})|_{\mathcal{U}_u^*} = \mathscr{B}_1 + \tilde{A}_1\mathscr{B}_1 + \cdots + (\tilde{A}_1)^{n-1}\mathscr{B}_1 \overset{(3.28)}{=} \mathcal{W}_n(\tilde{\Lambda}^{\tilde{u}}) \cap \mathcal{V}^*(\tilde{\Lambda}^{\tilde{u}}), \tag{3.29}$$

where $\mathscr{B}_1 = \operatorname{Im}[\tilde{B}_1 \quad 0 \quad 0 \quad 0]^T$. We can see from the above equation that the reachability space of $(\tilde{A}_1, \tilde{B}_1)$ is $\mathcal{W}^*(\Lambda) \cap \mathcal{V}^*(\Lambda) = \mathscr{X}_1$, which implies that $(\tilde{A}_1, \tilde{B}_1)$ is controllable. Since the proof of the observability of $(\tilde{C}_4, \tilde{A}_4)$ is completely dual to the above proof, we omit this part.

Subsequently, we prove that the system $\Lambda^3 = (\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$, given by (3.13), is prime. Using the system matrices of $\tilde{\Lambda}^u$ to calculate $\mathcal{W}^*|_{(\mathcal{U}_u^*)^\perp}$, we get

$$\mathcal{W}^*(\tilde{\Lambda}^u)|_{(\mathcal{U}_u^*)^\perp} = * \times \{0\} \times \mathcal{W}^*(\tilde{\Lambda}^3) \times \{0\},$$

where "$*$" denotes the terms which are irrelevant. From $\mathcal{W}^* = \mathcal{W}_n|_{(\mathcal{U}_u^*)} \oplus \mathcal{W}_n|_{(\mathcal{U}_u^*)^\perp}$ and equation (3.29), we can deduce that $\mathcal{W}^*(\tilde{\Lambda}^3) = \mathscr{X}(\tilde{\Lambda}^3) = \mathscr{X}_3(\tilde{\Lambda}^u)$. Moreover, by a direct calculation, we get

$$\mathcal{Y}^*(\tilde{\Lambda}^u) = \mathscr{Y}(\tilde{\Lambda}^3) = \tilde{C}_3\mathcal{W}^*(\tilde{\Lambda}^3) + \tilde{D}_3\mathscr{U}_w(\tilde{\Lambda}^3), \;\; \mathcal{V}^*(\tilde{\Lambda}^3) = 0, \;\; \mathcal{U}_w^*(\tilde{\Lambda}^3) = 0.$$

Finally, by Theorem 10 of [145], we conclude that $\tilde{\Lambda}^3 = (\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$ is prime. □

3.7.4 Proof of Proposition 3.3.2

Proof. First, by Algorithm 3.3.3 and a direct calculation, we have

$$\begin{aligned}
&\bar{A}_1 = \tilde{A}_1 + \tilde{B}_1F_{MN}^1, && \bar{A}_1^3 = \tilde{A}_1^3 + \tilde{B}_1^2F_{MN}^2 + K_{MN}^1\tilde{C}_3 + K_{MN}^1\tilde{D}_3F_{MN}^2, \\
&\bar{A}_4 = \tilde{A}_4 + K_{MN}^3\tilde{C}_3^4, && \bar{A}_3 = \tilde{A}_3 + K_{MN}^2\tilde{C}_3 + \tilde{B}_3F_{MN}^2 + K_{MN}^2\tilde{D}_3F_{MN}^2, \\
&\bar{B}_3 = \tilde{B}_3 + K_{MN}^2\tilde{D}_3, && \bar{A}_1^4 = \tilde{A}_1^4 + \tilde{B}_1^2F_{MN}^3 + K_{MN}^1\tilde{C}_3 + K_{MN}^1\tilde{D}_3F_{MN}^3, \\
&\bar{B}_1^2 = \tilde{B}_1^2 + K_{MN}^1\tilde{D}_3, && \bar{A}_3^4 = \tilde{A}_3^4 + \tilde{B}_3F_{MN}^3 + K_{MN}^2\tilde{C}_3^4 + K_{MN}^2\tilde{D}_3F_{MN}^3, \\
&\bar{C}_3 = \tilde{C}_3 + \tilde{D}_3F_{MN}^2, && \bar{C}_3^4 = \tilde{C}_3^4 + \tilde{D}_3F_{MN}^3.
\end{aligned}$$

We will show that we can always assume $\tilde{D}_3 = 0$. To this end, we can find a change of coordinates in the input and output spaces to obtain $\tilde{D}_3 = \begin{bmatrix} 0 & I_\delta \\ 0 & 0 \end{bmatrix}$. Then by suitable choice of feedback and output injection transformation, the 5-tuple $(\tilde{B}_1^2, \tilde{B}_3, \tilde{C}_3, \tilde{C}_3^4, \tilde{D}_3)$ can be brought into the following form:

$$\left[\begin{array}{cc|c} * & * & \tilde{B}_1^2 \\ * & * & \tilde{B}_3 \\ \hline \tilde{C}_3 & \tilde{C}_3^4 & \tilde{D}_3 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} * & * & \hat{B}_1^2 & 0 \\ * & * & \hat{B}_3 & 0 \\ \hline 0 & 0 & 0 & I_\delta \\ \hat{C}_3 & \hat{C}_3^4 & 0 & 0 \end{array}\right].$$

Now, by deleting the zero columns of $\hat{B}$ and the zero rows of $\hat{C}$, we get

$$\left[\begin{array}{cc|c} * & * & \hat{B}_1^2 \\ * & * & \hat{B}_3 \\ \hline \hat{C}_3 & \hat{C}_3^4 & \hat{D}_3 \end{array}\right],$$

whose $\tilde{D}_3$-matrix is $\hat{D}_3 = 0$.

Now with the assumption $\tilde{D}_3 = 0$, we show that the constrained Sylvester equations of (3.16) can be reduced to normal Sylvester equations by a suitable choice of F_{MN} and K_{MN}. We claim that the following matrix equation

$$\tilde{B}_1^2 = -\hat{T}_{MN}^2 \tilde{B}_3 \tag{3.30}$$

is solvable for $\hat{T}_{MN}^2$. This claim can be proved using the following observation,

$$\begin{bmatrix} \tilde{B}(\mathcal{U}_u^*)^\perp \\ \tilde{D}(\mathcal{U}_u^*)^\perp \end{bmatrix} \cap \begin{bmatrix} \mathcal{V}^* \\ 0 \end{bmatrix} = 0. \tag{3.31}$$

Note that the above equation is a consequence of the definition of $\mathcal{U}_u^*$ (see equation (3.45)). Now by (3.31), we have

$$\mathrm{Im}\left(\mathrm{col}\begin{bmatrix} \tilde{B}_1^2 & 0 & \tilde{B}_3 & 0 & \tilde{D}_3 & 0 \end{bmatrix}\right) \cap \begin{bmatrix} \mathcal{V}^* \\ 0 \end{bmatrix} = 0.$$

Since $\tilde{D}_3$ is already zero by assumption, the above equation proves that (3.30) is solvable for $\hat{T}_{MN}^2$. Consequently, substitute (3.30) into the upper equations of (3.16) and we get

$$\bar{A}_1 \bar{T}_{MN}^2 - \bar{T}_{MN}^2 \bar{A}_3 = -\bar{A}_1^3 + \bar{A}^1 \hat{T}_{MN}^2 - \hat{T}_{MN}^2 \bar{A}_3, \quad \bar{T}_{MN}^2 \bar{B}_3 = 0, \tag{3.32}$$

where $\bar{T}_{MN}^2 = T_{MN}^2 + \hat{T}_{MN}^2$.

Furthermore, since $(\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$ is prime (the result of Proposition 3.3.1), we can always assume $\tilde{B}_3 = [I_{m_3}, 0]^T$ and $\tilde{C}_3 = [I_{p_3}, 0]$ (if not, use coordinates transformations such that $\tilde{B}_3$ and $\tilde{C}_3$ are of that form), where $m_3 = \operatorname{rank} \tilde{B}_3 = \dim(\mathcal{U}_u^*)^\perp = p_3 =$

$\operatorname{rank} \tilde{C}_3 = \dim \mathcal{Y}^*$. Then, it is possible to choose K^1_{MN}, K^2_{MN}, F^2_{MN} such that the 4-tuple $(\hat{A}^3_1, \bar{A}_3, \bar{B}^2_1, \bar{C}_3)$ is transformed into the following form:

$$\left[\begin{array}{c|c} \hat{A}^3_1 & \\ \hline \bar{A}_3 & \bar{B}_3 \\ \hline \tilde{C}_3 & \end{array}\right] = \left[\begin{array}{cc|c} 0 & \hat{A}^{3'}_1 & \\ \hline 0 & 0 & I_{m_3} \\ 0 & \bar{A}'_3 & 0 \\ \hline I_{p_3} & 0 & \end{array}\right].$$

Thus $\bar{T}^2_{MN}$ in equation (3.32) is of the form $\bar{T}^2_{MN} = [\bar{G}'_2 \;\; 0]$ (since $\bar{T}^2_{MN}\bar{B}_3 = 0$). Hence, solving $\bar{T}^2_{MN}$ via equation (3.32) is equivalent to solving $\bar{G}'_2$ via

$$\bar{A}_1 \begin{bmatrix} 0 & \bar{G}'_2 \end{bmatrix} - \begin{bmatrix} 0 & \bar{G}'_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}'_3 \end{bmatrix} = \begin{bmatrix} 0 & \hat{A}^{3'}_1 \end{bmatrix}.$$

Therefore, the upper part of the constrained Sylvester equations of (3.16) can be reduced to the above normal Sylvester equation. The reduction of the lower part of (3.16) to a normal Sylvester equation follows dually from the above result and we will omit that proof.

Moreover, from Proposition 3.3.1, we have that the pair $(\tilde{A}_1, \tilde{B}_1)$ is controllable and the pair $(\tilde{C}_4, \tilde{A}_4)$ is observable. By the standard matrix theory, we can choose F_{MN} and K_{MN} such that the eigenvalues of $\bar{A}_1$, $\bar{A}_2$, $\bar{A}'_3$,and $\bar{A}^4_4$ are disjoint. Then there exist unique solutions for T^1_{MN}, T^2_{MN}, T^3_{MN}, T^4_{MN}, T^5_{MN} in (3.15) and (3.16). Furthermore, it is not hard to see that the state coordinates transformation matrix G brings $\tilde{\Lambda}^u$ into $\bar{\Lambda}^{\bar{u}}$. Feedback transformations preserve controllability, so controllability of $(\tilde{A}_1, \tilde{B}_1)$ implies controllability of $(\bar{A}_1, \bar{B}_1)$; output injection preserves observability, so observability of $(\tilde{C}_4, \tilde{A}_4)$ implies observability of $(\bar{C}_4, \bar{A}_4)$. We have $(\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3) \overset{M}{\sim} (\bar{A}_3, \bar{B}_3, \bar{C}_3, \bar{D}_3)$ and the fact that the 4-tuple $(\bar{A}_3, \bar{B}_3, \bar{C}_3, \bar{D}_3)$ is prime is inherited from the fact that $(\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$ is prime (see this property of prime systems in [145]). □

3.7.5 Proof of Theorem 3.4.2

Proof. By Theorem 3.3.5, for a given ODECS $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$, there exists an extended Morse transformation EM_{tran} such that $EM_{tran}(\Lambda^{uv})$ is in the **EMNF**. Therefore, the starting point of this proof is the **EMNF** given by (3.18). Since the system represented in the **EMNF** is already decoupled into four independent subsystems, we only need to transform each subsystem into its corresponding canonical form.

(i) We will prove that any controllable $\Lambda^{uv}_{n,m,s} = (A, B^u, B^v)$ can be transformed into the Brunovský canonical form with indices $(\epsilon_1, \ldots, \epsilon_m)$ and $(\bar{\epsilon}_1, \ldots, \bar{\epsilon}_s)$, then the transformation from $(\bar{A}_1, \bar{B}^u_1, \bar{B}^v_1)$ to $\left(\begin{bmatrix} A^{cu} & 0 \\ 0 & A^{cv} \end{bmatrix}, \begin{bmatrix} B^{cu} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ B^{cv} \end{bmatrix}\right)$ is straightforward to see. Since $\Lambda^{uv} = (A, B^u, B^v)$ is a control system without output, in view of the extended Morse equivalence of Definition 3.2.6, we just need to prove that there exist transformation matrices T_x, T_u, T_v, F_u, F_v, R such that the transformed system matrices

$$\left(T_x \left(A + B^u F_u + B^v \left(F_v + R F_u\right)\right) T_x^{-1}, T_x \left(B^u + B^v R\right) T_u^{-1}, T_x B^v T_v^{-1}\right)$$

are in the Brunovský canonical form. First, from the classical linear system theory (see, e.g., [31]), using only a state coordinates transformation and state feedback, i.e., choosing suitable T_x, F_v, F_u, and setting $T_u = I_m$, $T_v = I_s$, $R = 0$, we can transform Λ^{uv} into the following form:

$$\begin{cases} x_1^{(\kappa_1)} &= b_1^1 u_1 + \cdots + b_m^1 u_m + \bar{b}_1^1 v_1 + \cdots + \bar{b}_s^1 v_s, \\ x_2^{(\kappa_2)} &= b_1^2 u_1 + \cdots + b_m^2 u_m + \bar{b}_1^2 v_1 + \cdots + \bar{b}_s^2 v_s, \\ & \ldots \\ x_{m+s}^{(\kappa_{m+s})} &= b_1^{m+s} u_1 + \cdots + b_m^{m+s} u_m + \bar{b}_1^{m+s} v_1 + \cdots + \bar{b}_s^{m+s} v_s, \end{cases} \tag{3.33}$$

Moreover, without loss of generality, we assume $\operatorname{rank} B^w = m + s$ (since if not, we can always permute the variables of u and v such that the first m_1 columns of B^u and the first s_1 columns of B^v are independent, where $m_1 = \operatorname{rank} B^w$ and $s_1 = \operatorname{rank} B^v$, then we will work with the matrices with these independent columns), it can be deduced that the matrix

$$\Gamma = \begin{bmatrix} b_1^1 & \ldots & b_m^1 & \bar{b}_1^1 & \ldots & \bar{b}_s^1 \\ b_1^2 & \ldots & b_m^2 & \bar{b}_1^2 & \ldots & \bar{b}_s^2 \\ & \ldots & & & \ldots & \\ b_1^{m+s} & \ldots & b_m^{m+s} & \bar{b}_1^{m+s} & \ldots & \bar{b}_s^{m+s} \end{bmatrix}$$

is invertible. Furthermore, κ_i for $1 \le i \le m+s$ are the controllability indices of the pair $(A, \bar{B}^w)$, where $\bar{B}^w = \begin{bmatrix} \bar{B}^u & \bar{B}^v \end{bmatrix}$. Now without loss of generality, we may assume $\kappa_1 \ge \kappa_2 \ge \cdots \ge \kappa_{m+s}$. In the case of the Brunovský form for classical ODECS (with one kind of inputs), we can use Γ as an input coordinates transformation matrix. However, Δ^{uv} has two kinds of inputs and the input coordinates transformation matrix should have a triangular form (see Remark 3.2.7(ii)). In order to have such an input coordinates transformation matrix , we implement the following procedure:

Step 1: Starting from $i = 1$, search for the largest integer i, denoted by i_1^*, such that at least one of $\bar{b}_j^i$, $1 \le j \le s$, is not zero. Then we set

$$\epsilon_1 = \kappa_1, \quad \epsilon_2 = \kappa_2, \quad \ldots, \quad \epsilon_{i_1^*-1} = \kappa_{i_1^*-1}; \quad \bar{\epsilon}_1 = \kappa_{i_1^*}.$$

Now using the feedback transformation

$$\tilde{v}_1 = b_1^{i_1^*} u_1 + \cdots + b_m^{i_1^*} u_m + \bar{b}_1^{i_1^*} v_1 + \cdots + \bar{b}_s^{i_1^*} v_s,$$

we get

$$x_{i_1^*}^{(\bar{\epsilon}_1)} = \tilde{v}_1, \; x_{i_1^*+1}^{(\kappa_{i_1^*+1})} = \sum_{j=1}^{m} \tilde{b}_j^{i_1^*+1} \tilde{u}_j + \sum_{j=1}^{s} \tilde{\bar{b}}_j^{i_1^*+1} \tilde{v}_j, \ldots, \dot{x}_{m+s}^{(\kappa_{m+s})} = \sum_{j=1}^{m} \tilde{b}_j^{m+s} \tilde{u}_j + \sum_{j=1}^{s} \tilde{\bar{b}}_j^{m+s} \tilde{v}_j. \tag{3.34}$$

The terms $\tilde{b}_j^i$, $i_1^* + 1 \le i \le m+s$, $1 \le j \le m$ and $\tilde{\bar{b}}_j^i$, $i_1^* + 1 \le i \le m+s$, $1 \le j \le s$ in the above equation can be easily calculated. Note that since $\bar{b}_j^i = 0$, $1 \le i \le i_1^* - 1$, $1 \le j \le s$, the feedback transformation does not affect the subsystems whose states are x_i, $1 \le i \le i_1^* - 1$. Hence the remaining terms in equation (3.33) are kept the same. In the following, to simplify the notation, we will drop the tildes in equation (3.34).

The following construction of new coordinates is essential: for $1 \leq j \leq m+s-i_1^*$, set

$$\begin{cases} \tilde{x}^1_{i_1^*+j} &= x^1_{i_1^*+j} - \bar{b}_1^{i_1^*+j} x_{i_1^*}^{\kappa_{i_1^*}-\kappa_{i_1^*+j}+1}, \\ \tilde{x}^2_{i_1^*+j} &= x^2_{i_1^*+j} - \bar{b}_1^{i_1^*+j} x_{i_1^*}^{\kappa_{i_1^*}-\kappa_{i_1^*+j}+2}, \\ \dots \\ \tilde{x}^{\kappa_{i_1^*+j}}_{i_1^*+j} &= x^{\kappa_{i_1^*+j}}_{i_1^*+j} - \bar{b}_1^{i_1^*+j} x_{i_1^*}^{\kappa_{i_1^*}} \end{cases}$$

and we do not change the remaining state coordinates. It can be seen that by construction, all $\bar{b}_1^i = 0$ for $i_1^* + 1 \leq i \leq m+s$ and thus equation (3.33) becomes

$$\begin{cases} x_1^{(\epsilon_1)} &= b_1^1 u_1 + \cdots + b_m^1 u_m, \\ & \dots \\ x_{i_1^*-1}^{(\epsilon_{i_1^*-1})} &= b_1^{i_1^*-1} u_1 + \cdots + b_m^{i_1^*-1} u_m, \\ x_{i_1^*}^{(\bar{\epsilon}_1)} &= v_1, \\ \tilde{x}_{i_1^*+1}^{(\kappa_{i_1^*+1})} &= b_1^{i_1^*+1} u_1 + \cdots + b_m^{i_1^*+1} u_m + \bar{b}_2^{i_1^*+1} v_2 + \cdots + \bar{b}_s^{i_1^*+1} v_s, \\ & \dots \\ \tilde{x}_{m+s}^{(\kappa_{m+s})} &= b_1^{m+s} u_1 + \cdots + b_m^{m+s} u_m + \bar{b}_2^{m+s} v_2 + \cdots + \bar{b}_s^{m+s} v_s. \end{cases} \tag{3.35}$$

Then, drop all the tildes in equation (3.35) and go to next step.

Step k ($k > 1$): By constructions in former steps, we have $\bar{b}_j^i = 0$ for $i_{k-1}^* + 1 \leq i \leq m+s$, $1 \leq j \leq k-1$. Then staring from $i = i_{k-1}^* + 1$, we search for the largest i, denoted by i_k^*, such that at least one of $\bar{b}_j^i$, $k \leq j \leq s$ is not zero. Then we set

$$\epsilon_{i_{k-1}^*+1} = \kappa_{i_{k-1}^*+1}, \quad \epsilon_{i_{k-1}^*+2} = \kappa_{i_{k-1}^*+2}, \quad \dots, \quad \epsilon_{i_k^*-1} = \kappa_{i_k^*-1}; \quad \bar{\epsilon}_{i_k^*} = \kappa_{i_k^*}.$$

Now using the feedback transformation

$$\tilde{v}_k = b_1^{i_k^*} u_1 + \cdots + b_m^{i_k^*} u_m + \bar{b}_1^{i_k^*} v_1 + \cdots + \bar{b}_s^{i_k^*} v_s,$$

we get that

$$x_{i_k^*}^{(\bar{\epsilon}_k)} = \tilde{v}_k,\ \dot{x}_{i_k^*+1}^{(\kappa_{i_k^*+1})} = \sum_{j=1}^{m} \tilde{b}_j^{i_k^*+1} \tilde{u}_j + \sum_{j=1}^{s} \tilde{\bar{b}}_j^{i_k^*+1} \tilde{v}_j, \dots, \dot{x}_{m+s}^{(\kappa_{m+s})} = \sum_{j=1}^{m} \tilde{b}_j^{m+s} \tilde{u}_j + \sum_{j=1}^{s} \tilde{\bar{b}}_j^{m+s} \tilde{v}_j. \tag{3.36}$$

Note that the terms $\tilde{b}_j^i$, $i_k^* + 1 \leq i \leq m+s$, $1 \leq j \leq m$ and $\tilde{\bar{b}}_j^i$, $i_k^* + 1 \leq i \leq m+s$, $1 \leq j \leq s$ can be easily calculated. In the following, to simplify the notation, we will drop these tildes.

Then construct the following new coordinates, for $1 \leq j \leq m+s-i_k^*$, set

$$\begin{cases} \tilde{x}^1_{i_k^*+j} &= x^1_{i_k^*+j} - \bar{b}_1^{i_k^*+j} x_{i_k^*}^{\kappa_{i_k^*}-\kappa_{i_k^*+j}+1} \\ \tilde{x}^2_{i_k^*+j} &= x^2_{i_k^*+j} - \bar{b}_1^{i_k^*+j} x_{i_k^*}^{\kappa_{i_k^*}-\kappa_{i_k^*+j}+2} \\ \dots \\ \tilde{x}^{\kappa_{i_k^*+j}}_{i_k^*+j} &= x^{\kappa_{i_k^*+j}}_{i_k^*+j} - \bar{b}_1^{i_k^*+j} x_{i_k^*}^{\kappa_{i_k^*}}, \end{cases} \tag{3.37}$$

and keep the remaining ones unchanged. Again, it is seen that, by construction, all $\bar{b}_k^i = 0$ for $i_k^* + 1 \le i \le m + s$. Then, drop all the tildes for the transformed system and go to next step.

It is seen that after s steps, the matrix Γ becomes

$$\Gamma = \begin{bmatrix} T_u^1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ T_u^2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & & & & & \\ T_u^s & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 \\ T_u^{s+1} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad T_u^k = \begin{bmatrix} b_1^{i_{k-1}^*+1} & \dots & b_m^{i_{k-1}^*+1} \\ \dots & & \\ b_1^{i_k^*-1} & \dots & b_m^{i_k^*-1} \end{bmatrix} \in \mathbb{R}^{(i_k^* - i_{k-1}^* - 1)\times m}, \quad \text{for } 1 \le k \le s+1.$$

Note that in the above expression, $i_0^* = 0$ and $i_{s+1}^* = m + s$. Now set

$$T_u = \mathrm{col}[T_u^1, T_u^2, \dots, T_u^{s+1}],$$

and by calculating the dimensions, we see that T_u is an $m \times m$ matrix. Since the rank of Γ is invariant under coordinates and feedback transformations, we deduce that T_u is also invertible. Finally, using an input coordinates transformation $\tilde{u} = T_u u$, we get the Brunovský canonical form of Λ^{uv} with indices $(\epsilon_1, \dots, \epsilon_m)$ and $(\bar{\epsilon}_1, \dots, \bar{\epsilon}_s)$.

(ii) $A^{nn} = \bar{A}_2$.

(iii) First, we can find a Morse transformation M_{tran}^1 with a triangular T_w such that:

$$M_{tran}^1 \left(\begin{array}{c|c|c} \bar{A}_3 & \bar{B}_3^u & \bar{B}_3^v \\ \hline \bar{C}_3 & \bar{D}_3^u & \end{array} \right) = \left(\begin{array}{c|cc|c} A_p & B_p^u & 0 & B_p^v \\ \hline C_p & 0 & 0 & \\ 0 & 0 & I_\delta & \end{array} \right).$$

Since $(\bar{A}_3, \bar{B}_3^w, \bar{C}_3, \bar{D}_3^w)$ is prime, by Theorem 10 of [145], (A_p, B_p^w, C_p) enjoys the properties:

$$\mathcal{V}^*(A_p, B_p^w, C_p) = 0, \quad \mathcal{U}_w^*(A_p, B_p^w, C_p) = 0. \tag{3.38}$$

$$\mathcal{W}^*(A_p, B_p^w, C_p) = \mathbb{R}^{n_3}, \quad \mathcal{Y}^*(A_p, B_p^w, C_p) = \mathscr{Y}. \tag{3.39}$$

A little thought (or see Lemma 2 of [145]) and equation (3.38) gives that $\begin{bmatrix} A_p & B_p^w \\ C_p & 0 \end{bmatrix}$ is of full column rank. Then by $\mathcal{V}^*(A_p, B_p^w, C_p) = (\mathcal{W}^*((A_p)^T, (C_p)^T, (B_p^w)^T))^{\perp}$ (see the results of (3.49)) and equation (3.39), we have $\begin{bmatrix} A_p & B_p^w \\ C_p & 0 \end{bmatrix}$ is of full row rank. Thus $\begin{bmatrix} A_p & B_p^w \\ C_p & 0 \end{bmatrix}$ is square and invertible.

Moreover, by item (i) of this proof, there exists a Morse transformation M_{tran}^2 with triangular T_w such that the pairs $(\hat{A}^{pu}, \hat{B}^{pu})$ and (A^{pv}, B^{pv}) below are in the Brunovský form with indices $(\sigma_1, \dots, \sigma_c)$ and $(\bar{\sigma}_1, \dots, \bar{\sigma}_d)$ respectively.

$$M_{tran}^2 \left(\begin{array}{c|c|c} A_p & B_p^u & B_p^v \\ \hline C_p & 0 & \end{array}\right) = \left(\begin{array}{cc|c|c} \hat{A}^{pu} & 0 & \hat{B}^{pu} & 0 \\ 0 & A^{pv} & 0 & B^{pv} \\ \hline \hat{C}_p^u & C_p^v & 0 & \end{array}\right)$$

Then, according to the block-diagonal structure of $\hat{A}^{pu}$ and A^{pv}, the matrices $\hat{C}_p^u$ and C_p^v above have the form:

$$\hat{C}_p^u = \left[\hat{C}_1^u \mid \hat{C}_2^u \mid \cdots \mid \hat{C}_c^u\right], \quad C_p^v = \left[C_1^v \mid C_2^v \mid \cdots \mid C_d^v\right],$$

where $\hat{C}_i^u \in \mathbb{R}^{p_3 \times \sigma_i}$, $1 \le i \le c$ and $C_i^v \in \mathbb{R}^{p_3 \times \bar{\sigma}_i}$, $1 \le i \le d$.

Now every subsystem $(\hat{A}_{\sigma_i}^{pu}, \hat{B}_{\sigma_i}^{pu}, \hat{C}_i^u)$ and $(A_{\bar{\sigma}_i}^{pv}, B_{\bar{\sigma}_i}^{pv}, C_i^v)$ must have the properties that

$$\mathcal{W}^*(\hat{A}_{\sigma_i}^{pu}, \hat{B}_{\sigma_i}^{pu}, \hat{C}_i^u) = \mathbb{R}^{\sigma_i}, \quad \mathcal{W}^*(A_{\bar{\sigma}_i}^{pv}, B_{\bar{\sigma}_i}^{pv}, C_i^v) = \mathbb{R}^{\bar{\sigma}_i}, \tag{3.40}$$

since if not, equation (3.39) does not hold.

By a direct calculation, we have $\mathcal{W}_1(\hat{A}_{\sigma_i}^{pu}, \hat{B}_{\sigma_i}^{pu}, \hat{C}_i^u) = \operatorname{Im} \hat{B}_{\sigma_i}^{pu}$ and $\mathcal{W}_1(A_{\bar{\sigma}_i}^{pv}, B_{\bar{\sigma}_i}^{pv}, C_i^v) = \operatorname{Im} B_{\bar{\sigma}_i}^{pv}$. Then the subspaces $\mathcal{W}_2(\hat{A}_{\sigma_i}^{pu}, \hat{B}_{\sigma_i}^{pu}, \hat{C}_i^u, 0)$ and $\mathcal{W}_2(A_{\bar{\sigma}_i}^{pv}, B_{\bar{\sigma}_i}^{pv}, C_i^v, 0)$ coincide with $\operatorname{Im} \hat{B}_{\sigma_i}^{pu}$ and $\operatorname{Im} B_{\bar{\sigma}_i}^{pv}$ respectively, unless the last columns of $\hat{C}_i^u$ and C_i^v are zero vectors. By similar arguments, we can deduce that $\hat{C}_i^u$, $1 \le i \le c$ and C_i^v, $1 \le i \le d$ have the following form:

$$\hat{C}_i^u = \left[\hat{c}_i^u \mid 0 \mid \cdots \mid 0\right], \quad C_i^v = \left[c_i^v \mid 0 \mid \cdots \mid 0\right],$$

where $\hat{c}_i^u \in \mathbb{R}^{p_3}$ and $c_i^v \in \mathbb{R}^{p_3}$. Furthermore, since the columns of $\hat{A}_{\sigma_i}^{pu}$ and $A_{\bar{\sigma}_i}^{pv}$ corresponding to $\hat{c}_i^u$ and c_i^v are all zero, by the argument that $\begin{bmatrix} A_p & B_p^w \\ C_p & 0 \end{bmatrix}$ is invertible, we see that the following matrix

$$T_y^{-1} = \left[\hat{c}_1^u \;\; \hat{c}_2^u \;\; \dots \;\; \hat{c}_c^u \mid c_1^v \;\; c_2^v \;\; \dots \;\; c_d^v\right]$$

is invertible. Finally, using T_y as the output coordinates transformation matrix, we get the following canonical form for C_p

$$T_y C_p = T_y \left[\hat{C}_p^u \;\; C_p^v\right] = \begin{bmatrix} \hat{C}^{pu} & 0 \\ 0 & C^{pv} \end{bmatrix}.$$

(iv) The proof of $(\bar{A}_4^4, \bar{C}_2^4)$ implies (A^o, C^o) is omitted since it is well-known in the linear control theory. □

3.8 Appendix

We use the following notations in the present chapter.

$\mathcal{C}^k$	the class of k-times differentiable functions
$\mathbb{N}$	the set of natural numbers with zero and $\mathbb{N}^+ = \mathbb{N}\backslash\{0\}$
$\mathbb{R}^{n\times m}$	the set of real valued matrices with n rows and m columns
$Gl\,(n, \mathbb{R})$	the group of nonsingular matrices of $\mathbb{R}^{n\times n}$
$\ker A$	the kernel of the map given by a matrix A
$\operatorname{Im} A$	the image of the map given by a matrix A
$\operatorname{rank} A$	the rank of a matrix A
I_n	the identity matrix of size $n \times n$ for $n \in \mathbb{N}^+$
$0_{n\times m}$	the zero matrix of size $n \times m$ for $n, m \in \mathbb{N}^+$
A^T	the transpose of a matrix A
A^{-1}	the inverse of a matrix A
$A\mathscr{B}$	$\{Ax \mid x \in \mathscr{B}\}$, the image of $\mathscr{B}$ under matrix A
$A^{-1}\mathscr{B}$	$\{x \mid Ax \in \mathscr{B}\}$, the preimage of $\mathscr{B}$ under matrix A
$A^{-T}\mathscr{B}$	$(A^T)^{-1}\mathscr{B}$
$\mathscr{A}^{\perp}$	$\{x \mid \forall a \in \mathscr{A} : x^T a = 0\}$, the orthogonal complement of $\mathscr{A}$
$A^{\dagger}$	the right inverse of a full row rank matrix $A \in \mathbb{R}^{n\times m}$, i.e., $AA^{\dagger} = I_n$
$x^{(k)}$	k-th-order derivative of function $x(t)$

Recall the following geometric subspaces for DAECSs (see e.g. [152],[17]) of the form $\Delta^u : E\dot{x} = Hx + Lu$.

Definition 3.8.1. Consider a DAECS $\Delta^u_{l,n,m} = (E, H, L)$. A subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is called $(H, E; \operatorname{Im} L)$-invariant if

$$H\mathscr{V} \subseteq E\mathscr{V} + \operatorname{Im} L.$$

A subspace $\mathscr{W} \subseteq \mathbb{R}^n$ is called restricted $(E, H; \operatorname{Im} L)$-invariant if

$$\mathscr{W} \subseteq E^{-1}(H\mathscr{V} + \operatorname{Im} L).$$

Definition 3.8.2. For a DAECS $\Delta^u_{l,n,m} = (E, H, L)$, define the augmented Wong sequences as follows:

$$\mathscr{V}_0 = \mathbb{R}^n, \quad \mathscr{V}_{i+1} = H^{-1}(E\mathscr{V}_i + \operatorname{Im} L), \tag{3.41}$$

$$\mathscr{W}_0 = 0, \quad \mathscr{W}_{i+1} = E^{-1}(H\mathscr{W}_i + \operatorname{Im} L). \tag{3.42}$$

Additionally, define the sequence of subspaces $\hat{\mathscr{W}}_i$ as follows:

$$\hat{\mathscr{W}}_1 = \ker E, \;\; \hat{\mathscr{W}}_{i+1} = E^{-1}(H\hat{\mathscr{W}}_i + \operatorname{Im} L). \tag{3.43}$$

Remark 3.8.3. (i) The subspace sequences $\mathscr{W}_i$ and $\hat{\mathscr{W}}_i$ satisfy,

$$\mathscr{W}_0 \subseteq \hat{\mathscr{W}}_1 \subsetneq \mathscr{W}_1 \subsetneq \hat{\mathscr{W}}_2 \subsetneq \mathscr{W}_2 \cdots \subsetneq \hat{\mathscr{W}}_k \subsetneq \mathscr{W}_k \subsetneq \cdots \subsetneq \hat{\mathscr{W}}_{k^*} = \mathscr{W}_{k^*} = \hat{\mathscr{W}}_{k^*+j} = \mathscr{W}_{k^*+j},$$

or

$$\mathscr{W}_0 \subseteq \hat{\mathscr{W}}_1 \subsetneq \mathscr{W}_1 \subsetneq \hat{\mathscr{W}}_2 \subsetneq \mathscr{W}_2 \cdots \subsetneq \hat{\mathscr{W}}_k \subsetneq \mathscr{W}_k \subsetneq \cdots \subsetneq \hat{\mathscr{W}}_{k^*} \subsetneq \mathscr{W}_{k^*} = \hat{\mathscr{W}}_{k^*+j} = \mathscr{W}_{k^*+j},$$

where $j \geq 1$ and k^* is the smallest k such that $\mathscr{W}_{k^*} = \mathscr{W}_{k^*+1}$. Note that k^* may not be the smallest k such that $\hat{\mathscr{W}}_{k^*} = \hat{\mathscr{W}}_{k^*+1}$ (since $\hat{\mathscr{W}}_{k^*} \subsetneq \hat{\mathscr{W}}_{k^*+1}$ in the second case of the above). However, it is seen that $\mathscr{W}_i$ and $\hat{\mathscr{W}}_i$ always have the same limits.

(ii) From [17], we know that $\mathscr{V}^*$ is the largest $(A, E; \operatorname{Im} L)$-invariant and $\mathscr{W}^*$ is the smallest restricted $(E, A, \operatorname{Im} B)$-invariant, where $\mathscr{V}^*$ and $\mathscr{W}^*$ are the limits of the sequences $\mathscr{V}_i$ and $\mathscr{W}_i$, respectively.

(iii) In Chapter 2, we have proved that for the DAE $E\dot{x} = Hx$, $\mathscr{V}^*(E, H, 0)$ is also the largest such that $\mathscr{V}(E, H, 0) = H^{-1}E\mathscr{V}(E, H, 0)$, but $\mathscr{W}^*(E, H, 0)$ is not necessarily the smallest such that $\mathscr{W}(E, H, 0) = E^{-1}H\mathscr{W}(E, H, 0)$. It is easy to extend that result to the case of L not being zero, i.e., $\mathscr{V}^*(E, H, L)$ is the largest such that $\mathscr{V}(E, H, L) = H^{-1}(E\mathscr{V}(E, H, L) + \operatorname{Im} L)$, but $\mathscr{W}^*(E, H, L)$ is not necessarily the smallest such that $\mathscr{W}(E, H, L) = E^{-1}(H\mathscr{W}(E, H, L) + \operatorname{Im} L)$.

Consider an ODECS $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D)$ of the form

$$\Lambda^{uv} : \begin{cases} \dot{x} = Ax + B^u u + B^v v \\ y = Cx + D^u u. \end{cases}$$

The state, input and output space of Λ^{uv} will be denoted by $\mathscr{X}$, $\mathscr{U}_{uv}$ and $\mathscr{Y}$, respectively. The input subspaces of u and v will be denoted by $\mathscr{U}_u$ and $\mathscr{U}_v$, respectively. Thus we have $\mathscr{U}_{uv} = \mathscr{U}_u \oplus \mathscr{U}_v$. Recall that Λ^{uv} can be expressed as a classical ODECS $\Lambda^{w}_{n,m+s,p} = (A, B^w, C, D^w)$ of form (3.2). The input space of Λ^w is denoted by $\mathscr{U}_w$ and clearly, $\mathscr{U}_w = \mathscr{U}_{uv}$. We now recall the invariant subspaces $\mathcal{V}$ and $\mathcal{W}$ defined in [145] for Λ^w.

Definition 3.8.4. For an ODECS $\Lambda^{w}_{n,m+s,p} = (A, B^w, C, D^w)$, a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called a null-output (A, B^w)-controlled invariant subspace if there exists $F \in \mathbb{R}^{(m+s)\times n}$ such that

$$(A + B^w F)\mathcal{V} \subseteq \mathcal{V} \quad \text{and} \quad (C + D^w F)\mathcal{V} = 0$$

and a subspace $\mathcal{U}_w \subseteq \mathbb{R}^{s+m}$ is called a null-output (A, B^w)-controlled invariant input subspace if

$$\mathcal{U}_w = (B^w)^{-1}\mathcal{V} \cap \ker D^w.$$

Denote by $\mathcal{V}^*$ (respectively $\mathcal{U}^*_w$) the largest null-output (A, B^w) controlled invariant subspace (respectively input subspace).

Correspondingly, a subspace $\mathcal{W} \subseteq \mathbb{R}^n$ is called an unknown-input (C, A)-conditioned invariant subspace if there exists $K \in \mathbb{R}^{n\times p}$ such that

$$(A + KC)\mathcal{W} + (B^w + KD^w)\mathscr{U}_w = \mathcal{W}$$

and a subspace $\mathcal{Y} \subseteq \mathbb{R}^p$ is called an unknown-input (C, A)-conditioned invariant output subspace if

$$\mathcal{Y} = C\mathcal{W} + D^w\mathscr{U}_w.$$

Denote by $\mathcal{W}^*$ (respectively $\mathcal{Y}^*$) the smallest unknown-input (C, A)-conditioned invariant subspace (respectively output subspace).

Lemma 3.8.5. [144] *Initialize* $\mathcal{V}_0 = \mathscr{X} = \mathbb{R}^n$ *and, for* $i \in \mathbb{N}$, *define inductively*

$$\mathcal{V}_{i+1} = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\begin{bmatrix} I \\ 0 \end{bmatrix} \mathcal{V}_i + \operatorname{Im} \begin{bmatrix} B^w \\ D^w \end{bmatrix} \right) \tag{3.44}$$

and $\mathcal{U}_i \subseteq \mathscr{U}$ *for* $i \in \mathbb{N}$ *are given by*

$$\mathcal{U}_i = \begin{bmatrix} B^w \\ D^w \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{V}_i \\ 0 \end{bmatrix}. \tag{3.45}$$

Then $\mathcal{V}^* = \mathcal{V}_n$ *and* $\mathcal{U}^*_w = \mathcal{U}_n$ *.*

Correspondingly, initialize $\mathcal{W}_0 = \{0\}$, *and, for* $i \in \mathbb{N}$, *define inductively*

$$\mathcal{W}_{i+1} = \begin{bmatrix} A & B^w \end{bmatrix} \left(\begin{bmatrix} \mathcal{W}_i \\ \mathscr{U}_w \end{bmatrix} \cap \ker \begin{bmatrix} C & D^w \end{bmatrix} \right) \tag{3.46}$$

and $\mathcal{Y}_i \subseteq \mathscr{Y}$ *for* $i \in \mathbb{N}$ *are given by*

$$\mathcal{Y}_i = \begin{bmatrix} C & D^w \end{bmatrix} \begin{bmatrix} \mathcal{W}_i \\ \mathscr{U}_w \end{bmatrix}. \tag{3.47}$$

Additionally, define the subspace sequence $\hat{\mathcal{W}}_i$ *as*

$$\hat{\mathcal{W}}_1 = \operatorname{Im} B^v, \quad \hat{\mathcal{W}}_{i+1} = \begin{bmatrix} A & B^w \end{bmatrix} \left(\begin{bmatrix} \hat{\mathcal{W}}_i \\ \mathscr{U}_w \end{bmatrix} \cap \ker \begin{bmatrix} C & D^w \end{bmatrix} \right). \tag{3.48}$$

Then $\mathcal{W}^* = \mathcal{W}_n$, $\mathcal{Y}^* = \mathcal{Y}_n$ *and* $\hat{\mathcal{W}}_n = \mathcal{W}_n = \mathcal{W}^*$.

Note that when considering the above defined invariant subspaces for the dual system Λ^{wd} of Λ^w, given by $\Lambda^{wd} = (A^T, C^T, (B^w)^T, (D^w)^T)$, we have the following results [146],[145]:

$$\begin{aligned} \mathcal{V}^*(\Lambda^w) &= \mathcal{W}^*(\Lambda^{wd})^\perp, \quad \mathcal{W}^*(\Lambda^w) = \mathcal{V}^*(\Lambda^{wd})^\perp, \\ \mathcal{U}^*_w(\Lambda^w) &= \mathcal{Y}^*(\Lambda^{wd})^\perp, \quad \mathcal{Y}^*(\Lambda^w) = \mathcal{U}^*_w(\Lambda^{wd})^\perp. \end{aligned} \tag{3.49}$$

Chapter 4

Geometric Analysis and Normal Form of Nonlinear Differential-Algebraic Equations

Abstract: For nonlinear differential-algebraic equations DAEs, we define two kinds of equivalences, namely, the external and internal equivalence. The difference of the two notions will be illustrated by their relations with the existence and uniqueness of solutions. Roughly speaking, the word "external" means that we consider a DAE (locally) everywhere and "internal" means that we consider the DAE on its (locally) maximal invariant submanifold only. We show that this invariant manifold can be calculated by an algorithm iteratively. A procedure named explicitation with driving variables is proposed to connect nonlinear DAEs with nonlinear control systems. We then show that the driving variables of an explicitation system can be reduced under some involutivity conditions. Finally, due to the explicitation procedure, we will use the notion of zero dynamics from nonlinear control theory to derive a nonlinear generalization of the Weierstrass form.

Notation

$\mathbb{N}$	the set of natural numbers with zero and $\mathbb{N}^+ = \mathbb{N}\backslash\{0\}$
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}^{n\times m}$	the set of real valued matrices with n rows and m columns
I_n	$n \times n$ identity matrix
$\mathcal{C}^j(M;N)$	the class of maps of class $\mathcal{C}^j$, $j \in \mathbb{N}\cup\{\infty\}$, from M to N; if $j=\infty$, it is the set of $\mathcal{C}^\infty$-smooth functions
$Gl\,(n,\mathbb{R})$	the group of nonsigular matrices of $\mathbb{R}^{n\times n}$
T_xM	the tangent space of a submanifold M of $\mathbb{R}^n$ at $x \in M$
$\wedge$	exterior product

4.1 Introduction

Consider a nonlinear differential-algebraic equation DAE of the form

$$\Xi : E(x)\dot{x} = F(x), \tag{4.1}$$

where $x \in X$ is a vector of the "generalized" states and X is an open subset of $\mathbb{R}^n$ (or a n-dimensional manifold). The maps $E : TX \to \mathbb{R}^l$ and $F : X \to \mathbb{R}^l$ are smooth and the word "smooth" will mean throughout the chapter $\mathcal{C}^\infty$-smooth. We will denote a DAE of form (4.1) by $\Xi_{l,n} = (E, F)$ or, simply, Ξ. Equation (4.1) is affine with respect to the velocity $\dot{x}$, so sometimes it is called a quasi-linear DAE and can be considered as an affine Pfaffian system (for Pfaffian system, see any book on differential geometry, e.g. [112]). Note that some variables of x may perform like normal state-variables of differential equations and the others may play the role of an input, that is the reason why x is called the "generalized" state.

A *pure* semi-explicit PSE DAE is of the form

$$\Xi^{PSE} : \begin{cases} \dot{x}_1 = F_1(x_1, x_2) \\ 0 = F_2(x_1, x_2), \end{cases} \tag{4.2}$$

where $x_1 \in \mathbb{R}^q$ is a vector of state variables and $x_2 \in \mathbb{R}^{n-q}$ is a vector of algebraic variables (since there are no differential equations for x_2), the maps $F_1 : X_1 \times X_2 \to TX_1$ and $F_2 : X_1 \times X_2 \to \mathbb{R}^{l-q}$ are smooth, where X_1 and X_2 are open subsets of $\mathbb{R}^q$ and $\mathbb{R}^{n-q}$ (or a q- and $(n-q)$-dimensional manifolds), respectively. Comparing a DAE of form (4.2) with that of form (4.1), the function $E(x)$ becomes constant and is of the form $E(x) = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}$. A linear DAE of the form

$$\Delta : E\dot{x} = Hx \tag{4.3}$$

will be denoted by $\Delta_{l,n} = (E, H)$ or, simply, Δ, where $E \in \mathbb{R}^{l\times n}$ and $H \in \mathbb{R}^{l\times n}$. Apparently, both the PSE DAE Ξ^{PSE} and the linear DAE Δ can be seen as special cases of DAE Ξ.

The motivation of studying DAEs is their frequent presence in modelling of practical systems as electrical circuits [166], chemical processes [33, 154], mechanical systems [159, 22, 26] and mobile robots [90, 83], etc. A normal form or a canonical form of a DAE is the simplest possible form of the DAE under some predefined equivalence relations. The studies on normal forms and canonical forms of DAEs can be found in [186, 117, 131, 16, 20] for the linear case and in [169, 120, 13] for the nonlinear case. Two linear DAEs $E\dot{x} = Hx$ and $\tilde{E}\dot{\tilde{x}} = \tilde{H}\tilde{x}$ are called strictly equivalent [75] or externally equivalent in Chapter 2 and in [47], if there exist constant invertible matrices Q and P such that $QEP^{-1} = \tilde{E}$ and $QHP^{-1} = \tilde{H}$. Analogously, we define the external equivalence for nonlinear DAEs as follows.

Definition 4.1.1. (External equivalence) Two DAEs $\Xi_{l,n} = (E, F)$ and $\tilde{\Xi}_{l,n} = (\tilde{E}, \tilde{F})$ defined on X and $\tilde{X}$, respectively, are called externally equivalent, shortly ex-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$ and $Q : X \to Gl(l, \mathbb{R})$ such that

$$\tilde{F}(\psi(x)) = Q(x)F(x) \;\text{ and }\; \tilde{E}(\psi(x)) = Q(x)E(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}. \tag{4.4}$$

The ex-equivalence of two DAEs will be denoted by $\Xi \overset{ex}{\sim} \tilde{\Xi}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of x^0 and $\tilde{U}$ of $\tilde{x}^0$, and $Q(x)$ is defined on U, we will speak about local ex-equivalence.

There are three main results of this chapter. The first result concerns analyzing a DAE (locally) everywhere (i.e., externally) or considering the restriction of the DAE to a submanifold (i.e., internally), which corresponds to the external equivalence (see Definition 4.1.1) and the internal equivalence (see Definition 4.3.11), respectively. The difference between the two equivalences will be illustrated by their relations with the solutions of DAEs. In order to analyze solutions of DAEs, we use a concept named *locally maximal invariant submanifold* (see Definition 4.3.1), which can be calculated by an iterative reduction method shown in Algorithm 4.3.4. Actually, via this reduction method frequently appearing in the DAEs literature [161, 162, 164, 165], that works under some constant rank and smoothness assumptions, one can generate a sequence of submanifolds by analyzing the existence of solutions. If the sequence of submanifolds converges after a finite number of steps, then the solutions of the DAE are given by an ordinary differential equation ODE evolving on the limit (which is actually a maximal invariant submanifold) of that sequence of submanifolds. Then the word "internally" means that we consider the DAE restricted to its maximal invariant submanifold (i.e., where its solutions exist). Thus considering only the restriction of a DAE means that we only care about where and how the solutions of that DAE evolve. However, when the nominal point is not on the maximal invariant submanifold (which is common for practical systems, since an initial point could be anywhere), there are no solutions passing through the point but we still want to steer the solutions to the submanifold and this must follow the rules indicated by the "external" form of the DAE, thus considering DAEs everywhere is also important.

In Chapter 2, we have shown that one can associate a class of linear control systems to any linear DAE (by the procedure of the explicitation for linear DAEs). In this way, we can use the classical linear control theory to analyze linear DAEs. The second result of this chapter is a nonlinear counterpart of the result in Chapter 2. To any nonlinear DAE, by introducing extra variables (called driving variables), we can attach a class of nonlinear control systems. Moreover, we show that the driving variables in this explicitation procedure of nonlinear DAEs can be reduced under some involutivity conditions which explains when a DAE Ξ is ex-equivalent to a PSE DAE Ξ^{PSE}.

It is well-known (see e.g. [117],[75]) that any linear DAE Δ of form (4.3) is ex-equivalent (via linear transformations) to the Kronecker canonical form **KCF**. In particular, if Δ is regular, i.e., the matrices E and H are square ($l = n$) and $|sE - H| \not\equiv 0$ for $s \in \mathbb{C}$, then Δ is ex-equivalent (also via linear transformations) to the Weierstrass form **WF** [186], given by

$$\mathbf{WF}:\begin{bmatrix} N & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\dot\xi\\ \dot z\end{bmatrix}=\begin{bmatrix} I & 0\\ 0 & A\end{bmatrix}\begin{bmatrix}\xi\\ z\end{bmatrix}, \tag{4.5}$$

where $N = \operatorname{diag}(N_1, \ldots, N_m)$, and where N_i, $i = 1, \ldots, m$ are nilpotent matrices of index ρ_i, i.e., $N_i^j \neq 0$ for all $j = 1, \ldots, \rho_i - 1$ and $N_i^{\rho_i} = 0$. The last result of this chapter is to use such concept as zero dynamics of the nonlinear control theory [92],[151] to derive a nonlinear generalization of the **WF**. In the linear case, canonical forms as the **KCF** and the **WF** are closely related to a geometric concept named the Wong sequences (see Definition 4.1.2 below). As shown in [16], the relations between the **WF** and the Wong sequences has been built and in [20], the importance of the Wong sequences for the geometric analysis of linear DAEs are reconfirmed.

Definition 4.1.2. For a linear DAE of form (4.3), define the Wong sequences by

$$\begin{aligned}\mathscr{V}_0 &= \mathbb{R}^n, \quad \mathscr{V}_{i+1} = H^{-1}E\mathscr{V}_i, \quad i\in\mathbb{N},\\ \mathscr{W}_0 &= \{0\}, \quad \mathscr{W}_{i+1} = E^{-1}H\mathscr{W}_i, \quad i\in\mathbb{N}.\end{aligned}$$

In Chapter 2, we showed that the Wong sequences of linear DAEs have direct relations with the invariant subspaces of the explicitation systems and these invariant subspaces led us to the Morse canonical form of control systems. Thus generalizations of the Wong sequences for nonlinear DAEs are desired and possible candidates for a nonlinear version of Wong sequences would be invariant objects showing up in the procedure of explicitation of nonlinear DAEs.

This chapter is organized as follows. In Section 4.2, we discuss solutions of DAEs and show their relations with the external equivalence. In Section 4.3.1, we introduce the concept of locally maximal invariant submanifold, present an algorithm to calculate it (the reduction method), and we show that the internal regularity (existence and uniqueness of solutions) corresponds to the internal equivalence to an ODE without free variables. In Section 4.3.2, we show the explicitation (with driving variables) procedure and how DAEs are connected to nonlinear control systems. In Section 4.3.3, we show when a nonlinear DAE is externally equivalent to a pure semi-explicit one and how this problem is related to the explicitation. A nonlinear generalization of the Weierstrass form is given in Section 4.3.4. Finally, Section 4.5 and Section 4.4 contain the conclusions and the proofs of the results, respectively.

4.2 Preliminaries and problem statement

Definition 4.2.1. A solution of a DAE $\Xi_{l,n} = (E, F)$ is a $\mathcal{C}^1$ curve $\gamma : I \to \mathbb{R}^n$ defined on an open interval I such that for all $t \in I$, the curve $\gamma(t)$ satisfies $E(\gamma(t))\dot\gamma(t) = F(\gamma(t))$.

Throughout this chapter, we will be interested only in solutions of Ξ that are at least $\mathcal{C}^1$. If we fix (t_0, x^0), then a solution $\gamma(t)$ satisfying $\gamma(t_0) = x^0$ will be denoted by γ_{x^0}

and the maximal time-interval on which it exists by I_{x^0}. Clearly, I_{x^0} is an open interval that depends on x^0 and may be infinite or finite (depending on whether the trajectory γ_{x^0} escapes in finite time into infinity or not). We start from the following observation, which shows that the ex-equivalence preserves trajectories, but even if we can smoothly conjugate all trajectories of two DAEs, they are not necessarily ex-equivalent.

Observation 4.2.2. Consider two DAEs $\Xi \overset{ex}{\sim} \tilde{\Xi}$. If a $\mathcal{C}^1$-curve $\gamma(t)$, $t \in I_{x^0}$ is a solution of Ξ passing through $x^0 = \gamma(0)$, then $\psi \circ x : I_{\tilde{x}^0} \to \tilde{X}$ is a solution of $\tilde{\Xi}$ passing through $\tilde{x}^0 = \psi(x^0)$; The converse is, however, not true: even if there exists a diffeomorphism $\psi : X \to \tilde{X}$ which maps all solutions of Ξ into solutions of $\tilde{\Xi}$ and vice versa, the two DAEs are not necessarily ex-equivalent.

The following example illustrates the above observation. Consider two DAEs $\Xi_1 = (E_1, F_1)$ and $\Xi_2 = (E_2, F_2)$, where

$$E_1(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_1(x) = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}.$$

Although $\begin{bmatrix} ce^t, & 0, & 0 \end{bmatrix}^T$ is the only solution of both systems (with $c = x_{11}^0$ for the system Ξ_1 and $c = x_{21}^0$, for Ξ_2), the two DAEs are not ex-equivalent. The reason is that, due to algebraic constraints, solutions of Ξ_1 exist on $\{x_{12} = x_{13} = 0\}$ only, while those of Ξ_2 on $\{x_{22} = x_{23} = 0\}$ only, while the ex-equivalence requires to define the conjugating diffeomorphism ψ everywhere on X (on a whole neighborhood for the local ex-equivalence). The issue of identifying submanifolds, on which solutions exist, is crucial.

Lemma 4.2.3 (Solutions lemma)**.** *Consider a DAE $\Xi_{l,n} = (E, F)$. Let M be a smooth connected embedded s-dimensional submanifold of X and fix a point $x^0 \in M$. Assume that in a neighborhood U of x^0*

(A1) $\dim E(x)T_xM = const. = r$,

(A2) $F(x) \in E(x)T_xM$,

for all $x \in M \cap U$. Then, there exists a solution $\gamma_{x^0}(t)$ satisfying $\gamma(0) = x^0$ and $\gamma_{x^0}(t) \in M \cap U$ for $t \in I_{x^0}$. Moreover, the solution is unique if and only if $s = r$.

The proof is given in Section 4.4.1. In order to show that the constant rank assumption (A1) above is crucial, we give the following example.

Example 4.2.4. Consider the following DAE

$$\Xi_{1,1} : x\dot{x} = F(x),$$

where $x \in X = \mathbb{R}$, $F : X \to \mathbb{R}$ and $F(0) \neq 0$. Let $M = X$, clearly, $\dim E(x)T_xM$ equals 1 for $x \neq 0$ and is 0 for $x = 0$. It is seen that $\Xi_{1,1}$ has no solution for $x^0 = 0$ but has a solution $x(t)$ satisfying $\dot{x}(t) = \frac{F(x(t))}{x(t)}$, $x(0) = x^0$ for $x^0 \neq 0$.

For a given point x^0, if there exists at least one solution $\gamma(t)$ of Ξ satisfying $\gamma(0) = x^0$ (i.e., $E(x^0)\dot{\gamma}(0) = f(x^0)$), then x^0 is called an *admissible point* of Ξ. We will denote admissible points by x_a. The proof of Lemma 4.2.3 shows clearly the reason behind Observation 4.2.2: if we assume two DAEs to have corresponding solutions, this assumption only gives the information that there exists a (local) diffeomorphism between submanifolds on which solutions evolve (and maps solutions of one DAE into solutions of the other). We do not know, however, whether the diffeomorphism and the map Q can be extended outside the submanifolds. In fact, outside the manifolds, the two DAEs may have completely different behaviors or even different sizes of system matrices. This analysis gives a motivation to introduce the concept of *internal equivalence* of two DAEs (see the formal Definition 4.3.11). We will show that *internal equivalence* is useful when we only consider transformations and equivalences on the submanifolds on which solutions exist.

4.3 Main results

4.3.1 Maximal invariant submanifold and internal equivalence

For a DAE $\Xi_{l,n} = (E, F)$, given by (4.1), an *invariant submanifold* of Ξ is defined as follows.

Definition 4.3.1. (Invariant and locally invariant submanifold) Consider a DAE $\Xi_{l,n} = (E, F)$ defined on X. A smooth connected embedded submanifold M of X is called *invariant* if for any point $x^0 \in M$, there exists a solution $\gamma_{x^0} : I_{x^0} \to X$ of Ξ such that $\gamma_{x^0}(0) = x^0$ and $\gamma_{x^0}(t) \in M$ for all $t \in I_{x^0}$. Given an admissible point x_a, we will say that M is a *locally invariant submanifold* (around x_a) if there exists an open neighborhood $U \subseteq X$ of x_a such that $M \cap U$ is invariant. A locally invariant submanifold M^* is called maximal, if there exists a neighborhood U of x_a such that for any other locally invariant submanifold M, we have $M \cap U \subseteq M^* \cap U$.

Proposition 4.3.2. *Consider a DAE $\Xi_{l,n} = (E, F)$ and fix an admissible point x_a. Let M be a smooth connected embedded submanifold containing x_a. If M is a locally invariant submanifold, then $F(x) \in E(x)T_xM$ locally for all $x \in M$ around x_a. Conversely, assume that the dimension of $E(x)T_xM$ is constant locally around x_a, if $F(x) \in E(x)T_xM$ locally for all $x \in M$ around x_a, then M is a locally invariant submanifold.*

Proof. Suppose that M is a locally invariant submanifold around x_a. Then by Definition 4.3.1, there exists a neighborhood U of x_a such that for any point $x^0 \in M \cap U$, there exists a solution $\gamma_{x^0} : I_{x^0} \to M \cap U$ satisfying $\gamma(0) = x^0$. Since $x(t) = \gamma_{x^0}(t)$ is a solution, we have $f(x(t)) = E(x(t))\dot{x}(t) \in E(x(t))T_{x(t)}M$, which means that we have $f(x^0) \in E(x^0)T_{x^0}M$ for $t = 0$. Therefore we have $F(x) \in E(x)T_xM$ for all $x \in M \cap U$.

Conversely, choose a neighborhood U of x_a such that the dimension of $E(x)T_xM$ is constant and $F(x) \in E(x)T_xM$ for all $x \in M \cap U$. Then the assumption of Lemma 4.2.3

are satisfied for any point $x^0 \in M \cap U$ and hence a solution γ_{x^0} passing through x^0 exists and $\gamma_{x^0}(t) \in M \cap U$ for $t \in I_{x^0}$. Thus M is a locally invariant submanifold. □

Proposition 4.3.3. *For a DAE $\Xi_{l,n} = (E, F)$, assume that a point x^0 satisfies $F(x^0) \in \operatorname{Im} E(x^0)$. Set*

$$M_0 = \{x \in X : F(x) \in \operatorname{Im} E(x)\}; \tag{4.6}$$

Assume that $M_{k-1} \subsetneq \cdots \subsetneq M_0$, for a certain $k \geq 1$, have been constructed and for some open neighborhood $U_{k-1} \subseteq X$ of x^0 that the intersection $M_{k-1} \cap U_{k-1}$ is a smooth embedded submanifold and denote by M^c_{k-1} the connected component of $M_{k-1} \cap U_{k-1}$ satisfying $x^0 \in M^c_{k-1}$. Set

$$M_k = \left\{x \in M^c_{k-1} : F(x) \in E(x)T_x M^c_{k-1}\right\}. \tag{4.7}$$

Then there exists a smallest integer k, denoted by k^* ($k^* < n$), such that $M_{k^*+1} = M^c_{k^*}$. Moreover, assume that $\dim\ E(x)T_x M^c_{k^*}$ is constant locally for all $x \in M^c_{k^*}$, then x^0 is an admissible point and $M^* = M^c_{k^*}$ is a locally maximal invariant submanifold.

Proof. It is clear that by the assumption that $M_k \cap U_k$ is a smooth submanifold for $k > 0$, there exists a neighborhood U_{k^*} and a smallest $k^* \in \mathbb{N}$ such that $M_{k^*+1} = M^c_{k^*}$. Then by a dimensional argument, it can be deduced that $k^* < n$. Moreover, by equation (4.7) and $M_{k^*+1} = M^c_{k^*}$, we have $F(x) \in E(x)T_x M^c_{k^*}$ for all $x \in M^c_{k^*}$. Consequently, by the assumption that $\dim\ E(x)T_x M_{k^*}$ is constant locally for all $x \in M^c_{k^*}$ and Lemma 4.2.3, there exists at least one solution passing through x^0, i.e., x^0 is an admissible point and by Proposition 4.3.2, $M^* = M^c_{k^*}$ is a locally invariant submanifold.

Then we show by induction that any other invariant submanifold M' is locally contained in M^*. First, by Definition 4.3.1 and Proposition 4.3.2, M' satisfies that $F(x) \in E(x)T_x M'$ for any $x \in M'$ near x^0. Now by equation (4.6), we can deduce that $M' \subseteq M_0$ locally around x^0. Suppose $M' \subseteq M^c_{k-1}$, which implies that $F(x) \in E(x)T_x M^c_{k-1}$ locally for all $x \in M'$. Thus by equation (4.7), we have $x \in M_k$ for all $x \in M'$ around x^0, i.e., locally $M' \subseteq M_k$. Therefore we have locally $M' \subseteq M_k$ for all $k \geq 0$, which implies M' is locally contained in the limit $M^c_{k^*}$ of M_k, hence $M^* = M^c_{k^*}$ is locally maximal. □

The above sequence of submanifolds M_k can be constructed via the algorithm below under some constant rank assumptions. This algorithm can be seen as a nonlinear version of the shuffle algorithm given in [133] to verify the regularity of linear DAEs. In Algorithm 4.3.4, we will use the concept of restriction and reduction of a DAE (see Definition 4.3.7 and 4.3.8 below). Consider a DAE $\Xi_{l,n} = (E, F)$.

Algorithm 4.3.4. *Step 0: Set $E_0(x) = E(x)$ and $F_0(x) = F(x)$, assume that* $\operatorname{rank} E(x) = const. = r_0$ *in an open neighborhood $W_0 \subseteq X$ of x^0. Then there exists $Q_0 : W_0 \to Gl\,(l, \mathbb{R})$ such that $E_0^1(x)$ below is of full row rank, i.e.,* $\operatorname{rank} E_0^1(x) = r_0$*:*

$$Q_0(x)E_0(x) = \begin{bmatrix} E_0^1(x) \\ 0 \end{bmatrix}, \quad Q_0(x)F_0(x) = \begin{bmatrix} F_0^1(x) \\ F_0^2(x) \end{bmatrix},$$

where $E_0^1 : W_0 \to \mathbb{R}^{r_0\times n}$, $F_0^1 : W_0 \to \mathbb{R}^{r_0}$, $F_0^2 : W_0 \to \mathbb{R}^{l-r_0}$ *. By Proposition 4.3.3,* M_0 *is given by*

$$M_0 = \{x \in W_0 : F_0^2(x) = 0\}.$$

Then assume that $dF_0^2(x)$ *has* $n - s_0 \le l - r_0$ *independent rows for* $x \in W_0$*, denoted by* $d\varphi_0^1(x), ..., d\varphi_0^{n-s_0}(x)$*. We have*

$$M_0^c = \{x \in W_0 : \varphi_0^1(x) = ... = \varphi_0^{n-s_0}(x) = 0\}.$$

Now choose new coordinates:

$$z = \psi_0(x) = \mathrm{col}\big[\phi_0^1(x), ..., \phi_0^{s_0}(x), \varphi_0^1(x), ..., \varphi_0^{n-s_0}(x)\,\big],$$

where $\phi_0^1(x), ..., \phi_0^{s_0}(x)$ *are scalar functions chosen to complete* $\psi_0(x)$ *as a local diffeomorphism on* W_0*. It follows that locally* $\Xi_0 = (E_0, F_0) \overset{ex}{\sim} \tilde{\Xi}_0 = (\tilde{E}_0, \tilde{F}_0)$ *on* W_0*, via* $Q_0(x)$ *and* $\psi_0(x)$*, where*

$$\begin{aligned} \tilde{E}_0(z) &= Q_0(x)E_0(x)\left(\frac{\partial\psi_0(x)}{\partial x}\right)^{-1}\Big|_{x=\psi_0^{-1}(z)} = \begin{bmatrix} \tilde{E}_0^1(z) & \tilde{E}_0^2(z) \\ 0 & 0 \end{bmatrix}, \\ \tilde{F}_0(z) &= Q_0(x)F_0(x)\Big|_{x=\psi_0^{-1}(z)} = \begin{bmatrix} F_0^1(z) \\ F_0^2(z) \end{bmatrix}, \end{aligned}$$

where $\tilde{E}_0^1 : W_0 \to \mathbb{R}^{r_0\times s_0}$*. Thus*

$$\tilde{\Xi}_0 : \begin{bmatrix} \tilde{E}_0^1(z) & \tilde{E}_0^2(z) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\bar{z}}_1 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} F_0^1(z) \\ F_0^2(z) \end{bmatrix}.$$

where $\bar{z}_1 = [\phi_0^1(x), ..., \phi_0^{s_0}(x)]^T$ *and* $z_1 = \big[\varphi_0^1(x), ..., \varphi_0^{n-s_0}(x)\,\big]^T$*. Observe that* $z_1 = 0$ *and* $F_0^2(\bar{z}_1, 0) = 0$ *for all* $x \in M_0^c$*. Assume* $\mathrm{rank}\,[\tilde{E}_0^1(\bar{z}_1, 0), dF_0^1(\bar{z}_1, 0)] = r_0$*, then a reduction of local* M_0^c*-restriction of* $\tilde{\Xi}_0$ *is*

$$\tilde{\Xi}_0|_{M_0^c}^{red} : \tilde{E}_0^1\,(\bar{z}_1, 0)\,\dot{\bar{z}}_1 = F_0^1\,(\bar{z}_1, 0)\,.$$

If Ξ *is solvable for* x^0*, then* x^0 *has to be in* M_0^c *(since if so,* $F_0^2(x^0) = 0$*).*

Step k $(k > 0)$*: For all* $x \in M_{k-1}^c$*, set* $E_k(\bar{z}_k) = \tilde{E}_{k-1}^1(\bar{z}_k, 0)$*,* $F_k(\bar{z}_k) = \tilde{F}_{k-1}^1(\bar{z}_k, 0)$*, where* $E_k : M_{k-1}^c \to \mathbb{R}^{r_{k-1}\times s_{k-1}}$*,* $F_k : M_{k-1}^c \to \mathbb{R}^{r_{k-1}}$*. Assume that* $\dim\, E(x)T_xM_{k-1}^c = const. = r_k \le s_{k-1}$ *in a neighborhood* W_k *(*$W_k \subseteq M_{k-1}^c$*) of* x^0*, which implies that* $\mathrm{rank}\,E_k(\bar{z}_k) = r_k$*. Then there exists* $Q_k : W_k \to Gl(r_{k-1}, \mathbb{R})$ *such that the matrix* E_k^1 *below is of full row rank, i.e.,* $\mathrm{rank}\,E_k^1(\bar{z}_k) = r_k$ *for all* $x \in W_k$*:*

$$Q_k(\bar{z}_k)E_k(\bar{z}_k) = \begin{bmatrix} E_k^1(\bar{z}_k) \\ 0 \end{bmatrix}, \quad Q_k(\bar{z}_k)F_k(\bar{z}_k) = \begin{bmatrix} F_k^1(\bar{z}_k) \\ F_k^2(\bar{z}_k) \end{bmatrix},$$

where $E_k^1 : W_k \to \mathbb{R}^{r_k\times s_{k-1}}$*,* $F_k^1 : W_k \to \mathbb{R}^{r_k}$ *and* $F_k^2 : W_k \to \mathbb{R}^{r_{k-1}-r_k}$*. By Proposition 4.3.3,* M_k *is given by*

$$M_k = \left\{\bar{z}_k \in W_k : F_k^2(\bar{z}_k) = 0\right\}.$$

Then assume $dF_k^2(\bar{z}_k)$ has $s_{k-1} - s_k \le r_{k-1} - r_k$ independent rows in the neighborhood W_k of x^0, denoted by $d\varphi_k^1(\bar{z}_k), ..., d\varphi_k^{s_{k-1}-s_k}(\bar{z}_k)$. It follows that

$$M_k^c = \left\{ \bar{z}_k \in W_k : \varphi_k^1(\bar{z}_k) =, ..., = \varphi_k^{s_{k-1}-s_k}(\bar{z}_k) = 0 \right\}.$$

Now choose new coordinates

$$\mathrm{col}\,[z_{k+1}, \bar{z}_{k+1}] = \psi_k(\bar{z}_k) = \mathrm{col}\,[\varphi_k^1(\bar{z}_k), ..., \varphi_k^{s_{k-1}-s_k}(\bar{z}_k), \phi_k^1(\bar{z}_k), ..., \phi_k^{s_k}(\bar{z}_k)],$$

where $\bar{z}_{k+1} = [\phi_k^1, ..., \phi_k^{s_k}]^T$, $z_{k+1} = [\varphi_k^1, ..., \varphi_k^{s_{k-1}-s_k}]^T$ and ψ_k is a local diffeomorphism on W_k. Then we have $\Xi_k = (E_k, F_k) \overset{ex}{\sim} \tilde{\Xi}_k = (\tilde{E}_k, \tilde{F}_k)$ locally on W_k, where

$$\tilde{E}_k(\bar{z}_k) = Q_k(\bar{z}_k)E_k(\bar{z}_k)\left(\frac{\partial\psi(\bar{z}_k)}{\partial(\bar{z}_k)}\right)^{-1} = \begin{bmatrix} \tilde{E}_k^1\,(\bar{z}_{k+1}, z_{k+1}) & \tilde{E}_k^2(.) \\ 0 & 0 \end{bmatrix},$$

$$\tilde{F}_k(\bar{z}_k) = Q_k(\bar{z}_k)F_k(\bar{z}_k) = \begin{bmatrix} F_k^1\,(\bar{z}_{k+1}, z_{k+1}) \\ F_k^2\,(\bar{z}_{k+1}, z_{k+1}) \end{bmatrix}.$$

Thus $\tilde{\Xi}_k$ is locally of the form

$$\tilde{\Xi}_k : \begin{bmatrix} \tilde{E}_k^1\,(\bar{z}_{k+1}, z_{k+1}) & \tilde{E}_k^2(.) \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{\bar{z}}_{k+1} \\ \dot{z}_{k+1} \end{pmatrix} = \begin{bmatrix} F_k^1\,(\bar{z}_{k+1}, z_{k+1}) \\ F_k^2\,(\bar{z}_{k+1}, z_{k+1}) \end{bmatrix}.$$

Observe that $z_{k+1} = 0$ and $F_k^2(\bar{z}_{k+1}, 0) = 0$ for all $x \in M_k^c$. Assume

$$\mathrm{rank}\,[\tilde{E}_k^1(\bar{z}_{k+1}, 0), dF_0^1(\bar{z}_{k+1}, 0)] = r_k,$$

then a reduction of local M_k^c-restriction of $\tilde{\Xi}_k$ is

$$\tilde{\Xi}_k|_{M_k^c}^{red} : \tilde{E}_k^1\,(\bar{z}_{k+1}, 0)\dot{\bar{z}}_{k+1} = F_k^1\,(\bar{z}_{k+1}, 0)\,.$$

If Ξ is solvable for x^0, then $x^0 \in M_k^c$ (since if so, $F_k^2(x^0) = 0$).

Remark 4.3.5. (i) Algorithm 4.3.4 is a constructible algorithm for Proposition 4.3.3, but with more assumptions. That is, in order to assure M_0^c is a smooth connected submanifold, we assume that $\mathrm{rank}\, E(x)$ and $\mathrm{rank}\, dF_0^2(\bar{z}_0)$ are constant in the neighborhood $W_0 \subseteq X$ of x^0. Additionally, in order to assure M_{k-1}^c, $k > 0$ is a smooth connected submanifold, we assume that in every step k that $\dim\, E(x)T_xM_{k-1}^c$ and $\mathrm{rank}\, dF_k^2(\bar{z}_k)$ are constant in the neighborhood $W_k \subseteq M_{k-1}^c$ of x^0.

(ii) In the geometrical description of Proposition 4.3.3, U_k is not explicitly expressed as a neighborhood contained in M_{k-1}^c. However, as shown in the above algorithm that $W_k \subseteq M_{k-1}^c$, which implies that the constant rank assumptions are only required locally on the submanifold M_{k-1}^c.

(iii) The integers r_k, s_k of Algorithm 4.3.4, satisfy

$$\begin{cases} n \ge r_0 \ge r_1 \ge ... \ge r_k \ge ... \ge 0, & n \ge s_0 \ge s_1 \ge ... \ge s_k \ge ... \ge 0, \quad s_{k-1} \ge r_k, \\ n - s_0 \le l - r_0, & s_{k-1} - s_k \le r_{k-1} - r_k. \end{cases}$$

Proposition 4.3.6. *For $\Xi_{l,n} = (E, F)$, assume in Algorithm 4.3.4 that*

(A1) $\operatorname{rank} E(x)$ *and* $\operatorname{rank} dF_0^2(x)$ *are constant in the neighborhood* W_0 *of* x^0.

(A2) $\dim\ E(x)T_xM_{k-1}^c$ *and* $\operatorname{rank} dF_k^2(\bar{z}_k)$ *are constant in the neighborhood* W_k *of* x^0 *for* $k > 0$.

Then there exists a smallest k*, denoted by* $k^* < n$ *such that* $M_{k^*+1} = M_{k^*}^c$*,* x^0 *is an admissible point and* $M^* = M_{k^*}^c$ *is a locally maximal invariant submanifold around* $x_a = x^0$*.*

Proof. As mentioned in Remark 4.3.5, Algorithm 4.3.4 is an constructible algorithm for Proposition 4.3.3, which implies that under assumptions (A1) and (A2), there exists a smallest $k = k^* < n$ such that $r_{k^*+1} = r_{k^*}$, implying also $s_{k^*+1} = s_{k^*}$, i.e., $M_{k^*+1} = W_{k^*+1} \subseteq M_{k^*}^c$. Moreover, we have $x^0 \in M_{k^*}^c$ and $\dim\ M_{k^*}^c = s_{k^*} \geq 0$. By assumption (A2), $\dim\ E(x)M_{k^*}^c$ is constant locally for all $x \in M_{k^*}^c$, we denote $\dim\ E(x)T_xM_{k^*}^c = r^*$.

From Step k^* of Algorithm 4.3.4, it is seen that

$$\Xi|_{M^*}^{red} : \tilde{E}_{k^*+1}(\bar{z}_{k^*+1})\dot{\bar{z}}_{k^*+1} = F_{k^*+1}(\bar{z}_{k^*+1}), \tag{4.8}$$

where $E_{k^*+1} : M_{k^*}^c \to \mathbb{R}^{r_{k^*} \times s_{k^*}}$. Now come to Step $k^* + 1$, we have $\operatorname{rank} E_{k^*+1}(.) = \dim\ E_{k^*+1}(.)T_xM_{k^*}^c = r_{k^*+1} = r_{k^*}$ in a neighborhood $W_{k^*+1} \subseteq M_{k^*}^c$, hence $E_{k^*+1}(.)$ is of full row rank for $x \in W_{k^*+1}$. Thus we conclude that $F(x) \in E(x)T_xM_{k^*}^c$ for all $x \in W_{k^*+1}$, i.e., locally for all $x \in M_{k^*}^c$. Therefore, by Proposition 4.3.2, $M^* = M_{k^*}^c$ is a locally maximal invariant manifold, where

$$M^* = M_{k^*}^c = \{z : z_k = 0,\ \ i = 1, \ldots, k^* + 1\}.$$

□

Through the algorithm above, we consider a DAE more and more "internally", that is, at the end of every k step, we restrict the DAE to M_k^c and reduce its redundant equations (see the Definition 4.3.7 and 4.3.8 below) and terms which concern what is outside M_k^c vanish (by setting $z_{k+1} = 0$ and $\dot{z}_{k+1} = 0$). This observation motivates to define the internal equivalence for DAEs. Before giving a formal definition of the internal equivalence, we will define formally the restriction of a DAE to a smooth connected submanifold (invariant or not) as follows.

Definition 4.3.7 (Local restriction)**.** Consider a DAE $\Xi_{l,n} = (E, F)$ and fix a point $x^0 \in X$. Let R be a smooth connected embedded submanifold. Let $\psi(x) = z = (z_1, z_2)$ be local coordinates on a neighborhood U of x^0 such that $R \cap U = \{z_2 = 0\}$ and z_1 are thus coordinates on $R \cap U$. The restriction of Ξ to $R \cap U$, called local R-restriction of Ξ and denoted $\Xi|_R$ is

$$\tilde{E}(z_1, 0) \begin{bmatrix} \dot{z}_1 \\ 0 \end{bmatrix} = \tilde{F}(z_1, 0), \tag{4.9}$$

where $\tilde{E}(z) = E(\psi^{-1}(z)) \left(\frac{\partial \psi}{\partial x}(\psi^{-1}(z))\right)^{-1}$, $\tilde{F}(z) = F((\psi^{-1}(z)))$.

Note that, for any DAE $\Xi_{l,n} = (E, F)$, there may exist some redundant equations (in particular, some trivial algebraic equations $0 = 0$ and some dependent equations). In the linear case, we have defined the full rank reduction of a linear DAE (see Definition 2.6.4 of Chapter 2). We now generalize this notion of reduction to nonlinear DAEs Ξ to get rid of their redundant equations.

Definition 4.3.8 (Reduction)**.** For a DAE $\Xi_{l,n} = (E, F)$, assume $\operatorname{rank}[E(x), dF(x)] = const. = l^* \leq l$. Then there exists $Q : X \to Gl(l, \mathbb{R})$ such that

$$Q(x)\begin{bmatrix}E(x) & dF(x)\end{bmatrix} = \begin{bmatrix}Q_1(x)\\ Q_2(x)\end{bmatrix}\begin{bmatrix}E(x) & dF(x)\end{bmatrix} = \begin{bmatrix}Q_1(x)E(x) & Q_1(x)dF(x)\\ 0 & 0\end{bmatrix},$$

where $\operatorname{rank}[Q_1(x)E(x), Q_1(x)dF(x)] = l^*$, $Q_1 : X \to \mathbb{R}^{l^* \times l}$, $Q_2 : X \to \mathbb{R}^{(l-l^*)\times l}$, and the full row rank reduction, shortly reduction, of Ξ, denoted by Ξ^{red}, is a DAE $\Xi^{red}_{l^*,n} = (E^{red}, F^{red})$, where $E^{red}(x) = Q_1(x)E(x)$ and $F^{red}(x) = Q_1(x)F(x)$.

Remark 4.3.9. Clearly, since the choice of $Q(x)$ is not unique, the reduction of Ξ is not unique. Nevertheless, since $Q(x)$ preserves the solutions, each reduction Ξ^{red} has the same solutions as the original DAE Ξ.

For a locally invariant submanifold M, we consider local M-restriction $\Xi|_M$ of Ξ, and then we construct a reduction of $\Xi|_M$ and denote it by $\Xi|_M^{red}$. Notice that the order matters: to construct $\Xi|_M^{red}$, we first restrict and then reduce while reducing first and then restricting will, in general, not give $\Xi|_M^{red}$ but another DAE $\Xi^{red}|_M$.

Proposition 4.3.10. *Consider a DAE $\Xi_{l,n} = (E, F)$ and fix an admissible point x_a. Let M be a s-dimensional locally invariant submanifold of Ξ around x_a. Assume that*

$$\dim\, E(x)T_xM = const. = r$$

for all $x \in M$ around x_a. Then a reduction $\Xi|_M^{red}$ of local M-restriction of Ξ is a DAE of form (4.1) and the dimensions related to $\Xi|_M^{red}$ are r and s, i.e., $\Xi|_M^{red} = \Xi'_{r,s}$. Moreover, the matrix E' and $[E', dF']$ of $\Xi'_{r,s} = (E', F')$ are of the same full row rank equal to r.

Proof. Consider $\Xi|_M$, which is a DAE of the form (4.9). By the assumption that $\dim\, E(x)T_xM = const. = r$, there alway exists $Q : M \to Gl(l, \mathbb{R})$ such that $Q(z_1)\tilde{E}(z_1, 0) = \begin{bmatrix}E_1(z_1)\\ 0\end{bmatrix}$, where $E_1 : M \to \mathbb{R}^{r\times n}$ and $\operatorname{rank} E_1(z_1) = r$. Denote $Q(z_1)\tilde{F}(z_1, 0) = \begin{bmatrix}F_1(z_1)\\ F_2(z_1)\end{bmatrix}$, then $\Xi|_M$ is ex-equivalent via $Q(z_1)$ to $\begin{bmatrix}E_1(z_1)\\ 0\end{bmatrix}\begin{bmatrix}\dot{z}_1\\ 0\end{bmatrix} = \begin{bmatrix}F_1(z_1)\\ F_2(z_1)\end{bmatrix}$. Since M is locally invariant, by Proposition 4.3.3, we have locally for all $x \in M$, $F(x) \in E(x)T_xM \Rightarrow \begin{bmatrix}F_1(z_1)\\ F_2(z_1)\end{bmatrix} \in \operatorname{Im}\begin{bmatrix}E_1(z_1)\\ 0\end{bmatrix}$, thus $F_2(z_1) = 0$ locally for all z_1. So by Definition 4.3.8, a reduction $\Xi|_M^{red}$ of $\Xi|_M$ is locally of the form $E_1^1(z_1)\dot{z}_1 + E_1^2(z_1)0 = F_1(z_1)$, which is $\Xi'_{r,s} = (E', F')$, where $E' = E_1^1$ and $F' = F_1$. Clearly, E' and $[E', dF']$ of $\Xi'_{r,s} = (E', F')$ are of the same full row rank equal to r.

□

The definition of the internal equivalence of two DAEs is given as follows.

Definition 4.3.11. (Internal equivalence) Consider two DAEs $\Xi = (E, F)$ and $\tilde{\Xi} = (\tilde{E}, \tilde{F})$. Let M^* and $\tilde{M}^*$ be two smooth connected submanifolds and fix two admissible points $x_a \in M^*$, $\tilde{x}_a \in \tilde{M}^*$. Assume that

(A1) M^* and $\tilde{M}^*$ are locally maximal invariant submanifolds of Ξ and $\tilde{\Xi}$, respectively, around x_a and $\tilde{x}_a$.

(A2) $\dim E(x)T_xM^*$ is locally constant for $x \in M^*$ around x_a and $\dim \tilde{E}(\tilde{x})T_{\tilde{x}}\tilde{M}^*$ is locally constant for $\tilde{x} \in \tilde{M}^*$ around $\tilde{x}_a$.

Then, Ξ and $\tilde{\Xi}$ are called locally internally equivalent, shortly in-equivalent, if $\Xi|_{M^*}^{red}$ and $\tilde{\Xi}|_{\tilde{M}^*}^{red}$ are ex-equivalent, locally around x_a and $\tilde{x}_a$, respectively. Denote the in-equivalence of two DAEs by $\Xi \overset{in}{\sim} \tilde{\Xi}$.

Remark 4.3.12. (i) Note that assumptions (A1) and (A2) above are essential for the definition of the internal equivalence. Without those assumptions, the dimensions of $\tilde{\Xi}|_{\tilde{M}^*}^{red}$ and $\tilde{\Xi}|_{\tilde{M}^*}^{red}$ may not be constant. On the other hand, with these constant dimensional assumptions, by Proposition 4.3.10, we have $\Xi|_{M^*}^{red} = \Xi'_{r,s}$ and $\tilde{\Xi}|_{\tilde{M}^*}^{red} = \tilde{\Xi}'_{\tilde{r},\tilde{s}}$, where $r = \dim E(x)T_xM^*$, $s = \dim M^*$ and $\tilde{r} = \dim \tilde{E}(\tilde{x})T_{\tilde{x}}\tilde{M}^*$, $\tilde{s} = \dim \tilde{M}^*$.

(ii) The dimensions l and n, related to Ξ, and $\tilde{l}$ and $\tilde{n}$ related to $\tilde{\Xi}$ are not required to be the same. On the other hand, if Ξ and $\tilde{\Xi}$ are in-equivalent, then by definition, $\Xi|_{M^*}^{red} = \Xi'_{r,s}$ and $\tilde{\Xi}|_{\tilde{M}^*}^{red} = \tilde{\Xi}'_{\tilde{r},\tilde{s}}$ are locally ex-equivalent and thus the dimensions related to them have to be the same, i.e., $r = \tilde{r}$ and $s = \tilde{s}$.

Now we will study the existence and uniqueness of solutions of DAEs with the help of the notion of internal equivalence.

Definition 4.3.13. (Internal regularity) Consider a DAE $\Xi_{l,n} = (E, F)$, fix an admissible point x_a. Let M^* be a locally maximal invariant submanifold around x_a. Then Ξ is called internally regular around x_a, if there exists a neighborhood $U \subseteq X$ of x_a such that for any point $x^0 \in M^* \cap U$, there exists only one solution γ_{x^0} passing through x^0.

Theorem 4.3.14. *Consider a DAE $\Xi_{l,n} = (E, F)$, fix an admissible point x_a. Let M^* be a locally maximal invariant submanifold around x_a. Assume that $\dim E(x)T_xM^*$ is constant locally for all $x \in M^*$ around x_a. The following are equivalent locally around x_a:*

(i) Ξ is internally regular.

(ii) $\dim M^* = \dim E(x)T_xM^*$ *for all* $x \in M^*$.

(iii) Ξ *is internally equivalent to*

$$\Xi^* : \dot{z}^* = F^*(z^*), \tag{4.10}$$

where z^* *is a local, around* x_a*, system of coordinates on* M^**.*

The proof is given in Section 4.4.2.

Remark 4.3.15. (i) Theorem 4.3.14 illustrates that, under some constant dimension assumptions, Ξ is internally regular if and only if there are no free variables in $\Xi|^{red}_{M^*}$, which also means that $\Xi|^{red}_{M^*}$ is ex-equivalent to an ODE Ξ^* of form (4.10).

(ii) Assume that (A1)-(A2) of Proposition 4.3.6 are satisfied, if $\Xi_{l,n}$ is internally regular, then $n \leq l$. Indeed, first the internal regularity of Ξ implies that $s_{k^*} = r_{k^*}$. Then by $s_{k-1} - s_k \leq r_{k-1} - r_k$ shown in Remark 4.3.5(iii), we have $s_{k^*-1} \leq r_{k^*-1}$. By $s_{k-1} - s_k \leq r_{k-1} - r_k$ and an iterative argument, we can deduce $s_0 \leq r_0$. Finally, by $n - s_0 \leq l - r_0$ of Remark 4.3.5(iii), we get $n \leq l$.

(iii) Theorem 4.3.14 is a nonlinear generalization of the results on internal regularity of linear DAEs in Chapter 2. As stated in Proposition 2.6.12 of Chapter 2, a linear DAE Δ, given by (4.3), is internally regular if and only if the maximal invariant subspace $\mathscr{M}^*$ of Δ (i.e., the largest subspace such that $H\mathscr{M}^* \subseteq E\mathscr{M}^*$) satisfies $\dim \mathscr{M}^* = \dim E\mathscr{M}^*$. A nonlinear counterpart of the last condition is that of Theorem 4.3.14(ii) and thus M^* is a natural nonlinear generalization of $\mathscr{M}^*$.

(iv) The sequence of submanifolds M_k of Algorithm 4.3.4 can be seen as a nonlinear generalization of the Wong sequences $\mathscr{V}_i$ of Definition 4.1.2. Observe that M^* is the limit of M_k as $\mathscr{V}^*$ is the limit of $\mathscr{V}_i$. Moreover, we have shown in Chapter 2 that $\mathscr{M}^* = \mathscr{V}^*$.

4.3.2 Explicitation with driving variables of nonlinear DAEs

The explicitation (with driving variables) of a DAE Ξ is the following procedure.

- For a DAE $\Xi_{l,n} = (E, F)$, assume that $\operatorname{rank} E(x) = const. = q$ in a neighborhood $U \subseteq X$ of x^0. Then there exists $Q : U \to Gl(l, \mathbb{R})$ such that $Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$, where $E_1 : U \to \mathbb{R}^{q\times n}$, and $\operatorname{rank} E_1(x) = q$. It is seen that Ξ is ex-equivalent via $Q(x)$ to

$$\begin{cases} E_1(x)\dot{x} = F_1(x) \\ \quad 0 \quad = F_2(x), \end{cases} \tag{4.11}$$

where $Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$, and where F_1 and F_2 are smooth functions with values in $\mathbb{R}^q$ and $\mathbb{R}^{l-q}$, respectively.

- The matrix $E_1(x)$ is of full row rank q, so choose its right inverse $E_1^\dagger(x)$ and set $f(x) = E_1^\dagger(x)F_1(x)$. The collection of all $\dot{x}$ satisfying $E_1(x)\dot{x} = F_1(x)$ of (4.11) is given by the differential inclusion:

$$\dot{x} \in f(x) + \ker E_1(x) = f(x) + \ker E(x). \tag{4.12}$$

- Since $\ker E(x)$ is a distribution of constant rank $n-q$, so choose locally $m = n-q$ independent vector fields $g_1, \ldots, g_m$ on X such that

$$\ker E(x) = \operatorname{span}\{g_1, \ldots, g_m\}(x).$$

  Then by introducing *driving variables* v_i, $i = 1, \ldots, m$, we can parametrize the affine distribution $f(x) + \ker E_1(x)$ and thus all solutions of (4.12) are given by all solutions (corresponding to all controls $v_i(t)$) of

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)v_i. \tag{4.13}$$

- Form a matrix $g(x) = [g_1(x), \ldots, g_m(x)]$. Then, we rewrite equation (4.13) as $\dot{x} = f(x) + g(x)v$, where $v = [v_1, \ldots, v_m]^T$. Set $h(x) = F_2(x)$ and all solutions of DAE (4.11) can be expressed as all solutions (corresponding to all controls $v(t)$) of

$$\begin{cases} \dot{x} = f(x) + g(x)v \\ 0 = h(x). \end{cases} \tag{4.14}$$

  Compared with Ξ, equation (4.14) has an extra vector variable v, which we will call *the vector of driving variables.*

- To (4.14), we attach the control system $\Sigma = \Sigma_{n,m,p} = (f, g, h)$, given by

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)v \\ y = h(x), \end{cases} \tag{4.15}$$

  where $n = \dim x$, $m = \dim u$, $p = \dim y$. Clearly, $m = n - q$ and $p = l - q$ (we will use these dimensional relations in the following discussion). In the above way, we attach a control system Σ to a DAE Ξ.

Definition 4.3.16. (Explicitation with driving variables) Given a DAE $\Xi_{l,n} = (E, F)$, assume that the rank of $E(x)$ is constant locally around x^0. Then, by a (Q, v)-explicitation, we will call a control system $\Sigma = \Sigma_{n,m,p} = (f, g, h)$ given by

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)v \\ y = h(x), \end{cases}$$

with $f(x) = E_1^\dagger(x)F_1(x)$, $\operatorname{Im} g(x) = \ker E(x)$, $h(x) = F_2(x)$, where $Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$, $Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$. The class of all (Q, v)-explicitations will be called the explicitation with driving variables class, shortly, the explicitation class. If a particular control system Σ belongs to the explicitation class of Ξ, we will write $\Sigma \in \mathbf{Expl}(\Xi)$.

Remark 4.3.17. The constant rank assumption of $E(x)$ is essential for the above definition of explicitation and since we assume $E(x)$ is of constant rank around a point x^0, the matrices $Q(x)$, $f(x)$, $g(x)$, $h(x)$ are all defined locally and so is $\Sigma \in \mathbf{Expl}(\Xi)$.

Notice that a given Ξ has many (Q, v)-explicitations since the construction of $\Sigma \in \mathbf{Expl}(\Xi)$ is not unique: there is a freedom in choosing $Q(x)$, $E_1^\dagger(x)$, and $g(x)$. As a consequence of this non-uniqueness of construction, the explicitation Σ of Ξ is a system defined up to a *feedback transformation*, an *output multiplication* and a *generalized output injection* (or, equivalently, a class of systems).

Proposition 4.3.18. *Assume that a control system $\Sigma_{n,m,p} = (f, g, h)$ is a (Q, v)-explicitation of a DAE $\Xi = (E, F)$ corresponding to a choice of invertible matrix $Q(x)$, right inverse $E_1^\dagger(x)$, and matrix $g(x)$. Then a control system $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$ is a $(\tilde{Q}, \tilde{v})$- explicitation of Ξ corresponding to a choice of invertible matrix $\tilde{Q}(x)$, right inverse $\tilde{E}_1^\dagger(x)$, and matrix $\tilde{g}(x)$ if and only if Σ and $\tilde{\Sigma}$ are equivalent via a v-feedback transformation of the form $v = \alpha(x) + \beta(x)\tilde{v}$, a generalized output injection $\gamma(x)y = \gamma(x)h(x)$ and an output multiplication $\tilde{y} = \eta(x)y$, which map*

$$f \mapsto \tilde{f} = f + \gamma h + g\alpha, \quad g \mapsto \tilde{g} = g\beta, \quad h \mapsto \tilde{h} = \eta h, \tag{4.16}$$

where α, β, γ and η are smooth matrix-valued functions, and β and η are invertible.

The proof is given in Section 4.4.3. Since the explicitation of a DAE is a class of control systems, we will propose now an equivalence relation for control systems. An equivalence of two nonlinear control systems is usually defined by state coordinates transformations and feedback transformations (e.g. see [92],[151]), and sometimes output coordinates transformations [139]. In the present chapter, we define a more general system equivalence of two control systems as follows.

Definition 4.3.19. (System equivalence) Consider two control systems $\Sigma_{n,m,p} = (f, g, h)$ and $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$ defined on X and $\tilde{X}$, respectively. The systems Σ and $\tilde{\Sigma}$ are called system equivalent, or shortly sys-equivalent, denoted by $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$, if there exist a diffeomorphism $\psi : X \to \tilde{X}$, matrix-valued functions $\alpha : X \to \mathbb{R}^m$, $\gamma : X \to \mathbb{R}^{n\times p}$ and $\beta : X \to Gl(m, \mathbb{R})$, and $\eta : X \to Gl(p, \mathbb{R})$ such that

$$\tilde{f} \circ \psi = \frac{\partial \psi}{\partial x}\left(f + \gamma h + g\alpha\right), \ \ \tilde{g} \circ \psi = \frac{\partial \psi}{\partial x} g\beta, \ \ \tilde{h} \circ \psi = \eta h.$$

If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of x^0 and $\tilde{U}$ of $\tilde{x}^0$, and α, β, γ, η are defined locally on U, we will speak about local sys-equivalence.

Remark 4.3.20. The above defined sys-equivalence of two nonlinear control systems generalizes the Morse equivalence of two linear control systems (see [146],Chapter 2). In the linear case, the output multiplication $y \mapsto T_o y$, by a constant invertible matrix T_o, can be seen as a linear coordinates change in the output space, but in the nonlinear case, the transformation $h \mapsto \eta h$ is more general than a zero-preserving change of output coordinates $y \mapsto \varphi(y)$ by a diffeomorphism φ on the output space.

The following theorem is a fundamental result of the present chapter, which shows that sys-equivalence for explicitation systems (control systems) is a true counterpart of the ex-equivalence for DAEs.

Theorem 4.3.21. *Consider two DAEs $\Xi_{l,n} = (E, F)$ and $\tilde{\Xi}_{l,n} = (\tilde{E}, \tilde{F})$. Assume that* $\operatorname{rank} E(x)$ *and* $\operatorname{rank} \tilde{E}(\tilde{x})$ *are constant around two points x^0 and $\tilde{x}^0$, respectively. Then for any two control systems $\Sigma_{n,m,p} = (f, g, h) \in \mathbf{Expl}(\Xi)$ and $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h}) \in \mathbf{Expl}(\tilde{\Xi})$, we have that locally $\Xi \overset{ex}{\sim} \tilde{\Xi}$ if and only if $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$.*

The proof is given in Section 4.4.3. In order to show how the explicitation can be useful in the DAEs theory, we discuss below how the analysis of DAEs given in Section 4.3.1 is related to the notion of zero dynamics of nonlinear control theory.

For a nonlinear control system $\Sigma_{n,m,p} = (f, g, h)$, for a nominal point x^0, assume $h(x^0) = 0$. Recall its zero dynamics algorithm [92]:

Step 0: set $N_0 = h^{-1}(0)$. Step k: assume for some neighborhood $V_{k-1} \subseteq X$ of x^0, $N_{k-1} \cap V_{k-1}$ is a smooth submanifold and denote N_{k-1}^c the connected component of $N_{k-1} \cap V_{k-1}$ such that $x^0 \in N_{k-1}^c$. Set

$$N_k = \left\{x \in N_{k-1}^c : f(x) \in T_x N_{k-1}^c + \operatorname{span}\{g_1(x), \ldots, g_m(x)\}\right\}. \tag{4.17}$$

Remark 4.3.22. (i) It is shown in [92] that $N_k \cap V_k$ is invariant under feedback transformations. Then assume that a control system $\tilde{\Sigma} = (\tilde{f}, \tilde{g}, \tilde{h})$ is given by applying a *generalized output injection* and an *output multiplication* to Σ, i.e., $\tilde{f} = f + \gamma h$, $\tilde{g} = g$, $\tilde{h} = \eta h$, where $\gamma : X \to \mathbb{R}^{n\times p}$ and $\eta : X \to Gl(p, \mathbb{R})$. By $\tilde{N}_0 = \tilde{h}^{-1}(0) = h^{-1}(0)$ (since $\eta(x)$ is invertible) and

$$\begin{aligned}\tilde{N}_k &= \left\{x \in \tilde{N}_{k-1}^c : \tilde{f}(x) \in T_x \tilde{N}_{k-1}^c + \operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_m\}(x)\right\}\\ &= \left\{x \in \tilde{N}_{k-1}^c : (f + \gamma h)(x) \in T_x \tilde{N}_{k-1}^c + \operatorname{span}\{g_1, \ldots, g_m\}(x)\right\}\\ &= \left\{x \in \tilde{N}_{k-1}^c : f(x) \in T_x \tilde{N}_{k-1}^c + \operatorname{span}\{g_1, \ldots, g_m\}(x)\right\},\end{aligned}$$

we have $\tilde{N}_k = N_k$ for $k \geq 0$, which means that N_k of the zero dynamics algorithm is invariant under *generalized output injections* and *output multiplications*. From Definition 4.3.19, we know that sys-equivalence is defined by coordinates changes, feedback transformations, *generalized output injections* and *output multiplications*. Therefore, if two systems Σ and $\tilde{\Sigma}$ are sys-equivalent via the coordinates change $\tilde{x} = \psi(x)$, completed by transformations given by arbitrary α, β, γ, η, see Definition 4.3.19, then we have $\tilde{N}_k = \psi(N_k)$.

(ii) The sequence of submanifolds N_k of the zero dynamics algorithm is well-defined for the class $\mathbf{Expl}(\Xi)$, i.e., does not depend on the choice of $\Sigma \in \mathbf{Expl}(\Xi)$. Since by Proposition 4.3.18 any two systems $\Sigma, \Sigma' \in \mathbf{Expl}(\Xi)$ are equivalent via a v-feedback, a generalized output injection and an output multiplication, then by the argument in item (i) above we have $\tilde{N}_k = N_k$.

Proposition 4.3.23. *Consider a DAE $\Xi_{l,n} = (E, F)$ satisfying* $\operatorname{rank} E(x) = q = const.$ *around a point x^0 and a control system $\Sigma = (f, g, h) \in \mathbf{Expl}(\Xi)$. The following conditions, for each $k > 0$,* *(A1)* $M_k \cap U_k$ *of Proposition 4.3.3 is a smooth submanifold and* $\dim\, E(x)T_xM^*$ *is constant locally on* M^**,*

(A2) $N_k \cap V_k$ *of the zero dynamics algorithm is a smooth submanifold and the dimension of* $\mathrm{span}\{g_1, \ldots, g_m\}(x) \cap T_xN^*$ *is constant locally on* N^* *(the assumptions of Proposition 6.1.1 in [92]),*

can be concluded from each other (i.e., (A1) implies (A2) and vice versa). Assume that either (A1) or (A2) holds, then the maximal invariant submanifold $M^* = M^c_{k^*}$ *of* Ξ *coincides with the maximal output zeroing submanifold* $N^* = N^c_{k^*}$ *of* $\Sigma \in \mathbf{Expl}(\Xi)$*. Moreover,* Ξ *is internally regular if and only if* $\mathrm{span}\{g_1(x^0), \ldots, g_m(x^0)\} \cap T_{x^0}N^* = 0$ *(equation (6.4) of [92]).*

The proof is given in Section 4.4.3.

Remark 4.3.24. By Proposition 4.3.23, if there exists a unique $u = u(x)$ that renders N^* output zeroing and locally maximal control invariant for a control system $\Sigma \in \mathbf{Expl}(\Xi)$, then Ξ is internally regular. Since the zero dynamics does not depend on the choice of an explicitation, the internal regularity of Ξ corresponds to that fact that the zero output constraint $y(t) = 0$ of any control system $\Sigma \in \mathbf{Expl}(\Xi)$ can be achieved by a unique control $u(t)$.

4.3.3 Explicitation without driving variables and pure semi-explicit DAEs

Now we will show by an example that sometimes we can reduce some of the driving variables of a (Q, v)-explicitation.

Example 4.3.25. Consider a DAE $\Xi = (E, F)$, given by

$$\begin{bmatrix} \sin x_3 & -\cos x_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} F_1(x) \\ x_1^2 + x_2^2 - 1 \end{bmatrix},$$

where $F_1 : X \to \mathbb{R}$. By $\mathrm{rank}\, E(x) = 1$, the explicitation class $\mathbf{Expl}(\Xi)$ is not empty. A control system $\Sigma \in \mathbf{Expl}(\Xi)$ is:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{bmatrix} F_1(x) + \begin{bmatrix} 0 & \cos x_3 \\ 0 & -\sin x_3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ \quad y \quad = x_1^2 + x_2^2 - 1, \end{cases}$$

where $\begin{bmatrix} \sin x_3 & -\cos x_3 & 0 \end{bmatrix}^T$ is a right inverse of $E_1(x) = \begin{bmatrix} \sin x_3 & -\cos x_3 & 0 \end{bmatrix}$. Now consider the last equation in the dynamics of Σ, which is $\dot{x}_3 = v_1$. Observe that v_1 acts on

$\dot{x}_3$ only, which implies that v_1 is decoupled from the other part of the dynamics. Thus, we may get rid of v_1 and regard x_3 as a new control. Thus the dynamics of Σ become:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\sin x_3\\ -\cos x_3\end{bmatrix} F_1(x) + \begin{bmatrix}\cos x_3\\ -\sin x_3\end{bmatrix} v_2,$$

where x_1 and x_2 are the new states, x_3 and v_2 are the new control inputs. We are, however, not able to reduce v_2 in the same way.

From the above example, it can be observed that if we want to get rid of the i-th driving variable v_i of a control system Σ, then v_i should be present in one equation only (as $\dot{x}_j = \alpha_j(x) + \beta_j(x)v_i$) implying that $\tilde{g}_i = \frac{\partial}{\partial x_j}$, where $\tilde{v}_i = \alpha_j(x) + \beta_j(x)v_i$. Thus if we want to get rid of all driving variables, a necessary and sufficient condition is that the distribution $\operatorname{span}\{g_1,\dots,g_m\} = \operatorname{span}\{\tilde{g}_1,\dots,\tilde{g}_m\}$ is involutive (because the latter is given by $\operatorname{span}\left\{\frac{\partial}{\partial x_{j_1}},\dots,\frac{\partial}{\partial x_{j_m}}\right\}$). If so, Σ is always feedback equivalent to

$$\left\{\begin{array}{l}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}f_1(x_1,x_2)\\ 0\end{bmatrix} + \begin{bmatrix}0\\ I_{n-q}\end{bmatrix} v\\ \quad y \;\; = h(x_1,x_2).\end{array}\right.$$

The above system can be reduced to

$$\Sigma^r : \left\{\begin{array}{l}\dot{x}_1 = f_1(x_1,x_2)\\ \;y = h(x_1,x_2),\end{array}\right. \tag{4.18}$$

where x_2 is the new input. Observe that the above system Σ^r, given by (4.18), has the same number of variables as Ξ. Thus Σ^r is an *explicitation without driving variables* of Ξ. By setting $y = 0$, the control system Σ^r becomes a pure semi-explicit DAE:

$$\Xi^{PSE} : \left\{\begin{array}{l}\dot{x}_1 = f_1(x_1,x_2)\\ \;0 = h(x_1,x_2),\end{array}\right. \tag{4.19}$$

which is actually ex-equivalent to Ξ. Note that the procedure of setting $y = 0$ for a control system is called the *implicitation* of a control system, see [47] and Chapter 2, where we discuss it in detail for linear systems. Therefore Ξ^{PSE}, given by (4.19), is the implicitation of Σ^r, given by (4.18). Before giving the main result of this subsection, we formally define what we mean by "reducing" the variables of a control system Σ:

Definition 4.3.26. For a control system $\Sigma_{n,m,p} = (f,g,h)$, let $\mathscr{D}^{red}$ be an involutive subdistribution of constant rank k of the distribution $\mathscr{D} = \operatorname{span}\{g_1,\dots,g_m\}(x)$. There exists a feedback transformation and a coordinates change such that $\mathscr{D}^{red} = \operatorname{span}\left\{\frac{\partial}{\partial x_2^1},\dots,\frac{\partial}{\partial x_2^k}\right\}$ and Σ takes the form

$$\left\{\begin{array}{l}\dot{x}_1 = f_1(x_1,x_2) + \sum\limits_{i=1}^{m-k} g_1^i(x_1,x_2)v_1^i\\ \dot{x}_2 = v_2\\ \;y = h(x_1,x_2),\end{array}\right.$$

where $v_2 = [v_2^1, \ldots, v_2^k]^T$, we will say that Σ can be $\mathscr{D}^{red}$-reduced to the following control system

$$\begin{cases} \dot{x}_1 = f_1\left(x_1, x_2\right) + \sum\limits_{i=1}^{m-k} g_1^i\left(x_1, x_2\right) v_1^i \\ \ y \ = h\left(x_1, x_2\right), \end{cases}$$

where x_2 is a new control. We say that Σ can be fully reduced if $\mathscr{D}^{red} = \mathscr{D}$.

The above analysis motivates to connect the explicitation without driving variables with pure semi-explicit DAEs.

Theorem 4.3.27. *For a DAE $\Xi_{l,n} = (E, F)$, the following conditions are equivalent around a point x^0:*

(i) $\operatorname{rank} E(x)$ *is constant and the distribution* $\mathscr{D} = \ker E(x)$ *is involutive.*

(ii) Ξ *is ex-equivalent to a pure semi-explicit DAE* Ξ^{PSE} *of form (4.2).*

(iii) Any control system $\Sigma = (f, g, h) \in \mathbf{Expl}(\Xi)$ *can be fully reduced.*

The proof is given in Section 4.4.4.

Remark 4.3.28. (i) There are two kinds of explicitations for nonlinear DAEs, namely, explicitation with, or without, driving variables. Both of them need the constant rank assumption of $E(x)$. However, explicitation without driving variables requires also the involutivity of $\ker E(x)$. It means that if a DAE has an explicitation without driving variables, we can always get the one with driving variables by adding $\dot{x}_j = v$ to the dynamics (actually it is a $1-$fold prolongation of the variables x_j that enter statically into the dynamics). But we can not always reduce driving variables unless the involutivity condition is satisfied.

(ii) A linear DAE $\Delta = (E, H)$, given by (4.3), has always two kinds of explicitations, since the rank of E is always constant and the distribution $\mathscr{D} = \ker E$ is always involutive. The relations and differences of the two explicitations for linear DAEs are discussed in Chapter 3 and in [46] (note that the explicitation without driving variables for linear DAEs is called the (Q, P)-explicitation there).

4.3.4 Nonlinear generalization of the Weierstrass form

In this subsection, we will use the explicitation (with driving variables) procedure to transform an internally regular DAE $\Xi_{l,n} = (E, F)$ with $l = n$, to a normal form under external equivalence. A linear DAE Δ, given by (4.3), is regular (meaning $l = n$ and $\det(sE - H) \neq 0$, $s \in \mathbb{C}$, see e.g. [75]) if and only if E and H are square and Δ is internally regular, see Chapter 2. Moreover, if Δ is regular, then it is ex-equivalent (via linear transformations) to the Weierstrass form **WF**, given by (4.5). The following theorem generalizes this linear result and shows that any internally regular DAE (under the assumption that some dimensions are constant) is always ex-equivalent to a nonlinear Weierstrass form **NWF** (see (4.20) below).

Theorem 4.3.29. *Consider* $\Xi_{l,n} = (E, F)$ *with* $l = n$*, assume that* $\operatorname{rank} E(x) = const. = q$ *around an admissible point* x_a*. Also assume in Algorithm 4.3.4 that, locally around* x_a*:*

(A1) The rank of $dF_0^2(x)$ *and the ranks of the differentials of* $\frac{\partial \Phi_k(x)}{\partial x}$ *are constant for* $1 \leq k \leq k^* - 1$*, where* $\Phi_k(x) = \operatorname{col}[\varphi_0(x), \ldots, \varphi_{k-1}(x), F_k^2(x)]$*, and where*

$$\varphi_0 = \operatorname{col}[\varphi_0^1(x), ..., \varphi_0^{n-s_0}(x)] \text{ and } \varphi_i = \operatorname{col}[\varphi_i^1(x), ..., \varphi_i^{s_{i-1}-s_i}(x)] \text{ for } i > 0.$$

(A2) The dimensions of $E(x)T_xM_k$ *are constant for* $x \in M_k$*,* $0 \leq k \leq k^* - 1$*.*

(A3) $\dim\, E(x)T_xM^* = \dim\, M^*$*.*

Then Ξ *is internally regular and* Ξ *is locally ex-equivalent to the DAE (4.20), represented in the nonlinear Weierstrass form **NWF**:*

$$\left[\begin{array}{cccc|c} N_{\rho_1} & 0 & \cdots & 0 & \\ E_{2,1}(.) & N_{\rho_2} & \ddots & \vdots & \\ \vdots & \ddots & \ddots & 0 & 0 \\ E_{m,1}(.) & \cdots & E_{m,m-1}(.) & N_{\rho_m} & \\ \hline & G(\xi, z) & & & I \end{array}\right] \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_m \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \\ F^*(\xi, z) \end{bmatrix} + \begin{bmatrix} a_1 + b_1\dot{\xi}^\rho \\ a_2 + b_2\dot{\xi}^\rho \\ \vdots \\ a_m + b_m\dot{\xi}^\rho \\ 0 \end{bmatrix}, \tag{4.20}$$

where $\xi_i = [\xi_i^1, \ldots, \xi_i^{\rho_i}]^T$ *and* z *are the new coordinates, and where* $\dot{\xi}^\rho = [\dot{\xi}_1^{\rho_1}, \dot{\xi}_2^{\rho_2}, \ldots, \dot{\xi}_m^{\rho_m}]^T$*. The indices* ρ_i*,* $1 \leq i \leq m$*, with* $m = n - q$*, satisfy* $\rho_1 \leq \rho_2 \leq \ldots \leq \rho_m$*.*

More specifically, for $1 \leq i \leq m$*,* $1 \leq s < i$*, the* $\rho_i \times \rho_s$ *matrix-valued functions* $E_{i,s}$*, the* $\rho_i \times \rho_i$ *nilpotent matrix* N_{ρ_i} *and the* ρ_i*-dimensional vector-valued function* $a_i + b_i\dot{\xi}^\rho$ *are of the following form*

$$E_{i,s} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -E_{i,s}^{\rho_s}(\xi, z) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -E_{i,s}^{\rho_i - 1}(\xi, z) \end{bmatrix}, \quad N_{\rho_i} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad a_i + b_i\dot{\xi}^\rho = \begin{bmatrix} 0 \\ a_i^1 + \sum\limits_{l=1}^{m} b_{i,l}^1 \dot{\xi}_l^{\rho_l} \\ \vdots \\ a_i^{\rho_i - 1} + \sum\limits_{l=1}^{m} b_{i,l}^{\rho_i - 1} \dot{\xi}_l^{\rho_l} \end{bmatrix},$$

where the scalar functions $a_i^k, b_{i,l}^k \in \mathbf{I}^k$*,* $1 \leq k \leq \rho_i - 1$*, where* $\mathbf{I}^k$ *is the ideal generated by* ξ_i^j*,* $1 \leq i \leq m$*,* $1 \leq j \leq k$ *in the ring of smooth functions of* ξ_s^t *and* z_r*.*

Remark 4.3.30. (i) A more compact expression of the above **NWF** is

$$\mathbf{NWF}: \begin{cases} 0 = \xi_i^1, \ 1 \leq i \leq m, \ 1 \leq j \leq \rho_i - 1 \\ \dot{\xi}_i^j = \xi_i^{j+1} + a_i^j + \sum\limits_{l=1}^{m} b_{i,l}^j \dot{\xi}_l^{\rho_l} + E_i^j(\xi, z, \dot{\xi}^\rho), \\ \vdots \\ \dot{z} = F^*(\xi, z) - G(\xi, z)\dot{\xi}, \end{cases}$$

where $a_i^k, b_{i,l}^k \in \mathbf{I}^k$, $1 \le k \le \rho_i - 1$ and

$$E_i^j(\xi, z, \dot{\xi}^{\rho}) = \sum_{s=1}^{i-1} E_{i,s}^j(\xi, z)\dot{\xi}_s^{\rho_s}, \;\; j \ge \rho_s.$$

(ii) The submanifold sequences M_k of Algorithm 4.3.4 can be expressed as:

$$M_k = \left\{(\xi, z) : \xi_i^j = 0,\; 1 \le i \le m,\; 1 \le j \le k+1\right\},$$

the maximal invariant submanifold M^* is given by

$$M^* = \{(\xi, z) : \xi_i = 0,\; 1 \le i \le m\}.$$

(iii) By $a_i^k, b_{i,l}^k \in \mathbf{I}^k$ and $\mathbf{I}^k$ is the ideal generated by ξ_i^j, $1 \le i \le m$, $1 \le j \le k$ in the ring of smooth functions of ξ_s^t and z_r, it is not hard to see $a_i^k = b_{i,l}^k = 0$ locally for all $(\xi, z) \in M_{k-1}$.

(iv) We see that a solution $(\xi(t), z(t)$ passing through (ξ^0, z^0) exists if and only if $\xi^0 \in M^*$ and thus the solution $(0, z(t))$ is unique, where $z(t)$ is governed by the ODE $\dot{z} = F^*(0, z)$, which agrees with the result of Theorem 4.3.14(iii).

The proof is given in Section 4.4.5. This proof is closely related to the zero dynamics algorithm for nonlinear control systems shown in [92] and the construction procedure of the above normal form is not difficult but quite tedious, so in order to avoid reproducing the zero dynamics algorithm, we will use some results directly from [92] with small modifications. Then after the proof, we will show the construction procedure precisely by an example. Note that from the example below, it is not hard to deduce that under some extra rank assumptions, the terms a_i^k's and $b_{i,l}^k$'s vanish and the terms E_i^j's become constant, as shown in the following corollary. Denote by $\mathrm{rank}(A(x))$ the rank of the matrix $A(x)$ and denote by $\mathrm{rank}_{\mathbb{R}}(A(x))$ the dimension of the vector space spanned over $\mathbb{R}$ by the rows of $A(x)$. Use the notations as in Algorithm 4.3.4, set $H_k(x) = [\varphi_0, \ldots, \varphi_k]^T$ and $g = [g_1, \ldots, g_m]$ be a matrix such that $\mathrm{Im} g(x) = \ker E(x)$, denote by $L_g H_k$ the matrix $(L_{g_i} H_k^j)_{ij}$, $1 \le i \le m$, $1 \le j \le n - s_k$.

Corollary 4.3.31. *If, additionally to (A1)-(A3), we assume*

$$\text{(A4)} \quad \mathrm{rank}(L_g H_k(x)) = \mathrm{rank}(L_g H_k(x_a)),$$

*then in the **NWF** of (4.20),* $a_i^k = b_{i,l}^k = 0$ *for* $1 \le i \le m$, $1 \le j \le k+1$.

If, additionally to (A1)-(A3), we assume that on M_k,

$$\text{(A5)} \quad \mathrm{rank}_{\mathcal{C}^\infty(M_k)}(L_g H_k(x)) = \mathrm{rank}_{\mathbb{R}}(L_g H_k(x)),$$

*then in the **NWF** of (4.20),* $E_i^j = const.$ *for* $1 \le i \le m$, $1 \le j \le k+1$.

Example 4.3.32. Consider a DAE $\Xi_{n,n} = (E, F)$, assume that $\mathrm{rank} E(x) = q$ in a neighborhood U of an admissible point x_a, let $n = q+3$ and then suppose that the zero dynamics algorithm for a control system $\Sigma_{n,3,3} = (f, g, h) \in \mathbf{Expl}(\Xi)$ is implemented in the following way:

Step 1: Let $h = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} \xi_1^1 \\ \xi_2^1 \\ \xi_3^1 \end{bmatrix} = \xi^1$, suppose that the differentials dh_i of h_i for $i = 1, 2, 3$, are independent around x_a. Then define

$$M_0 = \{x : h_1(x) = h_2(x) = h_3(x) = 0\} = \{x : \xi_1^1 = \xi_2^1 = \xi_3^1 = 0\} = \{x : \xi^1 = 0\}.$$

Set $H_0(x) = h(x)$ and suppose that the matrix $L_g h = L_g H_0 = \begin{bmatrix} L_g h_1 \\ L_g h_2 \\ L_g h_3 \end{bmatrix}$ vanishes at all $x \in M_0$ around x_a, which implies that there exist smooth functions σ_i^1, $i = 1, 2, 3$ such that $\begin{bmatrix} L_g h_1 \\ L_g h_2 \\ L_g h_3 \end{bmatrix} = \begin{bmatrix} \sigma_1^1 \\ \sigma_2^1 \\ \sigma_3^1 \end{bmatrix} = \sigma^1$, where $\sigma^1 = 0$ for all $x \in M_0$ around x_a, i.e., $\sigma_i^1 \in \mathbf{I}^1$. Suppose that the differentials of $H_1 = \begin{bmatrix} H_0 \\ L_f H_0 \end{bmatrix}$ are independent around x_a, denote $L_f H_0 = L_f h = \begin{bmatrix} L_f h_1 \\ L_f h_2 \\ L_f h_3 \end{bmatrix} = \begin{bmatrix} \xi_1^2 \\ \xi_2^2 \\ \xi_3^2 \end{bmatrix} = \xi^2$. Then define

$$M_1 = \{x \in M_0 : \xi_1^2 = \xi_2^2 = \xi_3^2 = 0\} = \{x \in M_0 : \xi^2 = 0.\}$$

Step 2: Suppose that the matrix $L_g H_1 = \begin{bmatrix} L_g H_0 \\ L_g L_f H_0 \end{bmatrix} = \begin{bmatrix} L_g \xi^1 \\ L_g \xi^2 \end{bmatrix}$ has rank 1 for all $x \in M_1$ around x_a. Then, without loss of generality, we can assume $L_g \xi_1^2 \neq 0$ and there exists smooth functions $E_1(x)$,$E_2(x)$ and $\sigma_2^2(x), \sigma_3^2(x)$ such that

$$L_g \xi_2^2 = -E_1 L_g \xi_1^2 + \sigma_2^2, \quad L_g \xi_3^2 = -E_2 L_g \xi_1^2 + \sigma_3^2,$$

where $\begin{bmatrix} \sigma_2^2 \\ \sigma_3^2 \end{bmatrix} = \sigma^2 = 0$ for all $x \in M_1$ around x_a, i.e., $\sigma_i^2 \in \mathbf{I}^2$. Now set

$$R_1 = \begin{bmatrix} 0 & 0 & 0 & E_1 & 1 & 0 \\ 0 & 0 & 0 & E_2 & 0 & 1 \end{bmatrix}$$

and denote

$$R_1 L_f H_1 = \begin{bmatrix} E_1 L_f \xi_1^2 + L_f \xi_2^2 \\ E_2 L_f \xi_1^2 + L_f \xi_3^2 \end{bmatrix} = \begin{bmatrix} \xi_2^3 \\ \xi_3^3 \end{bmatrix} = \xi^3.$$

Suppose that the differentials of the matrix $H_2 = \begin{bmatrix} H_1 \\ R_1 L_f H_1 \end{bmatrix}$ are independent around x_a, thus we have

$$M_2 = \{x \in M_1 : \xi_2^3 = \xi_3^3 = 0\} = \{x \in M_1 : \xi^3 = 0\}.$$

Step 3: Suppose the matrix $L_gH_2 = \begin{bmatrix} L_g\xi^1 \\ L_g\xi^2 \\ L_g\xi^3 \end{bmatrix}$. has rank 2 for all $x \in M_2$ around x_a. Then, there exist smooth functions $E_3(x)$,$E_4(x)$ and $\sigma^3(x)$ such that

$$L_g\xi_3^3 = -E_3L_g\xi_1^2 - E_4L_g\xi_2^3 + \sigma_3^3,$$

where $\sigma_3^3(x) = 0$ for all $x \in M_2$ around x_a, i.e., $\sigma_3^3 \in \mathbf{I}^3$. Now set

$$R_2 = \begin{bmatrix} 0 & 0 & 0 & E_3 & 0 & 0 & E_4 & 1 \end{bmatrix}$$

and denote

$$R_2L_fH_2 = E_3L_f\xi_1^2 + E_4L_f\xi_2^3 + L_f\xi_3^3 = \xi_3^4.$$

Suppose that the differentials of matrix $H_3 = \begin{bmatrix} H_2 \\ R_2L_fH_2 \end{bmatrix}$ are independent around x_a and we thus have $M_3 = \{x \in M_2 : \xi_3^4 = 0\}$.

Step 4: Consider the matrix

$$L_gH_3 = [(L_g\xi^1)^T, (L_g\xi^2)^T, (L_g\xi^3)^T, (L_g\xi_3^4)^T]^T.$$

Suppose it has rank 3 at x_a, then the algorithm stops (since $m = p = 3$). Thus, by Proposition 6.1.3 of [92] (see also Claim 4.4.1 of Section 4.4.5), in $(z, \xi_1^1, \xi_1^2, \xi_2^1, \xi_2^2, \xi_2^3, \xi_3^1, \xi_3^2, \xi_3^3, \xi_3^4)$-coordinates (where z are complementary coordinates), Σ is brought into the following form:

$$\begin{cases} y_1 &= \xi_1^1 \\ \dot{\xi}_1^1 &= \xi_1^2 + \sigma_1^1(x)v \\ \dot{\xi}_1^2 &= L_f\xi_1^2 + L_g\xi_1^2 v \\ y_2 &= \xi_2^1 \\ \dot{\xi}_2^1 &= \xi_2^2 + \sigma_2^1(x)v \\ \dot{\xi}_2^2 &= \xi_2^3 - E_1(x)(L_f\xi_1^2 + L_g\xi_1^2 v) + \sigma_2^2(x)v \\ \dot{\xi}_2^3 &= L_f\xi_2^3 + L_g\xi_2^3 v \\ y_3 &= \xi_3^1 \\ \dot{\xi}_3^1 &= \xi_3^2 + \sigma_3^1(x)v \\ \dot{\xi}_3^2 &= \xi_3^3 - E_2(x)(L_f\xi_1^2 + L_g\xi_1^2 v) + \sigma_3^2(x)v \\ \dot{\xi}_3^3 &= \xi_3^4 - E_3(x)(L_f\xi_1^2 + L_g\xi_1^2 v) - E_4(x)(L_f\xi_2^3 + L_g\xi_2^3 v) + \sigma_3^3(x)v \\ \dot{\xi}_3^4 &= L_f\xi_3^4 + L_g\xi_3^4 v \\ \dot{z} &= \bar{F}(\xi, z) + \bar{g}(\xi, z)v, \end{cases} \tag{4.21}$$

where the matrix $\begin{bmatrix} L_g\xi_1^2 \\ L_g\xi_2^3 \\ L_g\xi_3^4 \end{bmatrix}$ is invertible at x_a. Then by the feedback transformation

$$\tilde{v} = \begin{bmatrix} L_f\xi_1^2 \\ L_f\xi_2^3 \\ L_f\xi_3^4 \end{bmatrix} + \begin{bmatrix} L_g\xi_1^2 \\ L_g\xi_2^3 \\ L_g\xi_3^4 \end{bmatrix} v = \alpha + \beta v,$$

we get

$$\left\{\begin{aligned}
y_1 &= \xi_1^1\\
\dot{\xi}_1^1 &= \xi_1^2 + \sigma_1^1(x)\beta^{-1}(x)\left(\tilde{v} - \alpha(x)\right)\\
\dot{\xi}_1^2 &= \tilde{v}_1\\
y_2 &= \xi_2^1\\
\dot{\xi}_2^1 &= \xi_2^2 + \sigma_2^1(x)\beta^{-1}(x)\left(\tilde{v} - \alpha(x)\right)\\
\dot{\xi}_2^2 &= \xi_2^3 - E_1(x)\tilde{v}_1 + \sigma_2^2(x)\beta^{-1}(x)\left(\tilde{v} - \alpha(x)\right)\\
\dot{\xi}_2^3 &= \tilde{v}_2\\
y_3 &= \xi_3^1\\
\dot{\xi}_3^1 &= \xi_3^2 + \sigma_3^1(x)\tilde{v}\\
\dot{\xi}_3^2 &= \xi_3^3 - E_2(x)\tilde{v}_1 + \sigma_3^2(x)\beta^{-1}(x)\left(\tilde{v} - \alpha(x)\right)\\
\dot{\xi}_3^3 &= \xi_3^4 - E_3(x)v_1 - E_4(x)\tilde{v}_2 + \sigma_3^3(x)\tilde{v}\\
\dot{\xi}_3^4 &= \tilde{v}_3\\
\dot{z} &= \bar{F}(\xi, z) + \bar{g}(\xi, z)\beta^{-1}(x)\left(\tilde{v} - \alpha(x)\right)
\end{aligned}\right. .$$

Now by setting $y = 0$ and deleting $\dot{\xi}_1^2 = \tilde{v}_1, \dot{\xi}_2^3 = \tilde{v}_2, \dot{\xi}_3^4 = \tilde{v}_3$, we get the following DAE:

$$\left[\begin{array}{cc|ccc|cccc|c}
0 & 0 & & & & & & & & \\
1 & 0 & & & & & & & & \\
\hline
0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 1 & 0 & 0 & & & & & \\
0 & E_1(.) & 0 & 1 & 0 & & & & & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
0 & E_2(.) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \\
0 & E_3(.) & 0 & 0 & E_4(.) & 0 & 0 & 1 & 0 & \\
\hline
0 & g_1^* & 0 & 0 & g_2^* & 0 & 0 & 0 & g_3^* & I
\end{array}\right]
\left[\begin{array}{c}
\dot{\xi}_1^1\\ \dot{\xi}_1^2\\ \hline \dot{\xi}_2^1\\ \dot{\xi}_2^2\\ \dot{\xi}_2^3\\ \hline \dot{\xi}_3^1\\ \dot{\xi}_3^2\\ \dot{\xi}_3^3\\ \dot{\xi}_3^4\\ \hline \dot{z}
\end{array}\right]
=
\left[\begin{array}{c}
\xi_1^1\\ \xi_1^2\\ \hline \xi_2^1\\ \xi_2^2\\ \xi_2^3\\ \hline \xi_3^1\\ \xi_3^2\\ \xi_3^3\\ \xi_3^4\\ \hline z
\end{array}\right]
+
\left[\begin{array}{c}
0\\ a_1^1 + b_1^1\dot{\xi}^\rho\\ \hline 0\\ a_2^1 + b_2^1\dot{\xi}^\rho\\ a_2^2 + b_2^2\dot{\xi}^\rho\\ \hline 0\\ a_3^1 + b_3^1\dot{\xi}^\rho\\ a_3^2 + b_3^2\dot{\xi}^\rho\\ a_3^3 + b_3^3\dot{\xi}^\rho\\ \hline F^*(\xi, z)
\end{array}\right],$$

where g_1^*, g_2^*, g_3^* are the nonzero columns of $-\bar{g}\beta^{-1}$ and $F^* = \bar{F} - \bar{g}\beta^{-1}\alpha$, where $a_i^k = -\sigma_i^k\beta^{-1}\alpha$, $b_i^k = \sigma_i^k\beta^{-1}$, for $1 \leq i \leq 3$, $1 \leq k \leq \rho_i - 1$. The indices $\rho_1 = 2$, $\rho_2 = 3$, $\rho_3 = 4$. Finally, by $\sigma_i^k \in \mathbf{I}^k$, we have $a_i^k, b_{i,l}^k \in \mathbf{I}^k$.

4.4 Proofs of the results

4.4.1 Proof of Lemma 4.2.3

Proof. Since M is a smooth connected embedded submanifold of dimension s, there exists a neighborhood U_0 of x^0 and $n-s$ smooth functions $\varphi^1, ..., \varphi^{n-s} : U_0 \to \mathbb{R}$ such that $M \cap U_0$ can be expressed as

$$M \cap U_0 = \left\{x \in U_0 : \varphi^1(x) = \varphi^2(x) =, ..., = \varphi^{n-s}(x) = 0\right\}.$$

Choose new coordinates:

$$z = \psi(x) = \left(\phi^1(x), ..., \phi^s(x), \varphi^1(x), ..., \varphi^{n-s}(x)\right)^T,$$

where $\phi^1(x),...,\phi^s(x)$ are smooth functions chosen to complete $\psi(x)$ as a local diffeomorphism. Then for all $x\in U_0$, we have

$$E(x)\dot{x}=E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1}\left(\frac{\partial\psi(x)}{\partial x}\right)\dot{x}=E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1}\dot{z}=\begin{bmatrix}E_1(x) & E_2(x)\end{bmatrix}\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix},$$

where $E_1:U_0\to\mathbb{R}^{l\times s}$, $E_2:U_0\to\mathbb{R}^{l\times(n-s)}$, $z_1=(\phi^1,...,\phi^s)^T$ and $z_2=(\varphi^1,...,\varphi^{n-s})^T$.

Since $z_2=0$ for all $x\in M\cap U_0$, we have

$$E_1(z_1,z_2)\dot{z}_1+E_2(z_1,z_2)\dot{z}_2\,|_{z_2=0}=E_1(z_1,0)\dot{z}_1.$$

By assumption (A1) that $\dim\, E(x)T_xM=r$ for all $x\in M\cap U$, we have $\operatorname{rank}E_1(z_1,0)=r$ for all $z\in M\cap U_1$, where $U_1=U_0\cap U$. Then there exists $Q':M\cap U_1\to Gl(l,\mathbb{R})$ such that :

$$Q'(z_1)E_1(z_1,0)\dot{z}_1=\begin{bmatrix}E_1^1(z_1) & E_1^2(z_1)\\ 0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{z}_1^1\\ \dot{z}_1^2\end{bmatrix},$$

where $\operatorname{rank}\begin{bmatrix}E_1^1(z_1) & E_1^2(z_1)\end{bmatrix}=r$ and $z_1=(z_1^1,z_1^2)$, $E_1^1:M\cap U_1\to\mathbb{R}^{r\times r}$ and $E_1^2:M\cap U_1\to\mathbb{R}^{r\times(s-r)}$. Without loss of generality, we can always assume that the matrix $E_1^1(z_1)$ is invertible, since if not, we can permute the variables of z_1 such that the first r columns of $E_1(z_1)$ are independent.

In view of the analysis above, there exist $Q(z_1,z_2)=Q'(z_1)$ and $z=\psi(x)$ defined on U_1 such that:

$$\tilde{E}(z)=Q(z)E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1},$$

$$\tilde{F}(z)=Q(z)F\left(\psi^{-1}(z)\right)=\begin{bmatrix}\tilde{F}_1^1(z)\\ \tilde{F}_1^2(z)\\ \tilde{F}_2(z)\end{bmatrix}=\begin{bmatrix}\tilde{F}_1^1(z_1,0)+\hat{F}_1^1(z)z_2\\ \tilde{F}_1^2(z_1,0)+\hat{F}_1^2(z)z_2\\ \tilde{F}_2(z_1,0)+\hat{F}_2(z)z_2\end{bmatrix},$$

where $\tilde{F}_1^1$, $\tilde{F}_1^2$, $\tilde{F}_2$ are smooth functions with values in $\mathbb{R}^r$, $\mathbb{R}^{s-r}$, $\mathbb{R}^{n-s}$, respectively, and they can be always represented as the above form by using some matrix-valued functions $\hat{F}_1^1(z),\hat{F}_1^2(z),\hat{F}_2(z)$. Thus by Definition 4.1.1, locally $\Xi=(E,F)\overset{ex}{\sim}\tilde{\Xi}=(\tilde{E},\tilde{F})$ on U_1.

Observe that by assumption (A2), we have $F(x)\in E(x)T_xM$ for all $x\in M\cap U_1$ (since $U_1\subseteq U$), which means $\tilde{F}(z)\in\tilde{E}(z)T_z\psi(M)$ for all $z\in\{z|z_2=0\}$:

$$\begin{bmatrix}\tilde{F}_1^1(z_1,0)\\ \tilde{F}_1^2(z_1,0)\\ \tilde{F}_2(z_1,0)\end{bmatrix}\in\operatorname{Im}\begin{bmatrix}E_1^1(z_1) & E_1^2(z_1)\\ 0 & 0\\ 0 & 0\end{bmatrix}.$$

It follows that $\tilde{F}_1^2(z_1,0)=0$ and $\tilde{F}_2(z_1,0)=0$. Thus $\tilde{\Xi}|_{z_2=0}$ (means a reduction of the restriction of $\tilde{\Xi}$ to $\{z:z_2=0\}$, compare Definition 4.3.7 and 4.3.8) has the following form:

$$\begin{bmatrix}E_1^1(z_1) & E_1^2(z_1)\end{bmatrix}\begin{bmatrix}\dot{z}_1^1\\ \dot{z}_1^2\end{bmatrix}=\tilde{F}_1(z_1). \tag{4.22}$$

Let $z_1^1 : I \to \mathbb{R}^r$ be the solution of the following ODE passing through $z_1^1(0) = z_1^{10}$ (note that $E_1^1(z_1)$ is invertible):

$$\dot{z}_1^1(t) = \left(E_1^1(z_1)\right)^{-1} \left(\tilde{F}_1(z_1) - E_1^2(z_1)\dot{z}_1^2(t)\right). \tag{4.23}$$

It is always possible to find such a solution because if we denote $\dot{z}_1^2 = u$, $f(z_1) = (E_1^1)^{-1}\tilde{F}_1(z_1)$ and $g(z_1) = (E_1^1)^{-1}E_1^2(z_1)$, then ODE (4.23) can be expressed as

$$\begin{cases} \dot{z}_1^1 = f(z_1) + g(z_1)\, u \\ \dot{z}_1^2 = u, \end{cases} \tag{4.24}$$

which can be seen as a control system with input u and it is always solvable for any u. Now assume $(z_1^1(t), z_1^2(t))$ is a solution of (4.24) passing through $z_1(0) = (z_1^1(0), z_1^2(0))$ for a fixed $u(t)$. Then for any $z^0 = (z_1^{10}, z_1^{20}, 0)$ (notice that $x^0 \in M$), $\tilde{\Xi}$ always has a solution $z(t) = (z_1^1(t) \quad z_2^1(t) \quad 0)$. Finally, by $\Xi \overset{ex}{\sim} \tilde{\Xi}$, we get $x(t) = \psi^{-1}\left(z_1^1(t) \quad z_2^1(t) \quad 0\right)$ is a solution of Ξ passing through $x^0 = \psi^{-1}(z^0)$. Clearly, $x_{x^0}(t) \in M \cap U_1 \subseteq M \cap U$ for all $t \in I_{x^0}$. Note that if $r \neq s \Rightarrow r < s$, there always exists a free variable u in equation (4.24) and then Ξ has infinite solutions. If $r = s$, then z_1^2 and u are absent in equation (4.24) and Ξ has just one solution. □

4.4.2 Proof of Theorem 4.3.14

Proof. By $\dim\, E(x)T_xM^*$ is constant and M^* is locally maximal invariant, we have $F(x) \in E(x)T_xM^*$ locally for all $x \in M^*$ (by Proposition 4.3.2). Then all the assumptions of Lemma 4.2.3 are satisfied for M^*. That is, $\dim\, E(x)T_xM^*$ is constant and $F(x) \in E(x)T_xM^*$ locally for all $x \in M^*$. By the proof of Lemma 4.2.3, we have $\Xi|_{M^*}^{red}$ is of the form (4.22).

(i)⇔(ii): Thus by Lemma 4.2.3, locally for any $x^0 \in M^*$, there exists one and only one solution passing through x^0 i.e., Ξ is internally regular (see Definition 4.3.13), if and only if $s = r$, that is $\dim\, M^* = \dim\, E(x)T_xM^*$.

(ii)⇔(iii): We can see from the the proof of Lemma 4.2.3 that $\Xi|_{M^*}^{red}$, given by (4.22), is externally equivalent to the ODE given by (4.23). Suppose that $\dim\, M^* = \dim\, E(x)T_xM^*$, i.e., $s = r$. It follows that z_1^2 is absent in (4.23). Rewrite ODE (4.23) as

$$\dot{z}_1^1 = \left(E_1^1(z_1^1)\right)^{-1}\tilde{F}_1\left(z_1^1\right). \tag{4.25}$$

Denote $z^* = z_1^1$ and $F^*(z^*) = \left(E_1^1(z_1^1)\right)^{-1}\tilde{F}_1\left(z_1^1\right)$. Thus $\Xi|_{M^*}^{red}$ is ex-equivalent to Ξ^*, given by (4.10), via $Q\left(z_1^1\right) = \left(E_1^1(z_1^1)\right)^{-1}$ and a local diffeomorphism $z^* = z_1^1$ defined on M^*. Therefore, Ξ is locally in-equivalent to Ξ^* by Definition 4.3.11.

Conversely, suppose (iii) holds. Ξ is locally in-equivalent to Ξ^* implies that $\Xi|_{M^*}^{red}$ is locally ex-equivalent to Ξ^*. Since z^* is local system of coordinates on M^*, we can directly see that Ξ^* satisfy (ii) since $\dim\, M^* = \dim\, I_{z^*}T_{z^*}M^*$, where I_{z^*} is an identical matrix of the same dimension as M^*. Finally, consider system $\Xi|_{M^*}^{red}$ given by (4.22), since ex-equivalence preserves the dimensions, we have $s = r$, implying that $\dim\, M^* = \dim\, E(x)T_xM^*$ locally for all $x \in M^*$. □

4.4.3 Proofs of Proposition 4.3.18, Theorem 4.3.21 and Proposition 4.3.23

Proof of Proposition 4.3.18. If. Suppose that Σ and $\tilde{\Sigma}$ are equivalent via the transformations given by (4.16). First, $\operatorname{Im}\tilde{g}(x) = \operatorname{Im} g(x)\beta(x) = \ker E_1(x) = \ker E(x)$ proves that $\tilde{g}(x)$ is another choice such that $\operatorname{Im}\tilde{g}(x) = \ker E(x)$. Moreover, we have

$$\tilde{\Sigma} : \begin{cases} \dot{x} = \tilde{f} + \tilde{g}\tilde{v} = f + g\alpha + \gamma h + g\beta v = E_1^{\dagger}F_1 + g\alpha + \gamma F_2 + g\beta v \\ \tilde{y} = \tilde{h} = \eta h, \end{cases}$$

Pre-multiply the differential part $\dot{x} = E_1^{\dagger}F_1 + g\alpha + \gamma F_2 + g\beta v$ of $\tilde{\Sigma}$ by $E_1(x)$, we get (note that $\operatorname{Im} g(x) = \ker E_1(x)$)

$$\begin{cases} E_1(x)\dot{x} = F_1(x) + E_1\gamma F_2(x) \\ \quad \tilde{y} \quad\; = \eta h(x). \end{cases}$$

Thus $\tilde{\Sigma}$ is an $(I, \tilde{v})$-explicitation of the following DAECS:

$$\begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F_1(x) + E_1\gamma F_2(x) \\ \eta(x)F_2(x) \end{bmatrix}.$$

Since the above DAE can be obtained from Ξ via $\tilde{Q}(x) = Q'Q(x)$, where $Q'(x) = \begin{bmatrix} I_q & E_1\gamma(x) \\ 0 & \eta(x) \end{bmatrix}$, it proves that $\tilde{\Sigma}$ is a $(\tilde{Q}, \tilde{v})$-explicitation of Ξ corresponding to the choice of invertible matrix $\tilde{Q}(x) = Q'(x)Q(x)$. Finally, by $E_1\tilde{f} = F_1 + E_1\gamma F_2$, we get $\tilde{f} = \tilde{E}_1^{\dagger}(F_1 + \gamma F_2)$ for the choice of right inverse $\tilde{E}_1^{\dagger}$ of E_1.

Only if. Suppose that $\tilde{\Sigma} \in \mathbf{Expl}(\Xi)$ via $\tilde{Q}(x)$, $\tilde{E}_1^{\dagger}(x)$ and $\tilde{g}(x)$. First by $\operatorname{Im}\tilde{g}(x) = \ker E(x) = \operatorname{Im} g(x)$, there exists an invertible matrix $\beta(x)$ such that $\tilde{g}(x) = g(x)\beta(x)$. Moreover, since $E_1^{\dagger}(x)$ is a right inverse of $E_1(x)$ if and only if any solution $\dot{x}$ of $E_1(x)\dot{x} = w$ is given by $E_1^{\dagger}(x)w$, we have $E_1E_1^{\dagger}F_1(x) = F_1(x)$ and $E_1\tilde{E}_1^{\dagger}F_1(x) = F_1(x)$. It follows that $E_1(\tilde{E}_1^{\dagger} - E_1^{\dagger})F_1(x) = 0$, so $(\tilde{E}_1^{\dagger} - E_1^{\dagger})F_1(x) \in \ker E_1(x)$. Since $\ker E_1(x) = \operatorname{Im} g(x)$, it follows that $(\tilde{E}_1^{\dagger} - E_1^{\dagger})F_1(x) = g(x)\alpha(x)$ for a suitable $\alpha(x)$. Furthermore, since $Q(x)$ is such that $E_1(x)$ of $Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$ is of full row rank, any other $\tilde{Q}(x)$, such that $\tilde{E}_1(x)$ of $\tilde{Q}(x)E(x) = \begin{bmatrix} \tilde{E}_1(x) \\ 0 \end{bmatrix}$ is full row rank, must be of the form $\tilde{Q}(x) = Q'(x)Q(x)$, where $Q'(x) = \begin{bmatrix} Q_1(x) & Q_2(x) \\ 0 & Q_4(x) \end{bmatrix}$. Thus via $\tilde{Q}(x)$, Ξ is ex-equivalent to

$$Q'(x)\begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}\dot{x} = Q'(x)\begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} \Rightarrow \begin{bmatrix} Q_1(x)E_1(x) \\ 0 \end{bmatrix}\dot{x} = \begin{bmatrix} Q_1(x)F_1(x) + Q_2(x)F_2(x) \\ Q_4(x)F_2(x) \end{bmatrix}.$$

The equation on the right-hand side of the above can be expressed (using $\tilde{E}_1^{\dagger}(x)$ and $\tilde{g}(x)$) as:

$$\begin{cases} \dot{x} = \tilde{E}_1^{\dagger}F_1 + \tilde{E}_1^{\dagger}Q_1^{-1}Q_2F_2 + \tilde{g}v = E_1^{\dagger}F_1 + g\alpha + E_1^{\dagger}Q_1^{-1}Q_2h + g\beta\tilde{v} \\ 0 = Q_4F_2 = Q_4h. \end{cases}$$

Thus the explicitation of Ξ via $\tilde{Q}(x)$, $\tilde{E}_1^{\dagger}(x)$ and $\tilde{g}(x)$ is

$$\tilde{\Sigma}: \begin{cases} \dot{x} = E_1^{\dagger}F_1 + g\alpha + \gamma h + g\beta\tilde{v} = f + \gamma h + g(\alpha + \beta\tilde{v}) = \tilde{f} + \tilde{g}\tilde{v} \\ \tilde{y} = \eta h = \tilde{h}. \end{cases}$$

where $\gamma(x) = E_1^{\dagger}Q_1^{-1}Q_2(x)$, $\eta(x) = Q_4(x)$. Now we can see that Σ and $\tilde{\Sigma}$ are equivalent via the transformations given in (4.16). □

Proof of Theorem 4.3.21. By the assumptions that $\operatorname{rank} E(x) = \operatorname{rank} \tilde{E}(\tilde{x}) = const. = q$ around x^0 and $\tilde{x}^0$, respectively, we have Ξ and $\tilde{\Xi}$ are locally ex-equivalent to

$$\Xi' : \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} \quad \text{and} \quad \tilde{\Xi}' : \begin{bmatrix} \tilde{E}_1(\tilde{x}) \\ 0 \end{bmatrix} \dot{\tilde{x}} = \begin{bmatrix} \tilde{F}_1(\tilde{x}) \\ \tilde{F}_2(\tilde{x}) \end{bmatrix},$$

respectively, where $E_1(x)$ and $\tilde{E}_1(\tilde{x})$ are full row rank matrices and their ranks are q. By Definition 4.3.16, we have

$$\begin{aligned} f(x) &= E_1^{\dagger}(x)F_1(x), \ \operatorname{Im} g(x) = \ker E_1(x), \ h(x) = F_2(x), \\ \tilde{f}(\tilde{x}) &= \tilde{E}_1^{\dagger}(\tilde{x})\tilde{F}_1(\tilde{x}), \ \operatorname{Im} \tilde{g}(\tilde{x}) = \ker \tilde{E}_1(\tilde{x}), \ \tilde{h}(\tilde{x}) = \tilde{F}_2(\tilde{x}). \end{aligned} \tag{4.26}$$

Note that the explicitation is defined up to a feedback, an output multiplication and a generalized output injection. Any two control systems belonging to $\mathbf{Expl}(\Xi)$ are sys-equivalent to each other and so are any two control systems belonging to $\mathbf{Expl}(\tilde{\Xi})$. Thus the choice of an explicitation system makes no difference for the proof of sys-equivalence. Without loss of generality, we will use $f(x)$, $g(x)$, $h(x)$ and $\tilde{f}(x), \tilde{g}(x), \tilde{h}(x)$, given in (4.26) for the remaining part of this proof.

If. Suppose $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$ locally in a neighborhood U of x^0. By Definition 4.3.19, there exists a diffeomorphism $\tilde{x} = \psi(x)$ and $\beta : U \to Gl(m, \mathbb{R})$ such that $\tilde{g} \circ \psi = \frac{\partial \psi}{\partial x} g\beta$, which implies

$$\ker(\tilde{E} \circ \psi) = \operatorname{span}\{\tilde{g}_1, ..., \tilde{g}_m\} \circ \psi = \operatorname{span}\left\{\frac{\partial \psi}{\partial x} g_1, ..., \frac{\partial \psi}{\partial x} g_m\right\} = \frac{\partial \psi}{\partial x} \ker E.$$

We can deduce from the above equation that there exists $Q_1 : U \to Gl(q, \mathbb{R})$ such that

$$\tilde{E}_1(\psi(x)) = Q_1(x)E_1(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}. \tag{4.27}$$

Subsequently, by $\tilde{f} \circ \psi = \frac{\partial \psi}{\partial x}(f + \gamma h + g\alpha)$ of Definition 4.3.19, we have

$$(\tilde{E}_1^{\dagger} \circ \psi)(\tilde{F}_1 \circ \psi) = \frac{\partial \psi}{\partial x}(E_1^{\dagger}F_1 + \gamma F_2 + g\alpha).$$

Premultiply the above equation by $\tilde{E}_1 \circ \psi = Q_1 E_1 \left(\frac{\partial \psi}{\partial x}\right)^{-1}$, to obtain

$$\tilde{F}_1(\psi(x)) = Q_1(x)F_1(x) + Q_1(x)E_1(x)\gamma(x)F_2(x). \tag{4.28}$$

Then by $\tilde{h} \circ \psi = \eta h$ of Definition 4.3.19, we immediately get

$$\tilde{F}_2(\psi(x)) = \eta(x)F_2(x). \tag{4.29}$$

Now combining (4.27), (4.28) and (4.29), we conclude that Ξ' and $\tilde{\Xi}'$ are ex-equivalent via $\tilde{x} = \psi(x)$ and $Q(x) = \begin{bmatrix} Q_1(x) & Q_1(x)E_1(x)\gamma(x) \\ 0 & \eta(x) \end{bmatrix}$, which implies that $\Xi \overset{ex}{\sim} \tilde{\Xi}$ (since $\Xi \overset{ex}{\sim} \Xi'$ and $\tilde{\Xi} \overset{ex}{\sim} \tilde{\Xi}'$).

Only if. Suppose that locally $\Xi \overset{ex}{\sim} \tilde{\Xi}$. It implies that locally $\Xi' \overset{ex}{\sim} \tilde{\Xi}'$. Assume that they are ex-equivalent via $Q : U \to Gl(l, \mathbb{R})$ and $\tilde{x} = \psi(x)$ defined on a neighborhood U of x^0. Let $Q(x) = \begin{bmatrix} Q_1(x) & Q_2(x) \\ Q_3(x) & Q_4(x) \end{bmatrix}$, where $Q_1(x)$, $Q_2(x)$, $Q_3(x)$ and $Q_4(x)$ are matrix-valued functions of sizes $q \times q$, $q \times m$, $p \times q$ and $p \times p$, respectively. Then by $\begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{E}_1 \circ \psi \\ 0 \end{bmatrix} \frac{\partial \psi}{\partial x}$, we can deduce that $Q_3(x) = 0$ and $Q_1(x)$, $Q_4(x)$ are invertible matrices. Then we have

$$\begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{E}_1 \circ \psi \\ 0 \end{bmatrix} \frac{\partial \psi}{\partial x}, \qquad \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1 \circ \psi \\ \tilde{F}_2 \circ \psi \end{bmatrix},$$

which implies

$$\tilde{E}_1 \circ \psi = Q_1 E_1 \left(\frac{\partial \psi}{\partial x} \right)^{-1}, \quad \tilde{F}_1 \circ \psi = Q_1 F_1 + Q_2 F_2, \quad \tilde{F}_2 \circ \psi = Q_4 F_2. \tag{4.30}$$

Thus by $\operatorname{Im} g(x) = \ker E(x) = \ker E_1(x)$ and $\operatorname{Im} \tilde{g}(x) = \ker \tilde{E}(\tilde{x}) = \ker \tilde{E}_1(\tilde{x})$, and using (4.30), we have

$$\tilde{g} \circ \psi = \frac{\partial \psi}{\partial x} g \beta \tag{4.31}$$

for some $\beta : U \to Gl(m, \mathbb{R})$. Moreover, there exists $\alpha : U \to \mathbb{R}^m$ such that

$$\begin{aligned} \tilde{f} \circ \psi = \tilde{E}_1^{\dagger} \circ \psi \tilde{F}_1 \circ \psi &\overset{(4.30)}{=} \frac{\partial \psi}{\partial x} E_1^{\dagger} Q_1^{-1} Q_1 F_1 + Q_2 F_2 \\ &= \frac{\partial \psi}{\partial x} E_1^{\dagger} Q_1^{-1} \left(Q_1 F_1 + Q_2 F_2 + Q_1 E_1 g \alpha \right) \\ &= \frac{\partial \psi}{\partial x} \left(f + E_1^{\dagger} Q_1^{-1} Q_2 y + g \alpha \right), \end{aligned} \tag{4.32}$$

In addition, we have

$$\tilde{h} \circ \psi = \tilde{F}_2 \circ \psi \overset{(4.30)}{=} Q_4 F_2 = Q_4 h. \tag{4.33}$$

Finally, it can be seen from (4.31), (4.32), and (4.33) that $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$ via $\tilde{x} = \psi(x)$, $\alpha(x)$, $\beta(x)$, $\gamma(x) = E_1^{\dagger} Q_1^{-1} Q_2(x)$ and $\eta(x) = Q_4(x)$. □

Proof of Proposition 4.3.23. We first show that the sequence of submanifolds $M_k \cap U_k$ of Proposition 4.3.3 of DAE Ξ and the sequence $N_k \cap V_k$ of the zero dynamics algorithm of any control system $\Sigma = (f, g, h) \in \mathbf{Expl}(\Xi)$ coincide. Suppose that $\operatorname{rank} E(x) = const. = q$ in a neighborhood U_0 of x^0. Then there always exists an invertible matrix $Q(x)$ defined

on U_0 such that $Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$, $Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$ and $\operatorname{rank} E_1(x) = q$ for all $x \in U_0$. Recall that N_k of the zero dynamics algorithm is well-defined for the class $\mathbf{Expl}(\Sigma)$ (see Remark 4.3.22) and N_k are the same for all control systems belonging to $\mathbf{Expl}(\Xi)$. Since the choice of an explicitation system makes no difference for N_k, we can assume $\Sigma = (f, g, h) \in \mathbf{Expl}(\Xi)$ is given by $f(x) = E_1^\dagger(x)F_1(x)$, $\operatorname{Im} g(x) = \ker E(x)$, $h(x) = F_2(x)$.
By the definition of M_0 (see (4.6) of Proposition 4.3.3) and $N_0 = h^{-1}(0)$, we have

$$\begin{aligned} M_0 \cap U_0 &= \{x \in U_0 : Q(x)F(x) \in \operatorname{Im} Q(x)E(x)\} \\ &= \left\{ x \in U_0 : \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix} \in \operatorname{Im} \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} \right\} = \{x \in U_0 : F_2(x) = 0\} \\ &= \{x \in U_0 : h(x) = 0\} = N_0 \cap U_0. \end{aligned}$$

For $k > 0$, suppose $M_k^c = N_k^c$. Then by equation (4.7), we have

$$\begin{aligned} M_k &= \left\{x \in M_{k-1}^c : Q(x)F(x) \in Q(x)E(x)T_xM_{k-1}^c\right\} \\ &= \left\{ x \in M_{k-1}^c : \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix} \in \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} T_xM_{k-1}^c \right\} \\ &= \left\{x \in M_{k-1}^c : F_1(x) \in \operatorname{Im} E_1(x)T_xM_{k-1}^c\right\} \\ &= \left\{x \in M_{k-1}^c : f(x) + \ker E_1(x) \subseteq T_xM_{k-1}^c + \ker E_1(x)\right\} \\ &= \left\{x \in N_{k-1}^c : f(x) \in T_xN_{k-1}^c + \operatorname{span}\{g_1(x), \dots, g_m(x)\}\right\} = N_k. \end{aligned}$$

If either one among (A1) and (A2) holds, then by the relations of N_k and M_k shown above, we can easily deduce the other one. If so, we have both (A1) and (A2) hold. Then by Proposition 4.3.3, $M^* = M_{k^*}^c$ is a locally maximal invariant submanifold and by Proposition 6.1.1 in [92], $N^* = N_{k^*}^c$ is a local maximal output zeroing submanifold. Moreover, we have locally $M^* = N^*$ (since locally $M_k = N_k$).

Now in view of Lemma 4.2.3 and Theorem 4.3.14, under the assumption that $\dim\ E(x)T_xM^*$ is constant locally for all $x \in M^*$, we can deduce the following equivalent statements around x^0 (using the result that $N^* = M^*$):

(a) For any point $x \in M^*$, there exists only one solution of Ξ passing through x (internal regularity of Definition 4.3.13);

(b) $\dim\ M^* = \dim\ E(x)T_xM^*$;

(c) the map $E(x)$ is one to one on T_xM^*;

(d) $\ker E(x) \cap T_xM^* = 0$;

(e) $\operatorname{span}\{g_1(x), \dots, g_m(x)\} \cap T_xN^* = 0$.

Thus we have Ξ is internally regular (condition (a)) if and only if

$$\operatorname{span}\{g_1(x^0), \dots, g_m(x^0)\} \cap T_{x^0}N^* = 0$$

(which is, equivalently, condition (c)). □

4.4.4 Proof of Theorem 4.3.27

Proof. $(i) \Rightarrow (ii)$: Suppose in a neighborhood U of x^0 that $\operatorname{rank} E(x) = q$ and $\mathscr{D}(x) = \ker E(x) = \operatorname{span}\{g_1(x), \ldots, g_m(x)\}$ is involutive, $m = n - q$, where $g_1, \ldots, g_m$ are independent vector fields on U. Then by the involutivity of $\mathscr{D}$, there exist local coordinates $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) = \psi(x)$, where $x_1 = (x_1^1, \ldots, x_1^q)$ and $x_2 = (x_2^1, \ldots, x_2^q)$, such that $\operatorname{span}\{d\tilde{x}_1^1, \ldots, d\tilde{x}_1^q\} = \operatorname{span}\{d\tilde{x}_1\} = \mathscr{D}^{\perp}$ (Frobenius theorem [112]). Note that in the $\tilde{x}$ coordinates, the distribution

$$\begin{aligned}\ker \tilde{E}(\tilde{x}) &= \ker\left(E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1}\right) = \frac{\partial\psi(x)}{\partial x}\mathscr{D}(x)\\ &= \operatorname{span}\left\{\frac{\partial\psi(x)}{\partial x}g_1(x), ..., \frac{\partial\psi(x)}{\partial x}g_m(x)\right\} = \operatorname{span}\{\tilde{g}_1(\tilde{x}), ..., \tilde{g}_m(\tilde{x})\}.\end{aligned}$$

where $\tilde{g}_i(\tilde{x}) = \frac{\partial\psi(x)}{\partial x}g_i(x)$, $i = 1, ..., m$. Now let $\tilde{g}(\tilde{x})$ be a matrix whose columns consist of $\tilde{g}_i(\tilde{x})$, for $i = 1, ..., m$. It follows that $\operatorname{rank}\tilde{g}(\tilde{x}) = m$ around x^0. By $d\tilde{x}_1 = \mathscr{D}^{\perp}$, we have $< d\tilde{x}_1, \tilde{g}_i >= 0$, for $i = 1, \ldots, m$. Thus $\tilde{g}(\tilde{x})$ is of the form $\tilde{g}(\tilde{x}) = \begin{bmatrix} 0 \\ \tilde{g}_2(\tilde{x}) \end{bmatrix}$, where $\tilde{g}_2 : \psi(U) \to \mathbb{R}^{m\times m}$. Since $\operatorname{rank} g(\tilde{x}) = m$, it can be seen that $\tilde{g}_2(\tilde{x})$ is an invertible matrix, which implies that $\tilde{E}(\tilde{x})$ has to be of the form $\tilde{E}(\tilde{x}) = \begin{bmatrix}\tilde{E}_1(\tilde{x}) & 0\end{bmatrix}$, where $\tilde{E}_1(\tilde{x}) : \psi(U) \to \mathbb{R}^{l\times m}$. Thus in $\tilde{x}$-coordinates, $\tilde{\Xi} = (\tilde{E}, \tilde{F})$ has the following form:

$$\begin{bmatrix}\tilde{E}_1(\tilde{x}) & 0\end{bmatrix}\begin{bmatrix}\dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2\end{bmatrix} = F(\tilde{x}).$$

Now by $\operatorname{rank} E(x) = q$, we get $\operatorname{rank}\begin{bmatrix}\tilde{E}_1(\tilde{x}) & 0\end{bmatrix} = \operatorname{rank} E(x) = q$ (the coordinate transformation preserves the rank). Thus there exists $Q : \psi(U) \to Gl(l, \mathbb{R})$ such that $Q(\tilde{x})\tilde{E}(\tilde{x}) = Q(\tilde{x})\begin{bmatrix}\tilde{E}_1(\tilde{x}) & 0\end{bmatrix} = \begin{bmatrix}\tilde{E}_1^1(\tilde{x}) & 0 \\ 0 & 0\end{bmatrix}$, where $\tilde{E}_1^1 : \psi(U) \to \mathbb{R}^{q\times q}$. Since $Q(\tilde{x})$ preserves the rank of $\tilde{E}(\tilde{x})$, we have $\operatorname{rank}\tilde{E}_1^1(\tilde{x}) = q$. Therefore, $\tilde{E}_1^1(\tilde{x})$ is an invertible matrix. Now let $Q'(\tilde{x}) = \begin{bmatrix}\left(\tilde{E}_1^1(\tilde{x})\right)^{-1} & 0 \\ 0 & I_{l-q}\end{bmatrix} Q(\tilde{x})$ and denote $Q'(\tilde{x})f(\tilde{x}) = \begin{bmatrix}F_1(\tilde{x}) \\ F_2(\tilde{x})\end{bmatrix}$. It is seen that, via $\tilde{x} = \psi(x)$ and $Q'(x)$, Ξ is locally ex-equivalent to $\tilde{\Xi} = (\tilde{E}, \tilde{F})$, where $\tilde{E}(\tilde{x}) = Q'(x)E(x)(\frac{\partial\psi(x)}{\partial x})^{-1} = \begin{bmatrix}I_q & 0 \\ 0 & 0\end{bmatrix}$ and $\tilde{F}(\tilde{x}) = Q'(\psi(x))F(x) = \begin{bmatrix}F_1(\tilde{x}) \\ F_2(\tilde{x})\end{bmatrix}$. Clearly, $\tilde{\Xi}$ is a pure semi-explicit DAE.

$(ii) \Rightarrow (iii)$: Suppose that Ξ is locally ex-equivalent to Ξ^{PSE}. Then, any control system $\Sigma \in \mathbf{Expl}(\Xi)$ is sys-equivalent to $\Sigma' \in \mathbf{Expl}(\Xi^{PSE})$ below (by Theorem 4.3.14):

$$\Sigma' : \begin{cases}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}F_1(x_1, x_2) \\ 0\end{bmatrix} + \begin{bmatrix}0 \\ I_m\end{bmatrix} v \\ \quad y \;\; = F_2(x_1, x_2).\end{cases}$$

Suppose that $\Sigma \overset{sys}{\sim} \Sigma'$ via $z = (z_1, z_2) = \psi(x)$, α, β and $\gamma = \begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$, then

$$\Sigma : \begin{cases} \begin{bmatrix}\dot z_1\\ \dot z_2\end{bmatrix} &= \frac{\partial\psi(x)}{\partial x}\left(\begin{bmatrix}F_1(x)\\ 0\end{bmatrix} + \begin{bmatrix}\gamma_1(x)\\ \gamma_2(x)\end{bmatrix} y + \begin{bmatrix}0\\ I_m\end{bmatrix}(\alpha(x)+\beta(x)\tilde v)\right) \\ \tilde y &= \eta(x)F_2(x) \end{cases} .$$

By Definition 4.3.26, Σ can be always fully reduced to (by a coordinates change and a feedback transformation)

$$\begin{cases} \dot x_1 &= F_1(x_1,x_2) + \gamma_1(x_1,x_2)F_2(x_1,x_2) \\ y &= \eta(x_1,x_2)F_2(x_1,x_2), \end{cases}$$

where x_2 is the new control.

$(iii) \Rightarrow (i)$: Suppose (iii) holds. Then $\mathbf{Expl}(\Xi)$ is not empty implies that locally $E(x)$ has constant rank. By Definition 4.3.26, any control system $\Sigma \in \mathbf{Expl}(\Xi)$ can be fully reduced implies $\mathscr{D} = \ker E(x) = \operatorname{span}\{g_1, ..., g_m\}$ is involutive. □

4.4.5 Proof of Theorem 4.3.29

$$\begin{cases} y_1 &= \xi_1^1 \\ \dot\xi_1^1 &= \xi_1^2 + \sigma_1^1 v \\ &\cdots \\ \dot\xi_1^{\rho_1-1} &= \xi_1^{\sigma_1} + \sigma_1^{\rho_1-1} v \\ \dot\xi_1^{\rho_1} &= \alpha_1 + \beta_1 v \\ y_2 &= \xi_2^1 \\ \dot\xi_2^1 &= \xi_2^2 + E_{2,1}^1(\alpha_1+\beta_1 v) + \sigma_2^1 v \\ &\cdots \\ \dot\xi_2^{\rho_2-1} &= \xi_1^{\rho_2} + E_{2,1}^{\rho_2-1}(\alpha_1+\beta_1 u) + \sigma_2^{\rho_2-1} v \\ \dot\xi_1^{\rho_2} &= \alpha_2 + \beta_2 v \\ &\vdots \\ y_i &= \xi_i^1, \quad i = 2,\dots,m \\ \dot\xi_i^1 &= \xi_i^2 + \sum_{s=1}^{i-1} E_{i,s}^1(\alpha_s+\beta_s v) + \sigma_i^1 v \\ &\cdots \\ \dot\xi_i^{\rho_i-1} &= \xi_i^{\rho_2} + \sum_{s=1}^{i-1} E_{i,s}^{\rho_i-1}(\alpha_s+\beta_s v) + \sigma_i^{\rho_i-1} v \\ \dot\xi_i^{\rho_i} &= \alpha_i + \beta_i v \\ &\vdots \\ \dot z &= \bar F(\xi,z) + \bar g(\xi,z)v. \end{cases} \tag{4.34}$$

where $E_{i,s}^j = 0$ for $1 \le i \le m$, $1 \le j < \rho^s$.

Claim 4.4.1. *If assumptions (A1)-(A3) of Theorem 4.3.29 are satisfied, then the admissible point x_a is a regular point of the zero dynamics algorithm (rank conditions (i), (ii), (iii) of Proposition 6.1.3 of [92] are satisfied) for any control system $\Sigma \in \mathbf{Expl}(\Xi)$. If so, we use Proposition 6.1.5 of [92] with a small modification: there exist local coordinates $(\xi, z) = (\xi_1, \dots, \xi_m, z)$ such that Σ is in the form of (4.34) above.*

Remark 4.4.2. (i) Note that in equation (4.34), $\rho_1 \leq \rho_2 \leq \ldots \leq \rho_m$, the matrix $\beta = [\beta_1^T, \ldots, \beta_m^T]^T$ is invertible at x_a. Denote $\xi^j = [\xi_1^j, \ldots, \xi_m^j]$, where $\xi_i^j = 0$ for $j \geq \rho_i$, then the functions $\sigma^k|_{N_{k-1}}^{red} = 0$ for $k = 1, \ldots, \rho_i - 1$, where

$$N_{k-1} = \{(\xi, z) : \xi_i^j = 0,\ 1 \leq i \leq m,\ 1 \leq j \leq k.\}$$

(ii) There are two differences between system (4.34) and the zero dynamics form of Proposition 6.1.3 of [92], where the functions $\sigma_1^1, \ldots, \sigma_1^{\rho_1 - 1}$ are not present and the functions $E_{i,s}^j$ for $1 \leq j < s$ are not necessarily zero. However, in (4.34), $\sigma_1^1, \ldots, \sigma_1^{\rho_1-1}$ vanish on $M_0, \ldots, M_{\rho_1-2}$, respectively, but may not outside, and the functions $E_{i,s}^j$ for $1 \leq j < s$ are zero.

Proof of Claim 4.4.1. We will prove that assumptions (A1), (A2), (A3) of Theorem 4.3.29 correspond to the rank conditions (i), (ii), (iii) of Proposition 6.1.3 in [92]. By the assumption of Theorem 4.3.29 that the rank of $E(x)$ is constant in a neighborhood $U = U_0$, we have **Expl**(Ξ) is not empty. Now, in order to compare the two algorithms (Algorithm 4.3.4 for Ξ and the zero dynamics algorithm for control system $\Sigma \in$ **Expl**(Ξ)), we use the same notations as in Algorithm 4.3.4.

Then for a control system $\Sigma = (f, g, h) \in$ **Expl**(Ξ), we have $f(x) = (E_0^1)^\dagger F_0^1(x)$, $\operatorname{Im} g(x) = \ker E(x)$, $h(x) = F_0^2(x)$. The zero dynamics algorithm for Σ can be implemented in the following way:

Step 0: by assumption (A1) of Theorem 4.3.29 that $dF_0^2(x)$ has constant rank $n - s_0$ around x_a, we get $dh(x) = dF_0^2(x)$ has constant rank $n - s_0$ around x_a (condition (i) of Proposition 6.1.3 in [92]). Thus $h^{-1}(0)$ can be locally expressed as $N_0 = \{x : H_0(x) = 0\}$, where $H_0 = \varphi_0(x) = \operatorname{col}[\varphi_0^1, ..., \varphi_0^{n-s_0}]$.

Step k ($k > 0$): From the proof of Proposition 4.3.23, we have locally $N_{k-1} = M_{k-1}$, which is

$$N_{k-1} = M_{k-1} = \{x : H_{k-1}(x) = 0\},$$

where $H_{k-1} = \operatorname{col}[\varphi_0, \ldots, \varphi_{k-1}]$. By the zero dynamic algorithms, N_k can be calculated by all $x \in N_{k-1}^c$ such that

$$L_f H_{k-1}(x) + L_g H_{k-1}(x) u = 0.$$

Then by assumption (A2) of Theorem 4.3.29, we can deduce $E(x) \ker dH_k$ is constant rank for all $x \in M_k$ around x_a, we have that

$$\dim \ker E(x) \cap \ker dH_{k-1} = \dim \operatorname{span}\{g_1, \ldots, g_m\} \cap \ker dH_{k-1} = const., \quad (4.35)$$

for all $x \in M_{k-1}$ around x_a. Now by $\dim \ker E(x) = const.$ around x_a (since $E(x)$ is of constant rank), we get

$$\dim \operatorname{span}\{g_1, \ldots, g_m\} = const. \quad (4.36)$$

locally around x_a. By (4.35) and (4.36), we get $\operatorname{rank} L_gH_{k-1}(x) = const.$ for all $x \in M_{k-1}$ around x_a (condition (ii) of Proposition 6.1.3 in [92]).

Since the rank of $L_gH_{k-1}(x)$ is constant, there exists a basis matrix $R_{k-1}(x)$ of the annihilator of the image of $L_gH_{k-1}(x)$, that is $R_{k-1}(x)L_gH_{k-1}(x) = 0$. Thus N_k can be defined by

$$N_k = \{x \in U_k : H_{k-1}(x) = 0, \quad R_{k-1}(x)L_fH_{k-1}(x) = 0\}.$$

Notice that by Algorithm 4.3.4, we have

$$M_k = \{x \in U_k : H_{k-1}(x) = 0, \quad F_k^2(x) = 0\}.$$

By $N_k = M_k$ and ranks of the differentials of $\Phi_k(x) = \operatorname{col}[\varphi_0(x), \ldots, \varphi_{k-1}(x), F_k^2(x)]$ are constant for all x around x_a (assumption (A1) of Theorem 4.3.29), it follows that the rank of the differentials of $\begin{bmatrix} H_{k-1}(x) \\ R_{k-1}(x)L_fH_{k-1}(x) \end{bmatrix}$ has constant rank around x_a (condition (i) of Proposition 6.1.3 in [92]).

Finally, the assumption (A3) of Theorem 4.3.29 that $\dim\, E(x)T_xM^* = \dim\, M^*$ locally around x_a implies

$$\operatorname{span}\{g_1(x_a), \ldots, g_m(x_a)\} \cap T_{x_a}N^* = 0.$$

Finally, by $N^* = \{x : H_{k^*} = 0\}$, it follows that the matrix $L_gH_{k^*}(x_a)$ has rank m (condition (iii) of Proposition 6.1.3 in [92]). □

Proof of Theorem 4.3.29. Observe that by assumption (A3) and Theorem 4.3.14(iii), we have Ξ is internally regular. Then by Claim 4.4.1, we have x_a is a regular point of the zero dynamics algorithm for any control system $\Sigma \in \mathbf{Expl}(\Xi)$. Then there exists local coordinates (ξ, z) such that Σ is in the form of (4.34) around x_a. Notice that the matrix $\beta = [\beta_1^T, \ldots, \beta_m^T]^T$ is invertible at x_a and the sequence of submanifolds N_k in the zero dynamics algorithm can be expressed as $N_k = \{(\xi, z) : \xi^j = 0, \ 1 \le j \le k+1\}$. Moreover, locally for all $x \in N_k$, we have $\sigma_i^k = 0$ for $1 \le i \le m$, $1 \le k \le \rho_i - 1$, which implies $\sigma_i^k \in \mathbf{I}^k$. Then for system (4.34), using the feedback transformation $\tilde{v} = \alpha + \beta v$, where $\alpha = \operatorname{col}[a_1, \ldots, a_m]$, we get :

$$\begin{cases} y_i &= \xi_i^1, \qquad i = 1, \ldots, m, \\ \dot{\xi}_i^1 &= \xi_i^2 + \sum\limits_{s=1}^{i-1} E_{i,s}^1 \tilde{v}_s + \sigma_i^1\left(\beta^{-1}\left(\tilde{v} - \alpha\right)\right) \\ & \cdots \\ \dot{\xi}_i^{\rho_i-1} &= \xi_i^{\rho_i} + \sum\limits_{s=1}^{i-1} E_{i,s}^{\rho_i-1} \tilde{v}_s + \sigma_i^{\rho_i-1}\left(\beta^{-1}\left(\tilde{v} - \alpha\right)\right) \\ \dot{\xi}_i^{\rho_i} &= \tilde{v}_i \\ \dot{z} &= \bar{F}(\xi, z) + \bar{g}(\xi, z)\left(\beta^{-1}\left(\tilde{v} - \alpha\right)\right). \end{cases}$$

Denote $a_i^k = -\sigma_i^k \beta^{-1}\alpha$, $b_i^k = \sigma_i^k\beta^{-1}$, for $1 \le i \le m$, $1 \le k \le \rho_i - 1$, then we get

$$\tilde{\Sigma} : \begin{cases} y_i &= \xi_i^1, \quad i = 1, \dots, m \\ \dot{\xi}_i^1 &= \xi_i^2 + \sum\limits_{s=1}^{i-1} E_{i,s}^1 \tilde{v}_s + a_i^1 + b_i^1 \tilde{v} \\ & \cdots \\ \dot{\xi}_i^{\rho_i - 1} &= \xi_i^{\rho_i} + \sum\limits_{s=1}^{i-1} E_{i,s}^{\rho_i - 1} \tilde{v}_s + a_i^{\rho_i - 1} + b_i^{\rho_i - 1}\tilde{v} \\ \dot{\xi}_i^{\rho_i} &= \tilde{v}_i \\ & \vdots \\ \dot{z} &= F^*(\xi, z) + G^*(\xi, z)\tilde{v}, \end{cases}$$

where $F^* = \bar{F} - \bar{g}\beta^{-1}\alpha$ and $G^* = \bar{g}\beta^{-1}$. Denote the above control system by $\tilde{\Sigma}$, we have $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$.

Then consider the last row of every subsystem of $\tilde{\Sigma}$, which is $\dot{\xi}_i^{\rho_i} = \tilde{v}_i$. By deleting this equation in every subsystem and setting $y_i = 0$ for $i = 1, \dots, m$ and replacing the variable $\tilde{v}_i$ by $\dot{\xi}_i^{\rho_i}$, we transform $\tilde{\Sigma}$ to DAE $\tilde{\Xi}$ below. Note that this transformation from $\tilde{\Sigma}$ to $\tilde{\Xi}$ is called the driving variables reduction and implicitation of a control system, which has been discussed in Section 4.3.3.

$$\tilde{\Xi} : \begin{cases} & \vdots & \\ \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_i^1 \\ \dot{\xi}_i^2 \\ \vdots \\ \dot{\xi}_i^{\rho_i} \end{bmatrix} - \begin{bmatrix} 0 \\ \sum\limits_{s=1}^{i-1} E_{i,s}^1 \dot{\xi}_s^{n_s} \\ \vdots \\ \sum\limits_{s=1}^{i-1} E_{i,s}^{\rho_i - 1} \dot{\xi}_s^{n_s} \end{bmatrix} & = & \begin{bmatrix} \xi_i^1 \\ \xi_i^2 \\ \vdots \\ \xi_i^{\rho_i} \end{bmatrix} + \begin{bmatrix} 0 \\ a_i^1 + b_i^1 \dot{\xi}^{\rho} \\ \vdots \\ a_i^{\rho_i - 1} + b_i^{\rho_i - 1}\dot{\xi}^{\rho} \end{bmatrix} \\ & \vdots & \\ -G^*(\xi, z)\,\dot{\xi}^{\rho_i} + \dot{z} & = & F^*(\xi, z), \end{cases}$$

where $E_{i,s}^j = 0$ for $1 \le j < \rho^s$. By $\sigma_i^k \in \mathbf{I}^k$, we have $a_i^k, b_{i,l}^k \in \mathbf{I}^k$. Finally, by Theorem 4.3.21 and $\Sigma \overset{sys}{\sim} \tilde{\Sigma}$, we have that $\Xi \overset{ex}{\sim} \tilde{\Xi}$ and that $\tilde{\Xi}$ is in the **NWF** of (4.20). □

4.5 Conclusion

In this chapter, for a nonlinear DAE $\Xi = (E, F)$, we define the internal and external equivalence, their differences are discussed by analyzing their relations with solutions. We show that the internal regularity (existence and uniqueness of solutions) of a DAE is equivalent to the fact that the DAE is in-equivalent to an ODE on its maximal invariant submanifold. A procedure named explicitation with driving variables is proposed to connect nonlinear DAEs with nonlinear control systems. We show that the external equivalence for two DAEs is the same as the system equivalence for their explicitation systems. Moreover, we show that Ξ is externally equivalent to a pure semi-explicit DAE if and only if the distribution defined by $\ker E(x)$ is of constant rank and involutive. If so, the driving variables of a control system $\Sigma \in \mathbf{Expl}(\Xi)$ can be fully reduced. Finally, a nonlinear generalization of the Weierstrass form **WF** is proposed and an example is given to show its construction procedure.

Chapter 5

Feedback Linearization of Nonlinear Differential-Algebraic Control Systems

Abstract: In this chapter, we study feedback linearizability for nonlinear differential-algebraic control systems DAECSs under two kinds of feedback equivalence, namely, the external and internal feedback equivalence. Necessary and sufficient conditions are given for the internal and external feedback linearization problems with the help of an explicitation procedure. This explicitation procedure attaches a class of ODE control systems with two kinds of inputs to any DAECS. We prove that feedback linearizability of a DAECS is closely related to the involutivity of some distributions of a system given by the explicitation. Moreover, two normal forms are proposed based on the notion of maximal controlled invariant submanifold of DAECSs. These two normal forms facilitate understanding the role of the variables in DAECSs. Finally, we illustrate the results of this chapter with examples (from both practical and academical systems).

Notation

$\mathbb{N}$	the set of natural numbers with zero and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}^{n\times m}$	the set of real valued matrices with n rows and m columns
$\mathcal{C}^j(M;N)$	the class of maps of class $\mathcal{C}^j$, $j \in \mathbb{N}\cup\{\infty\}$, from M to N; if $j=\infty$, it is the set of smooth maps
$Gl\,(n,\mathbb{R})$	the group of nonsigular matrices of $\mathbb{R}^{n\times n}$
T_xM	the tangent space of a submanifold M of $\mathbb{R}^n$ at $x \in M$
Id	identity matrix
$\wedge$	exterior product
$d\xi_1 \wedge d\xi_2$	$d\xi_1^1 \wedge \cdots \wedge \xi_1^{n_1} \wedge d\xi_2 \wedge \cdots \wedge \xi_2^{n_2}$, where $\xi_1 = (\xi_1^1, \dots, \xi_1^{n_1})$ and $\xi_2 = (\xi_2^2, \dots, \xi_2^{n_2})$

5.1 Introduction

Consider a nonlinear control system, given by a differential algebraic equation DAE of quasi-linear form:

$$\Xi^u : E(x)\dot{x} = F(x) + G(x)u, \tag{5.1}$$

where $x \in X$ is the "generalized" state, with X an open subset of $\mathbb{R}^n$, the vector of control inputs $u \in \mathbb{R}^m$, and where $E : TX \to \mathbb{R}^l$, $F : X \to \mathbb{R}^l$ and $G : X \to \mathbb{R}^{l\times m}$ are smooth maps and the word "smooth" will always mean $\mathcal{C}^\infty$-smooth throughout the chapter. Note that $E(x)$ is not necessarily square and invertible. A differential-algebraic control system DAECS of form (5.1) will be denoted by $\Xi^u_{l,n,m} = (E, F, G)$ or, simply, Ξ^u. We call x in (5.1) the "generalized" state because it is different from the state of a classical ODE control system ODECS, which is

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{5.2}$$

where $f, g_1, \ldots, g_m : X \to TX$. Note that the variables of the "generalized" states play two different roles for the system. More specifically, non-invertibility of $E(x)$ may imply the existence of algebraic constraints and some variables of x (even some u-variables) are constrained by the algebraic constraints. On the other hand, some other variables of x are free and they play the role of an input (since they enter the system statically). Note that although the free variables of x may perform "like" inputs, we will emphasize their differences with the original control input u.

A linear DAECS is of the form

$$\Delta^u : E\dot{x} = Hx + Lu, \tag{5.3}$$

where $E, H \in \mathbb{R}^{l\times n}$ and $L \in \mathbb{R}^{l\times m}$ and will be denoted by $\Delta^u_{l,n,m} = (E, H, L)$ or, simply, Δ^u. Linear DAECSs have been studied for decades, there is a rich literature devoted to them (see, e.g., the surveys [127, 128] and textbook [59]). In the context of this chapter, we will need results about canonical forms [131],[124] and Chapter 3, controllability [17],[55],[74], and geometric subspaces [79],[152]. The motivation of studying linear and nonlinear DAECSs is their frequent presence in mathematical models of practical systems e.g., constrained mechanics [159, 199, 22, 141, 177], chemical processes [119, 33, 154], electrical circuits [165, 166, 67], etc.

The feedback linearization problem for nonlinear ODECSs (i.e., when there exists a local change of coordinates in the state space and a feedback transformation such that the transformed system has a linear form in the new coordinate) has drawn attention of researchers for decades (e.g. see survey papers [163],[180] and books [151],[92]). The solution of the feedback linearization problem of ODECSs was first given in Brockett's paper [29] and developed by Jakubczyk and Respondek [98], Su [178], Hunt et Su [88]. Compared to the ODEs, fewer results on the linearization problem of DAE systems can be found. Xiaoping [195] transformed a nonlinear DAECS into a linear one by state space transformations, Kawaji [111] gave sufficient conditions for the feedback linearization of a special class of DAECSs, Jie Wang and Chen Chen [185] considered a semi-explicit DAE and linearized the differential part of the DAE. The linearization of semi-explicit DAEs under equivalence of different levels [49] is studies in Chapter 6.

There are mainly two contributions of this chapter. One is to find when a given DAECS of form (5.1) is locally equivalent to a linear completely controllable one (see the definition of complete controllability in [17]). In particular, we will consider two kinds of equivalence, namely, the external feedback equivalence given in Definition 5.2.2 and the internal feedback equivalence given in Definition 5.3.8. Note that the words "external" and "internal", appearing throughout this chapter, basically mean that we consider the DAECS on an open neighborhood of X and on the *locally maximal controlled invariant submanifold* (see [13]), respectively. We discuss in detail the difference and relations of the two equivalence relations for linear DAEs in Chapter 2, or see [47], and for nonlinear DAEs in Chapter 4, or see [48]. In this chapter, we will use a procedure named *explicitation with driving variables* (proposed in Chapter 3 and Chapter 4) to connect nonlinear DAECSs with nonlinear ODECSs. By this explicitation procedure, we interpret linearizability of DAECSs under internal or external feedback equivalence with the help of linearizability of the explicitation systems under system feedback equivalence (see Definition 5.2.7). The other contribution is to propose two normal forms based on the notion of maximal controlled invariant submanifold. These normal forms are helpful in understanding the role of the variables in a DAECS, e.g., to see which variables of the “generalized” state are actually free and which control inputs are actually constrained by algebraic constraints.

The chapter is organized as follows. In Section 5.2, we give the definition of external feedback equivalence and describe the explicitation with driving variables procedure step by step. In Section 5.3, we show a DAECS can be externally equivalent to two normal forms under different assumptions. In Section 5.4, we give necessary and sufficient conditions for the linearization of DAECSs under external and internal feedback equivalence. In Section 5.5, we illustrate the results of Section 5.3 and Section 5.4 by some examples. Section 5.6 contains the proofs. In Section 5.7, we give conclusions and some perspectives of this chapter.

5.2 Explicitation of differential algebraic control systems

We define the solution of a DAECS as follows:

Definition 5.2.1. (Solution) For $\Xi^u_{l,n,m} = (E, F, G)$, a curve $(\gamma, u) : I \to X \times \mathscr{U}$ defined on an open interval $I \in \mathbb{R}$ with $\gamma(t) \in \mathcal{C}^1$ and $u(t) \in \mathcal{C}^0$ is called a solution of Ξ^u, if for all $t \in I$, $E(\gamma(t)) = F(\gamma(t)) + G(\gamma(t))u(t)$.

If we fix (t_0, x^0) and $u(t)$, then a solution $\gamma(t)$ satisfying $\gamma(t_0) = x^0$ will be denoted by γ_{x^0} and the maximal time-interval on which it exists by I_{x^0}. Clearly, I_{x^0} is an open interval that depends on x^0 and $u(t)$, and may be infinite or finite (depending whether the trajectory

γ_{x^0} escapes in finite time into infinity or not). Note that for a given point x^0, there may not exist any solution passing through x^0 due to the existence of algebraic constraints or a solution γ_{x^0} may not be unique even for a fixed control input $u(t)$. We call a point $x^0 \in X$ an admissible point of Ξ^u, if there exists at least one solution $(\gamma(t), u(t))$ of Ξ^u satisfying $\gamma(t_0) = x^0$. We will denote admissible points by x_a.

Definition 5.2.2. (External feedback equivalence) Two DAECSs $\Xi^u_{l,n,m} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on X and $\tilde{X}$, respectively, are called external feedback equivalent, shortly ex-fb-equivalent, if there exists a diffeomorphism $\psi : X \to \tilde{X}$ and smooth functions $Q : X \to Gl(l, \mathbb{R})$, $\alpha^u : X \to \mathbb{R}^m$, $\beta^u : X \to Gl(m, \mathbb{R})$ such that

$$\begin{aligned} \tilde{E}(\psi(x)) &= Q(x)E(x)\left(\tfrac{\partial\psi(x)}{\partial x}\right)^{-1}, \\ \tilde{f}(\psi(x)) &= Q(x)\left(F(x) + G(x)\alpha^u(x)\right), \\ \tilde{g}(\psi(x)) &= Q(x)G(x)\beta^u(x). \end{aligned} \tag{5.4}$$

The ex-fb-equivalence of two DAECSs is denoted by $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of x^0 and $\tilde{U}$ of $\tilde{x}^0$, and $Q(x)$, $\alpha^u(x)$, $\beta^u(x)$ are defined locally on U, we will talk about local ex-fb-equivalence.

Remark 5.2.3. If two DAECSs are ex-fb-equivalent, then the diffeomorphism ψ establishes one to one correspondence between their solutions. Notice, however, that the control-parameterizing solutions are not the same but are related via the feedback transformation $\tilde{u}(t) = F(x(t)) + G(x(t))\alpha^u(x(t))$.

Consider a DAECS $\Xi^u_{l,n,m} = (E, F, G)$, given by (5.1). The *explicitation with driving variables* of Ξ^u is the following procedure.

Step 1: Assume in a neighborhood U of a given point x^0 that $\operatorname{rank} E(x) = const. = r$. Then there exists a matrix-valued function $Q(x) \in Gl(l, \mathbb{R})$ defined on U such that

$$Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix},$$

where $E_1 : U \to \mathbb{R}^{r\times n}$ and $\operatorname{rank} E_1(x) = r$. Thus via $Q(x)$, Ξ^u is locally ex-fb-equivalent to the following DAECS:

$$\begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} + \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix} u, \tag{5.5}$$

where $Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$, $Q(x)G(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix}$, and F_1,F_2,G_1,G_2, are smooth matrix-valued functions of appropriate sizes.

Step 2: The matrix $E_1(x)$ is of full row rank, so let $E_1^\dagger(x)$ be a right inverse of $E_1(x)$ and set $f(x) = E_1^\dagger(x)F_1(x)$, $g^u(x) = E_1^\dagger(x)G_1(x)$. The collection of all $\dot{x}$ satisfying $E_1(x)\dot{x} = F_1(x) + G_1(x)u$ is given by the following differential inclusion

$$\dot{x} \in f(x) + g^u(x)u + \ker E_1(x) = f(x) + g^u(x)u + \ker E(x). \tag{5.6}$$

Since $\ker E(x)$ is a distribution of rank $n-r$ (because $\operatorname{rank} E(x) = r$), there exist linear independent vector-valued functions $g_1^v, \ldots, g_s^v : U \to \mathbb{R}^n$, where $s = n - r$, such that locally $\ker E(x) = \operatorname{span}\{g_1^v(x), \ldots, g_s^v(x)\}$. Thus by introducing *driving variables* $v_1, \ldots, v_s$, we parametrize the affine distribution $f(x) + g^u(x)u + \ker E(x)$, and all solutions of (5.6) correspond to all solutions (generated by all controls $v_i(t)$) of

$$\dot{x} = f(x) + g^u(x)u + \sum_{i=1}^{s} g_i^v(x)v_i. \tag{5.7}$$

Let $g^v : U \to \mathbb{R}^{n\times s}$ be a smooth matrix-valued function whose columns are g_i^v, $i = 1, \ldots, s$. Then, equation (5.5) can be expressed as the following equation

$$\begin{cases} \dot{x} = f(x) + g^u(x)u + g^v(x)v \\ 0 = h(x) + l^u(x)u, \end{cases} \tag{5.8}$$

where $h(x) = F_2(x)$ and $l^u(x) = G_2(x)$.

Step 3: Now we introduce the following control system

$$\Sigma^{uv} : \begin{cases} \dot{x} = f(x) + g^u(x)u + g^v(x)v \\ y = h(x) + l^u(x)u, \end{cases} \tag{5.9}$$

denoted by $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ or, simply, Σ^{uv}. Note that $s = n - r$ and $p = l - r$, which will be used throughout the chapter to denote $\dim v$ and $\dim y$, respectively. Clearly, equation (5.8) can be seen as an ODECS Σ^{uv} by setting the output $y = 0$. In the above way, we attach an ODECS Σ^{uv} to a DAECS Ξ^u.

Definition 5.2.4. (Explicitation with driving variables) Given a DAECS $\Xi^u_{l,n,m} = (E, F, G)$, fix a point x^0. Assume that the rank of $E(x)$ is constant around x^0. Then, by a (Q, v)-explicitation, we will call a control system $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ with

$$f(x) = E_1^\dagger(x)F_1(x), \;\; g^u(x) = E_1^\dagger(x)G_1(x), \;\; \operatorname{Im} g^v(x) = \ker E(x),$$
$$h(x) = F_2(x), \quad l^u(x) = G_2(x),$$

where

$$Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}, \;\; Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}, \;\; Q(x)G(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix}.$$

The class of all (Q, v)-explicitations will be called the explicitation with driving variables class, shortly the explicitation class. For a particular control system Σ^{uv} belonging to the explicitation class **Expl**(Ξ^u) of Ξ^u, we will write $\Sigma^{uv} \in$ **Expl**(Ξ^u).

Apparently, in the above explicitation procedure, the choice of $Q(x)$, $E_1^\dagger(x)$ and $g^v(x)$ is not unique. The following proposition shows that a given Ξ^u has many (Q, v)-explicitations and any two explicitation systems of Ξ^u are equivalent via a feedback transformation of v, an *output multiplication* and a *generalized output injection*.

Proposition 5.2.5. *Assume that an ODECS $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ is a (Q, v)-explicitation of a DAECS $\Xi^u = (E, F, G)$ corresponding to the choice of invertible matrix $Q(x)$, right inverse $E_1^\dagger(x)$ and matrix $g^v(x)$. Then a control system $\tilde{\Sigma}^{u,\tilde{v}}_{n,m,p} = (\tilde{f}, \tilde{g}^u, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^u)$ is a $(\tilde{Q}, \tilde{v})$-explicitation of Ξ^u corresponding to the choice of invertible matrix $\tilde{Q}(x)$, right inverse $\tilde{E}_1^\dagger(x)$ and matrix $\tilde{g}^{\tilde{v}}(x)$ if and only if Σ^{uv} and $\tilde{\Sigma}^{u,\tilde{v}}$ are equivalent via a v-feedback transformation of the form $v = \alpha^v(x) + \lambda(x)u + \beta^v(x)\tilde{v}$, a generalized output injection $\gamma(x)y = \gamma(x)(h(x) + l^u(x))$ and an output multiplication $\tilde{y} = \eta(x)y$, which map*

$$\begin{aligned} f \mapsto \tilde{f} = f + \gamma h + g^v\alpha^v, \quad g^u \mapsto \tilde{g}^u = g^u + \gamma l^u + g^v\lambda, \\ g^v \mapsto \tilde{g}^{\tilde{v}} = g^v\beta^v, \quad h \mapsto \tilde{h} = \eta h, \quad l^u \mapsto \tilde{l}^u = \eta l^u. \end{aligned} \tag{5.10}$$

where $\alpha^v(x)$, $\beta^v(x)$, $\gamma(x)$, $\lambda(x)$, $\eta(x)$ are smooth matrix-valued functions, and $\beta^v(x)$ and $\eta(x)$ are invertible.

The proof is given in Section 5.6.

Remark 5.2.6. The constant rank assumption of $E(x)$ (around x^0) is essential for the explicitation of Ξ^u. Because without this assumption, we may not have a smooth $Q(x)$ and/or a smooth right inverse $E_1^\dagger(x)$ of $E_1(x)$. Since we assume $E(x)$ is of constant rank around a point x^0, the matrices Q, f, g^u, g^v, h, l^u are all defied locally, and so is $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u)$. Note that x^0 is not necessarily an admissible point, i.e., there may not exist solutions passing through x^0. However, the explicitation around x^0 always exists as long as the constant rank assumption of $E(x)$ is satisfied.

Now we will define an equivalence relation for two ODECSs of form (5.9), which can be seen as a generalization of the notion of sys-equivalence given as Definition 4.3.19 in Chapter 4.

Definition 5.2.7. (System feedback equivalence) Consider two control systems $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}})$ defined on X and $\tilde{X}$, respectively. Then Σ^{uv} and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$ are called system feedback equivalence, shortly sys-fb-equivalent, if there exists a diffeomorphism $\psi : X \to \tilde{X}$, smooth functions $\alpha^u(x)$, $\alpha^v(x)$ $\lambda(x)$ and $\gamma(x)$ with values in $\mathbb{R}^m$,$\mathbb{R}^s$,$\mathbb{R}^{s\times m}$ and $\mathbb{R}^{n\times p}$, respectively, and invertible smooth matrix-valued functions $\beta^u(x)$, $\beta^v(x)$ and $\eta(x)$ with values in $Gl(m, \mathbb{R})$, $Gl(s, \mathbb{R})$ and $Gl(p, \mathbb{R})$, respectively, such that

$$\begin{bmatrix} \tilde{f} \circ \psi & \tilde{g}^{\tilde{u}} \circ \psi & \tilde{g}^{\tilde{v}} \circ \psi \\ \tilde{h} \circ \psi & \tilde{l}^{\tilde{u}} \circ \psi & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial x}\gamma\eta \\ 0 & \eta \end{bmatrix} \begin{bmatrix} f & g^u & g^v \\ h & l^u & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ \alpha^u & \beta^u & 0 \\ \alpha^v + \lambda\alpha^u & \lambda\beta^u & \beta^v \end{bmatrix}. \tag{5.11}$$

The sys-fb-equivalence of two control systems will be denoted by $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of x_0 and $\tilde{U}$ of $\tilde{x}_0$, and α^u, α^v, λ, γ, β^u, β^v, η are defined locally on U, we will speak about local sys-fb-equivalence.

Remark 5.2.8. (i) Observe that, in equation (5.11), there are two kinds of feedback transformations. Namely, $u = \alpha^u(x) + \beta^u(x)\tilde{u}$ and $v = \alpha^v(x) + \lambda(x)u + \beta^v(x)\tilde{v}$, which can be written together as:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha^u \\ \alpha^v \end{bmatrix} + \begin{bmatrix} \beta^u & 0 \\ \lambda & \beta^v \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}. \tag{5.12}$$

It implies that there are two kinds of inputs in ODECSs of form (5.9). Moreover, one input (the driving variable v) is more "powerful" than the other input (the original control variable u), since when transforming v, we can use both u and x in feedback tranformation, but when transforming u, we are not allowed to use v. Another difference between u and v is that the input u is injected into the output y via $l^u(x)u$, but the input v is not directly injected into the output y.

(ii) Recall that we denote a control system of form (5.9) by $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$. Throughout, for simplicity when needed, we will denote it by $\Sigma^{w}_{n,m+s,p} = (f, g^w, h, l^w)$, or shortly Σ^w, where $g^w = [g^u, g^v]$, $l^w = [l^u, 0]$, $w = (u, v)$. Moreover, if we denote

$$\alpha^w = \begin{bmatrix} \alpha^u \\ \alpha^v + \lambda\alpha^u \end{bmatrix}, \quad \beta^w = \begin{bmatrix} \beta^u & 0 \\ \lambda\beta^u & \beta^v \end{bmatrix}, \quad \gamma^w = \frac{\partial \psi}{\partial x}\gamma\eta,$$

then we have the following equivalent expression of equation (5.11):

$$\begin{bmatrix} \tilde{f} \circ \psi & \tilde{g}^{\tilde{w}} \circ \psi \\ \tilde{h} \circ \psi & \tilde{l}^{\tilde{w}} \circ \psi \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial x} & \gamma^w \\ 0 & \eta \end{bmatrix} \begin{bmatrix} f & g^w \\ h & l^w \end{bmatrix} \begin{bmatrix} I & 0 \\ \alpha^w & \beta^w \end{bmatrix}. \tag{5.13}$$

Observe that in equation (5.13), $\gamma^w(x)$ and $\alpha^w(x)$ can be arbitrary since $\gamma(x)$, $\alpha^v(x)$ and $\alpha^u(x)$ are arbitrary. The matrix $\beta^w(x)$ is invertible since $\beta^u(x)$ and $\beta^v(x)$ are invertible and, which is crucial, $\beta^w(x)$ has a *lower block-triangular form*. This triangular form is a consequence of two kinds of feedback transformations as explained in item (i) of this remark.

(iii) The transformations in equation (5.13) can be seen as a nonlinear generalization of the Morse transformation (see [146],[145], or Definition 2.2.3 of Chapter 2) of linear ODECSs. In the linear case, the transformations $\psi(x)$, $\alpha^w(x)$ and $\eta(x)$ in equation (5.13) correspond to, respectively, coordinates changes in the state, input, and output space of a linear ODECS. Moreover, α^w and γ^w correspond to the feedback transformation and output injection matrix, respectively.

The following theorem connects the ex-fb-equivalence of two DAECSs with the sys-fb-equivalence of two ODECSs (explicitations), which can be seen as a generalization of Theorem 4.3.21 of Chapter 4.

Theorem 5.2.9. (Extension of Theorem 4.3.21 of Chapter 4) *Consider two DAE control systems* $\Xi^u_{l,n,m} = (E, F, G)$ *and* $\tilde{\Xi}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$ *defined on* X *and* $\tilde{X}$*, respectively. Assume* $\operatorname{rank} E(x) = r$ *in a neighborhood* U *of a point* $x^0 \in X$ *and* $\operatorname{rank} \tilde{E}(\tilde{x}) = r$ *in a neighborhood* $\tilde{U}$ *of a point* $\tilde{x}^0 \in \tilde{X}$*. Then, given any ODECSs* $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u) \in$ $\mathbf{Expl}(\Xi^u)$ *and* $\tilde{\Sigma}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}}) \in \mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$*, we have locally* $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ *if and only if* $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$.

The proof is given in Section 5.6.

5.3 Maximal controlled invariant submanifold form

In this section, we extend the internal analysis of nonlinear DAEs presented in Chapter 4 to DAECSs using the concept of controlled invariant submanifolds (see e.g., [92],[151] for ODECSs, and [12] for DAECSs). In this chapter, we define locally controlled invariant submanifold of DAECSs as follows.

Definition 5.3.1. (Locally controlled invariant submanifold) Consider a DAECS $\Xi^u_{l,n,m} = (E, F, G)$ and fix an admissible point x_a. A smooth connected embedded submanifold M, such that $x_a \in M$, is called a locally controlled invariant submanifold (around x_a) of Ξ^u if there exists a neighborhood U of x_a such that for any point $x^0 \in M \cap U$, there exist a $\mathcal{C}^0$-control $u(t)$ and a $\mathcal{C}^1$-solution $\gamma_{x^0} : I_{x^0} \to M \cap U$ such that $\gamma_{x^0}(t) \in M \cap U$ for all $t \in I_{x^0}$. A locally controlled invariant submanifold M^* is called maximal, if there exists a neighborhood U of x_a such that for any other locally controlled invariant submanifold, we have $M \cap U \subseteq M^* \cap U$.

Remark 5.3.2. Recall that solutions $\gamma(t)$ of Ξ^u are not unique, even for a fixed initial point x^0 and a fixed control $u(t)$. Thus it is possible that a solution passing through $x^0 \in M \cap U$ stays in $M \cap U$ for $t \in I_{x^0}$, however, other solutions may escape from $M \cap U$ even for the same $u(t)$.

Consider a DAECS $\Xi^u_{l,n,m} = (E, F, G)$. Let M be a smooth connected embedded submanifold and fix a point $x^0 \in M$. We introduce the following regularity condition

(Reg) there exists a neighborhood $U \subseteq X$ such that the dimensions of $E(x)T_xM$ and of $E(x)T_xM + \operatorname{Im} G(x)$ are constant for all $x \in M \cap U$.

Proposition 5.3.3. *Consider a DAECS $\Delta^u = (E, F, G)$, fix an admissible point x_a, and a smooth connected embedded submanifold M containing x_a. Then if M satisfies the regularity condition **(Reg)** and $F(x) \in E(x)T_xM + \operatorname{Im} G(x)$ locally for all $x \in M$ around x_a, then M is a locally controlled invariant submanifold. On the other hand, if M is a locally controlled invariant submanifold, then $F(x) \in E(x)T_xM + \operatorname{Im} G(x)$ locally for all $x \in M$ around x_a.*

The above statement is a generalization of Proposition 4.3.2 of Chapter 4 for nonlinear DAEs, and was stated as Theorem 9 in [12] for DAECSs. We omit the proof of this statement because it follows exactly the same line as that of Proposition 4.3.2 of Chapter 4. Then, we introduce the concept of *restriction* of a DAECS to a controlled invariant submanifold as follows.

Definition 5.3.4. (Restriction) Consider a DAECS $\Xi^u_{l,n,m} = (E, F, G)$ and a controlled invariant submanifold M, of dimension n_1, satisfying the regularity condition **(Reg)** in a neighborhood U of x_a. Let $\psi(x) = z = (z_1, z_2)$ be local coordinates on U such that

$$M \cap U = \{z_2 = 0\} = \left\{z_2^1 = \cdots = z_2^{n_2} = 0\right\},$$

where $n_1 + n_2 = n$. Thus $z_1 = (z_1^1, \ldots, z_1^{n_1})$ form coordinates on $M \cap U$. Then, in U, the restriction of Ξ^u to $M \cap U$, called local M-restriction of Ξ^u, denoted $\Xi^u|_M$ is the following DAECS

$$\tilde{E}(z_1, 0)\begin{bmatrix}\dot{z}_1\\0\end{bmatrix} = \tilde{F}(z_1, 0) + \tilde{G}_1(z_1, 0)u_1,$$

where $G_1 : M \cap U \to \mathbb{R}^{l\times m_1}$ is such that

$$\operatorname{Im} G_1(\psi^{-1}(z)) = E(\psi^{-1}(z))T_zM \cap \operatorname{Im} G(\psi^{-1}(z)),$$

and where

$$\tilde{E}(z) = E(\psi^{-1}(z))\left(\frac{\partial\psi}{\partial x}(\psi^{-1}(z))\right)^{-1}, \ \tilde{F}(z) = F((\psi^{-1}(z))), \ \tilde{G}_1(z) = G_1((\psi^{-1}(z))),$$

with $u_1 \in \mathbb{R}^{m_1}$ and $m_1 = \dim(E(x)T_xM \cap \operatorname{Im} G(x))$.

Remark 5.3.5. (i) M satisfies the regularity condition **(Reg)** in a neighborhood U of x_a implies that the dimension of $E(x)T_xM \cap \operatorname{Im} G(x)$ is constant for $x \in M \cap U$.

(ii) Notice that G_1 is not unique. In fact, it is given up to a u-feedback transformation $u_1 = \beta(x)\tilde{u}_1$ that maps G_1 into $G_1\beta$, where β is invertible.

Moreover, we introduce the notion of reduction of DAECSs, which is an extension of reductions of nonlinear DAEs shown in Definition 4.3.8 of Chapter 4.

Definition 5.3.6 (Reduction)**.** For a DAECS $\Xi^u_{l,n,m} = (E, F, G)$, assume

$$\operatorname{rank}[E(x), dF(x), G(x)] = const. = l^* \le l.$$

Then there exists $Q : X \to Gl(l, \mathbb{R}^n)$ such that

$$Q\begin{bmatrix}E & dF & G\end{bmatrix} = \begin{bmatrix}Q_1\\Q_2\end{bmatrix}\begin{bmatrix}E & dF & G\end{bmatrix} = \begin{bmatrix}Q_1E & Q_1dF & Q_1G\\0 & 0 & 0\end{bmatrix},$$

where $\operatorname{rank}[Q_1E(x), Q_1dF(x), Q_1G(x)] = l^*$, $Q_1 : X \to \mathbb{R}^{l^*\times l}$, $Q_2 : X \to \mathbb{R}^{(l-l^*)\times l}$, and the full row rank reduction, shortly reduction, of Ξ^u, is a DAECS $\Xi^{u,red} = (E^{red}, F^{red}, G^{red})$, where $E^{red}(x) = Q_1(x)E(x)$, $F^{red}(x) = Q_1(x)F(x)$ and $G^{red}(x) = Q_1(x)G(x)$.

For a locally invariant submanifold M, we consider the M-restriction $\Xi^u|_M$ of Ξ^u, and then we construct a reduction of $\Xi^u|_M$ and denote it by $\Xi^u|_M^{red}$. Notice that the order matters: to construct $\Xi^u|_M^{red}$, we first restrict and then reduce while reducing first and then restricting will, in general, not give $\Xi^u|_M^{red}$ but another DAECS $\Xi^{u,red}|_M$.

Proposition 5.3.7. *Consider a DAECS $\Xi^u_{l,n,m} = (E, F, G)$ and fix an admissible point x_a. Let M be a n_1-dimensional locally controlled invariant submanifold satisfying the regularity condition **(Reg)** around x_a. Denote $\dim E(x)T_xM = r$ and $\dim(E(x)T_xM + \operatorname{Im} G(x)) = r + m_2$. Then $\Xi^u|_M^{red}$ is a DAECS $\hat{\Xi}^{u_1}$ of form (5.1) and the dimensions related to $\Xi^u|_M^{red}$ are r, n_1, m_1, where $m_1 = m - m_2$, i.e., $\Xi^u|_M^{red} = \hat{\Xi}^{u_1}_{r,n_1,m_1}$. Moreover,* **Expl**$(\Xi^u|_M^{red})$ *is not empty and consists of ODECSs without outputs.*

The proof is given in Section 5.6. The definition below is based on just introduced concepts.

Definition 5.3.8. (Internal feedback equivalence) Consider two DAECSs $\Xi^u = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on X and $\tilde{X}$, respectively. Fix two admissible points $x_a \in X$ and $\tilde{x}_a \in \tilde{X}$. Assume that

(A1) M^* and $\tilde{M}^*$ are locally maximal controlled invariant submanifolds of Ξ^u and $\tilde{\Xi}^{\tilde{u}}$, respectively, such that $x_a \in M^*$, $\tilde{x}_a \in \tilde{M}^*$.

(A2) M^* and $\tilde{M}^*$ satisfy the regularity condition **(Reg)** around x_a and $\tilde{x}_a$, respectively.

Then, Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ are called internally feedback equivalent, shortly in-fb-equivalent, if $\Xi^u|^{red}_{M^*}$ and $\tilde{\Xi}^{\tilde{u}}|^{red}_{\tilde{M}^*}$ are ex-fb-equivalent. We will denote the in-fb-equivalence of two DAECSs by $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$.

Remark 5.3.9. In the above definition, the dimensions of two in-fb-equivalent DAECSs Ξ^u and $\tilde{\Xi}^{\tilde{u}}$ are not necessarily the same. However, since $\Xi^u|^{red}_{M^*}$ and $\tilde{\Xi}^{\tilde{u}}|^{red}_{\tilde{M}^*}$ are required to be external feedback equivalent, their dimensions have to be the same.

Theorem 5.3.10. (Maximal controlled invariant submanifold form **MCISF**) *Consider a DAE control system* $\Xi^u_{l,n,m} = (E, F, G)$ *and fix a point* x^0. *Assume that* $F(x^0) \in \operatorname{Im} E(x^0) + \operatorname{Im} G(x^0)$. *Set*

$$M_0 = \{x \in X : F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x)\}.$$

Assume that $M_{k-1} \subsetneq \cdots \subsetneq M_0$*, for a certain* $k \geq 1$*, have been constructed and that for some neighborhood* U_{k-1} *of* x^0 *the intersection* $M_{k-1} \cap U_{k-1}$ *is a smooth embedded submanifold, and denote by* M^c_{k-1} *the connected component of* $M_{k-1} \cap U_{k-1}$ *satisfying* $x^0 \in M^c_{k-1}$. *Set*

$$M_k = \left\{x \in M^c_{k-1} : F(x) \in E(x) T_x M^c_{k-1} + \operatorname{Im} G(x)\right\}. \tag{5.14}$$

Then there exists a smallest integer k*, denoted by* $k^* < n$ *such that* $M_{k^*+1} = M^c_{k^*}$ *and assume that* M^* *satisfies the regularity condition* ***(Reg)*** *around* x^0*, where* $M^* = M^c_{k^*}$*, then* x^0 *is an admissible point and* $M^* = M^c_{k^*}$ *is a locally maximal controlled invariant submanifold. Moreover, if additionally, for any* $x \in U_{k^*}$,

(A1) $\operatorname{rank} E(x) = const. = r$ *and* $\operatorname{rank} \begin{bmatrix} E(x) & G(x) \end{bmatrix} = const. = r + m_2$,

then there exists a neighborhood U *of* x^0 *such that* Ξ^u *is locally ex-fb-equivalent to a DAECS represented in the following maximal controlled invariant submanifold form*

$$\textbf{\textit{MCISF}}: \begin{bmatrix} I_{r_1} & E_1^2(z) & 0 & E_1^4(z) \\ 0 & E_2^2(z) & I_{r_2} & E_2^4(z) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} F_1(z) \\ F_2(z) \\ 0 \\ F_4(z) \end{bmatrix} + \begin{bmatrix} G_1(z) & 0 \\ G_2(z) & 0 \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{5.15}$$

where (z_1, z_2) are local coordinates on M^*, and E_1^2, E_1^4, E_2^2, E_2^4 are smooth matrix-valued functions defined on U with values in $\mathbb{R}^{r_1\times(n_1-r_1)}$, $R^{r_1\times(n_2-r_2)}$, $\mathbb{R}^{r_2\times(n_1-r_1)}$, $\mathbb{R}^{r_2\times(n_2-r_2)}$, respectively, and $E_2^2(z) = 0$ and $F_4(z) = 0$ for $z \in M^*$, where $r_1 = \dim E(x)T_xM^*$, $r_2 = r - r_1$, $n_1 = \dim M^*$, $n_2 = n - n_1$.

Furthermore, if the above (A1) holds and additionally there exists an involutive distribution $\mathcal{D}$ on U_{k^*} satisfying $\mathcal{D}(x) = T_xM^*$ for $x \in M^*$ such that for any $x \in U_{k^*}$,

(A2) $\dim E(x)\mathcal{D}(x) = const. = r_1$ *and* $\dim\left(E(x)\mathcal{D}(x) + \operatorname{Im} G(x)\right) = const. = r_1 + m_2$,

then there exist a neighborhood U of x^0 such that Ξ^u is locally ex-fb-equivalent to a DAECS represented in the following special maximal controlled invariant submanifold form

$$\textbf{SMCISF}: \begin{bmatrix} I_{r_1} & E_1^2(z) & 0 & E_1^4(z) \\ 0 & 0 & I_{r_2} & E_2^4(z) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} F_1(z) \\ F_2(z) \\ 0 \\ F_4(z) \end{bmatrix} + \begin{bmatrix} G_1(z) & 0 \\ 0 & 0 \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{5.16}$$

where $F_2(z) = 0$ and $F_4(z) = 0$ for $z \in M^*$.

The proof is given in Section 5.6.

Remark 5.3.11. (i) If M^* exists and only the constant rank assumption (A1) holds, then Ξ^u is locally ex-fb-equivalent to the **MCISF** given by (5.15). If Ξ^u satisfies the involutivity and constant dimension condition (A2), then it is locally ex-fb-equivalent the **SMCISF**, given by (5.16). Compared to (5.15), the matrices $E_2^2(z) \equiv 0$ and $G_2(z) \equiv 0$ on U, and $F_2(z) = 0$ for $z \in M^* \cap U$ in (5.16).

(ii) In the above **SMCISF**, $M^* \cap U = \{z : z_3 = 0, z_4 = 0\}$ and $F_3(z) = F_3^1(z)z_3 + F_3^2(z)z_4$, $F_4(z) = F_4^1(z)z_3 + F_4^2(z)z_4$, where F_3^1, F_3^2, F_4^1, F_4^2 are matrix-valued functions of appropriate sizes.

(iii) The above are two external equivalence normal forms for Ξ^u that are constructed under assumption (A1), for the first one, or (A1)-(A2) for the second one. The word external means that we consider the DAECS Ξ^u locally everywhere around x^0, not just on its maximal controlled invariant manifold M^*. For the points around x^0 but out of M^*, the system does not have solutions, nevertheless, the system admits the above normal forms.

(iv) The above two normal forms facilitate understanding the actual role of the variables in a DAECS Ξ^u. It is easy to see that some "generalized" state variables, namely (z_1, z_3) behave like state variables of differential equations and some "generalized" state variables, namely (z_2, z_4), are free and perform like inputs. Moreover, some control variables, e.g. u_2, are constrained and not free to be chosen (u_2 is forced to be 0 by the algebraic constraints).

(v) The above normal forms are also convenient for the internal analysis of DAECSs. For instance, the result of Proposition 5.3.7, can be easily seen from the **SMCISF** by setting z_3 and z_4 to zero. Moreover, we can use the **SMCISF** to analyze the existence and uniqueness of solutions. A solution $z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))$ of the **SMCISF** should satisfy $z_3(t) = 0$ and $z_4(t) = 0$. Using the definition of internal feedback equivalence of Definition 5.3.8, we have the **SMCISF** is in-fb-eq to

$$\begin{bmatrix} I_{r_1} & E_1^2(z_1, z_2) \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = F_1(z_1, z_2) + G_1(z_1, z_2)u_1.$$

It is seen that for a fixed $u(t)$, Ξ^u has a unique solution if and only if $m_1 = r_1$ (since in this case, the z_2-variables are absent).

5.4 Feedback linearizations of nonlinear DAECSs

In this subsection, we will discuss the problem of when a nonlinear DAECS of form (5.1) is external or internal feedback equivalent to a linear DAECS of form (5.3). First, we review some definitions and criteria of the controllability of linear DAECSs. The augmented Wong sequences (see [17] and Chapter 3) of a linear DAECS $\Delta^u_{l,n,m} = (E, H, L)$, given by (5.3), are

$$\mathscr{V}_0 := \mathbb{R}^n, \quad \mathscr{V}_{i+1} := H^{-1}(E\mathscr{V}_i + \operatorname{Im} L); \tag{5.17}$$

$$\mathscr{W}_0 := 0, \quad \mathscr{W}_{i+1} := E^{-1}(H\mathscr{W}_i + \operatorname{Im} L). \tag{5.18}$$

Additionally, recall the following sequence of subspace (see e.g. [128]):

$$\hat{\mathscr{W}}_1 := \ker E, \quad \hat{\mathscr{W}}_{i+1} := E^{-1}(H\hat{\mathscr{W}}_i + \operatorname{Im} L). \tag{5.19}$$

Now for simplicity of notation, we denote

$$K_k = \begin{bmatrix} 0 & I_{k-1} \end{bmatrix} \in \mathbb{R}^{(k-1)\times k}, \qquad L_k = \begin{bmatrix} I_{k-1} & 0 \end{bmatrix} \in \mathbb{R}^{(k-1)\times k},$$
$$N_k = \begin{bmatrix} 0 & 0 \\ I_{k-1} & 0 \end{bmatrix} \in \mathbb{R}^{k\times k}, \qquad N_\beta = \operatorname{diag}\{N_{\beta_1}, \ldots, N_{\beta_k}\} \in \mathbb{R}^{|\beta|\times|\beta|},$$
$$K_\beta = \operatorname{diag}\{K_{\beta_1}, \ldots, K_{\beta_k}\} \in \mathbb{R}^{(|\beta|-k)\times|\beta|}, \quad L_\beta = \operatorname{diag}\{L_{\beta_1}, \ldots, L_{\beta_k}\} \in \mathbb{R}^{(|\beta|-k)\times|\beta|},$$
$$\mathcal{E}_\beta = \operatorname{diag}\{e_{\beta_1}, \ldots, e_{\beta_k}\} \in \mathbb{R}^{|\beta|\times k} \qquad e_{\beta_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\beta_i\times 1},$$

where β is a multi-index $\beta = (\beta_1, \ldots, \beta_k)$, and where $|\beta| = \sum\limits_{i=1}^{k} \beta_i$. Definition 5.2.2 applied to linear systems says that two linear SE DAEs $\Delta^u_{l,n,m} = (E, H, L)$ and $\tilde{\Delta}^u_{l,n,m} = (\tilde{E}, \tilde{H}, \tilde{L})$ are ex-fb-equivalent if there exists constant invertible matrices Q, P, S and a matrix R such that $\tilde{E} = QEP^{-1}$, $\tilde{H} = Q(H + LR)P^{-1}$, $\tilde{L} = QLS$.

Definition 5.4.1. (Complete controllability in [17]) A linear DAECS $\Delta^u_{l,n,m} = (E, H, L)$ is completely controllable if for any $x^0, x^f \in \mathbb{R}^n$, there exist a solution (x, u) of Δ^u and $t \in \mathbb{R}^+$ such that $x(0) = x^0$ and $x(t) = x^f$.

Lemma 5.4.2. [17] *For a linear DAECS* $\Delta^u_{l,n,m} = (E, H, L)$*, the following are equivalent:*

(i) Δ^u *is completely controllable.*

(ii) $\operatorname{Im} E + \operatorname{Im} H + \operatorname{Im} L = \operatorname{Im} E + \operatorname{Im} L$ *and* $\operatorname{Im}_{\mathbb{C}} E + \operatorname{Im}_{\mathbb{C}} H + \operatorname{Im}_{\mathbb{C}} L = \operatorname{Im}_{\mathbb{C}}(\lambda E - H) + \operatorname{Im}_{\mathbb{C}} L,\ \forall\, \lambda \in \mathbb{C}$.

(iii) $\mathscr{V}^* \cap \mathscr{W}^* = \mathbb{R}^n$*, where* $\mathscr{V}^*$ *and* $\mathscr{W}^*$ *are the limits of the augmented Wong sequences (5.17) and (5.18), respectively;*

(iv) Δ^u *is ex-fb-equivalent (under linear transformations) to*

$$\begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar\rho} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} N^T_\rho & 0 \\ 0 & K_{\bar\rho} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho & 0 \\ 0 & 0 \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $\rho = (\rho_1, \dots, \rho_k)$ *and* $\bar\rho = (\bar\rho_1, \dots, \bar\rho_k)$ *are multi-indices.*

In view of the two feedback equivalence relations for DAECSs (external feedback equivalence of Definition 5.2.2 and internal feedback equivalence of Definition 5.3.8), we give the following definition for the feedback linearization problems of DAECSs.

Definition 5.4.3. (Feedback linearization of DAECSs) For a DAECS $\Xi^u_{l,n,m} = (E, F, G)$,

(i) assume that M^* is a locally maximal controlled invariant submanifold of Ξ^u. Then Ξ^u is called locally completely internal feedback linearizable, if Ξ^u is locally in-fb-equivalent to a linear DAECS with complete controllability;

(ii) Ξ^u is called locally completely external feedback linearizable, if Ξ^u is locally ex-fb-equivalent to a linear DAECS with complete controllability.

Now consider a nonlinear ODECS $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$, given by (5.9). If ODECS Σ^{uv} has no outputs, we denote it by $\Sigma^{uv}_{n,m,s} = (f, g^u, g^v)$. Then for $\Sigma^{uv}_{n,m,s} = (f, g^u, g^v)$, define the following two sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$, called the linearizability distributions of Σ^{uv},

$$\begin{cases} \mathcal{D}_0 := \{0\}, \\ \mathcal{D}_1 := \operatorname{span}\{g^u_1, \dots, g^u_m, g^v_1, \dots, g^v_s\} \\ \mathcal{D}_{i+1} := \mathcal{D}_i + [f, \mathcal{D}_i], \quad i = 1, 2, \dots, \end{cases} \qquad \begin{cases} \hat{\mathcal{D}}_1 := \operatorname{span}\{g^v_1, \dots, g^v_s\} \\ \hat{\mathcal{D}}_{i+1} := \mathcal{D}_i + [f, \hat{\mathcal{D}}_i], \quad i = 1, 2, \dots. \end{cases}$$

Remark 5.4.4. (i) The distribution sequences $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ satisfy:

$$\mathcal{D}_0 \subsetneq \hat{\mathcal{D}}_1 \subsetneq \mathcal{D}_1 \subsetneq \hat{\mathcal{D}}_2 \subsetneq \mathcal{D}_2 \cdots \subsetneq \hat{\mathcal{D}}_k \subsetneq \mathcal{D}_k \subsetneq \cdots \subsetneq \hat{\mathcal{D}}_{k^*},$$

and either

$$\hat{\mathcal{D}}_{k^*} = \mathcal{D}_{k^*} = \hat{\mathcal{D}}_{k^*+j} = \mathcal{D}_{k^*+j}$$

or

$$\hat{\mathcal{D}}_{k^*} \subsetneq \mathcal{D}_{k^*} = \hat{\mathcal{D}}_{k^*+j} = \mathcal{D}_{k^*+j},$$

where $j \geq 1$ and k^* is the smallest k such that $\mathcal{D}_{k^*} = \mathcal{D}_{k^*+1}$. Note that k^* is not necessarily the smallest k such that $\hat{\mathcal{D}}_{k^*} = \hat{\mathcal{D}}_{k^*+1}$ (as seen in the second case, where $\hat{\mathcal{D}}_{k^*} \subsetneq \hat{\mathcal{D}}_{k^*+1}$). However, $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ always have the same limit.

(ii) For a linear DAECS $\Delta^u = (E, H, L)$, denote $\mathscr{W}_i(\Delta^u)$ and $\hat{\mathscr{W}}_i(\Delta^u)$ as the subspace $\mathscr{W}_i$ and $\hat{\mathscr{W}}_i$ of Δ^u, respectively. For a linear ODECS $\Lambda^{uv} = (A, B^u, B^v, C, D^u)$ (of form (5.9) but with constant system matrices), denote $\mathcal{W}_i(\Lambda^{uv})$ and $\hat{\mathcal{W}}_i(\Lambda^{uv})$ as the subspace $\mathcal{W}_i$ and $\hat{\mathcal{W}}_i$ of Λ^{uv}, respectively, where

$$\mathcal{W}_0 = \{0\}, \ \ \mathcal{W}_{i+1} = \begin{bmatrix} A & B^w \end{bmatrix} \left(\begin{bmatrix} \mathcal{W}_i \\ \mathscr{I} \end{bmatrix} \cap \ker \begin{bmatrix} C & D^w \end{bmatrix} \right),$$

$$\hat{\mathcal{W}}_1 = \operatorname{Im} B^v, \ \ \hat{\mathcal{W}}_{i+1} = \begin{bmatrix} A & B^w \end{bmatrix} \left(\begin{bmatrix} \hat{\mathcal{W}}_i \\ \mathscr{I} \end{bmatrix} \cap \ker \begin{bmatrix} C & D^w \end{bmatrix} \right),$$

where $B^w = [B^u, B^v]$ and $D^w = [D^u, 0]$. We have proved in Proposition 3.2.9 of Chapter 3 that if $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$, then for $i \in \mathbb{N}$,

$$\mathscr{W}_i(\Delta^u) = \mathcal{W}_i(\Lambda^{uv}), \quad \hat{\mathscr{W}}_i(\Delta^u) = \hat{\mathcal{W}}_i(\Lambda^{uv}).$$

Apparently, $\mathcal{W}_i$ and $\hat{\mathcal{W}}_i$ are the linear counterparts of $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$, respectively, but they are for linear systems with outputs.

Theorem 5.4.5. *Consider a DAECS $\Xi^u = \Xi^u_{l,n,m} = (E, F, G)$, fix an admissible point x_a. Let M^* be the n^*-dimensional maximal controlled invariant submanifold of Ξ^u around x_a. Assume that there exists a neighborhood $U \subseteq X$ of x_a such that in $M^* \cap U$, we have*

(A1) the dimensions of $E(x)T_xM^*$ and $E(x)T_xM^* + \operatorname{Im} G(x)$ are constant,

(A2) the rank of $G(x)$ is m.

Then $\mathbf{Expl}(\Xi^u|^{red}_{M^*})$ is not empty and Ξ^u is locally completely internal feedback linearizable if and only if for one (and thus any) ODECS $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u|^{red}_{M^*})$, the linerizability distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ of Σ^{uv} satisfy in a neighborhood $W \subseteq M^*$ of x_a:

(FL1) $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ are of constant rank for $1 \leq i \leq n^*$.

(FL2) $\mathcal{D}_{n^*} = \hat{\mathcal{D}}_{n^*} = TM^*$.

(FL3) $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ are involutive for $1 \leq i \leq n^* - 1$.

Theorem 5.4.6. *Consider a DAECS $\Xi^u_{l,n,m} = (E, F, G)$, fix a point x^0. Then Ξ^u is locally completely external feedback linearizable, locally around x^0, if and only if there exists a neighborhood $U \subseteq X$ of x^0 in which the following conditions are satisfied.*

(EFL1) $\operatorname{rank} E(x)$ *and* $\operatorname{rank} [E(x), G(x)]$ *are constant.*

(EFL2) $F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x)$.

(EFL3) *For one (and thus any) control system $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u|^{red}_{M^*})$, which is a system with no outputs on $M^* = U$, a neighborhood of x^0, the linerizability distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ satisfy conditions (FL1)-(FL3) of Theorem 5.4.5.*

The proofs of Theorem 5.4.5 and Theorem 5.4.6 are given in Section 5.6.

Remark 5.4.7. (i) By $\operatorname{rank} E(x) = const.$, the explicitation class $\mathbf{Expl}(\Xi^u|^{red}_{M^*})$ is well-defined. Moreover, by conditions (EFL1)-(EFL2), for any $x^0 \in X$, the locally maximal controlled invariant submanifold M^* through x^0 is a neighborhood U of x^0. So condition (EFL3) is actually, satisfied if and only if the condition (FL1)-(FL3) are satisfied on $M^* = U$, i.e., locally around x^0.

(ii) We do not assume the point x^0 in Theorem 5.4.6 to be admissible. However, by conditions (EFL1)-(EFL2), any point x^0 is always admissible.

(iii) Note that condition (EFL2) and the condition $\hat{\mathcal{D}}_{n^*} = \mathcal{D}_{n^*} = TM^*$ of (FL2) are nonlinear counterparts of the condition $\mathscr{V}^* \cap \mathscr{W}^* = \mathbb{R}^n$ of Lemma 5.4.2. However, in order to guarantee feedback linearizability, involutivity and some constant rank conditions are needed.

(iv) The distributions sequences $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ can thus be seen as nonlinear generalizations of the augmented Wong sequence $\mathscr{W}_i$ of equation (5.18) and the sequence $\hat{\mathscr{W}}_i$ of (5.19), respectively.

5.5 Examples

In the section, we will illustrate the results of the present chapter by some examples.

Example 5.5.1. (Model of a 2-D crane) Consider the model of a 2-D crane taken from [68], which is described by a DAECS of the following form:

$$\begin{cases} m\ddot{x} = -T\sin\theta \\ m\ddot{z} = -T\cos\theta + mg \\ \ \ x \ = R\sin\theta + D \\ \ \ z \ = R\cos\theta, \end{cases} \tag{5.20}$$

where (x, z) is the position of a load m, and T is the tension of the rope, and together with θ, they are variables of the "generalized" state, which is thus $(x, \dot{x}, z, \dot{z}, \theta, T)$. The predefined control variables are D and R, which represent the position of the trolley and the length of the rope, respectively.

Rewrite system (5.20) in the form of DAECS (5.1) to get the following system:

$$\Xi^u : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{\theta} \\ \dot{T} \end{bmatrix} = \begin{bmatrix} x_2 \\ -T\sin\theta \\ z_2 \\ -T\cos\theta + mg \\ -x_1 \\ -z_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sin\theta & 1 \\ \cos\theta & 0 \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}, \tag{5.21}$$

where $x_1 = x$, $x_2 = \dot{x}$ and $z_1 = z$, $z_2 = \dot{z}$. The above DAECS is denoted by Ξ^u. We consider Ξ^u around the following admissible point (it is also an equilibrium):

$$x_{1a} = 0, \;\; x_{2a} = 0, \;\; z_{1a} = 1, \;\; z_{2a} = 0, \;\; \theta_a = 0, \;\; T_a = mg.$$

It is easy to verify that conditions (EFL1) and (EFL2) of Theorem 5.4.6 are satisfied for Ξ^u in a neighborhood U ($\cos\theta \neq 0$ for all points in U). Then by using the following feedback transformation for Ξ^u:

$$\begin{bmatrix} R \\ D \end{bmatrix} = \begin{bmatrix} 0 & 1/\cos\theta \\ 1 & -\sin\theta/\cos\theta \end{bmatrix} \tilde{u} + \begin{bmatrix} z_1/\cos\theta \\ x_1 - (z_1\sin\theta)/\cos\theta \end{bmatrix}, \tag{5.22}$$

the algebraic constraint of Ξ^u becomes $0 = \tilde{u}$.

Now by Definition 5.3.4, a reduction of $M^* = U$-restriction of Ξ^u is

$$\Xi^u|_{M^*}^{red} : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{\theta} \\ \dot{T} \end{bmatrix} = \begin{bmatrix} x_2 \\ -T\sin\theta \\ z_2 \\ -T\cos\theta + mg \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tilde{u}.$$

By the explicitation procedure described in Section 5.2, we can find an ODECS $\Sigma^{uv} = (f, g^u, g^v) \in \mathbf{Expl}(\Xi^u|_{M^*}^{red})$

$$\Sigma^{uv} : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{\theta} \\ \dot{T} \end{bmatrix} = \begin{bmatrix} x_2 \\ -(T\sin\theta)/m \\ z_2 \\ -(T\cos\theta)/m + g \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tilde{u} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v,$$

where v is a vector of the driving variables (notice that in the present example, $v = [\dot{\theta}, \dot{T}]^T$ is also the prolongation of (θ, T)).

Now calculating the distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ for the system Σ^{uv}, we get

$$\begin{aligned} &\mathcal{D}_1 = \hat{\mathcal{D}}_1 = \operatorname{span}\{g_1^v, g_2^v\}, \;\; \mathcal{D}_2 = \hat{\mathcal{D}}_2 = \operatorname{span}\{g_1^v, g_2^v, ad_f g_1^v, ad_f g_2^v\}, \\ &\mathcal{D}_3 = \hat{\mathcal{D}}_3 = \operatorname{span}\left\{g_1^v, g_2^v, ad_f g_1^v, ad_f g_2^v, ad_f^2 g_1^v, ad_f^2 g_2^v\right\}, \end{aligned} .$$

where

$$\begin{array}{l} g_1^v = \frac{\partial}{\partial \theta}, \quad g_2^v = \frac{\partial}{\partial T}, \quad ad_f g_1^v = \frac{\sin\theta}{m}\frac{\partial}{\partial x_2} + \frac{\cos\theta}{m}, \\ ad_f g_2^v = \frac{T\cos\theta}{m}\frac{\partial}{\partial x_2} - \frac{T\sin\theta}{m}\frac{\partial}{\partial z_2}, \quad ad_f^2 g_1^v = \frac{-\sin\theta}{m}\frac{\partial}{\partial x_1} + \frac{-\cos\theta}{m}\frac{\partial}{\partial z_1}, \\ ad_f^2 g_2^v = \frac{-T\cos\theta}{m}\frac{\partial}{\partial x_1} + \frac{T\sin\theta}{m}\frac{\partial}{\partial z_1}. \end{array}$$

It is seen that $\hat{\mathcal{D}}_1 = \mathcal{D}_1 \subset \hat{\mathcal{D}}_2 = \mathcal{D}_2 \subset \hat{\mathcal{D}}_3 = \mathcal{D}_3 = TU$. Thus $\hat{\mathcal{D}}_i$ and $\mathcal{D}_i$ satisfy conditions (FL1) and (FL2) of Theorem 5.4.5. Moreover, a direct calculation of Lie brackets gives that $\hat{\mathcal{D}}_1 = \mathcal{D}_1$, $\hat{\mathcal{D}}_2 = \mathcal{D}_2$, $\hat{\mathcal{D}}_3 = \mathcal{D}_3$ are all involutive, which implies that condition (FL3) of Theorem 5.4.5 is also satisfied. Therefore, (EFL3) is satisfied and Ξ^u is completely external feedback linearizable by Theorem 5.4.6.

In fact, use the following coordinates change and feedback transformation for Σ^{uv}:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -(T\sin\theta)/m \\ z_1 - z_{1a} \\ z_2 \\ -(T\cos\theta)/m + g \end{bmatrix}, \quad \begin{bmatrix} \tilde{u} \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & -(m\cos\theta)/T & (m\sin\theta)/T \\ 0 & -m\sin\theta & -m\cos\theta \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix},$$

where $z_{1a} = 1$, to get the following linear ODECS:

$$\Lambda^{\tilde{v}}: \ \dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = \xi_3, \quad \dot{\xi}_3 = \tilde{v}_1, \quad \dot{\xi}_4 = \xi_5, \quad \dot{\xi}_5 = \xi_6, \quad \dot{\xi}_6 = \tilde{v}_2.$$

Thus we have $\Sigma^{uv} \overset{sys-fb}{\sim} \Lambda^{\tilde{v}}$. Moreover, the following linear DAECS $\Delta^{\tilde{u}}$, given by (5.23), satisfies $\Lambda^{\tilde{v}} \in \mathbf{Expl}(\Delta^{\tilde{u}})$.

$$\Delta^{\tilde{u}}: \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \\ \dot{\xi}_5 \\ \dot{\xi}_6 \end{bmatrix} = \begin{bmatrix} 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} + \begin{bmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \end{bmatrix} \tilde{u}. \tag{5.23}$$

Thus by Theorem 5.2.9, we have $\Xi^u|_{M^*}^{red}$ is ex-fb-equivalent to the linear DAECS $\Delta^{\tilde{u}}$ (since $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u|_{M^*}^{red})$, $\Lambda^{\tilde{v}} \in \mathbf{Expl}(\Delta^{\tilde{u}}$ and $\Sigma^{uv} \overset{sys-fb}{\sim} \Lambda^{\tilde{v}})$. Finally, it is seen that Ξ^u is ex-fb-equivalent to the following completely controllable linear DAECS:

$$\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \\ \dot{\xi}_5 \\ \dot{\xi}_6 \end{bmatrix} = \begin{bmatrix} 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} + \begin{bmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 1&0 \\ 0&1 \end{bmatrix} \tilde{u} \tag{5.24}$$

via the following transformation:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -(T\sin\theta)/m \\ z_1 - z_{1a} \\ z_2 \\ -(T\cos\theta)/m + g \end{bmatrix},$$

$$\begin{bmatrix} R \\ D \end{bmatrix} = \begin{bmatrix} 0 & 1/\cos\theta \\ 1 & -\sin\theta/\cos\theta \end{bmatrix} \tilde{u} + \begin{bmatrix} z_1/\cos\theta \\ x_1 - (z_1\sin\theta)/\cos\theta \end{bmatrix}.$$

Remark 5.5.2. (i) In [68], it is shown that the above model of crane (with state space representation) is flat with outputs $y_1 = x$, $y_2 = z$. It raises interest for further studies about the connections between linearizability of DAECSs and flatness of systems in the state space representation.

(ii) The above model of crane appears also in [72], whose authors construct nonlinear control law based on the linearization of the system after eliminating the variables θ and T. From the view point of the present chapter, this eliminating procedure actually means constructing the internal system (restricting the original system to M^*). Thus the present chapter offers another interpretation for the results in [72].

(iii) Since the system is linearizable, we can easily design control laws for such problems as tracking or stabilization. Since ξ_3 and ξ_6 are free variables in the dynamics of system (5.24), we can regard them as some artificial controls. Consider the stabilization problem for example and take $\xi_3 = k_1\xi_1 + k_2\xi_2$, $\xi_6 = k_3\xi_4 + k_4\xi_5$ such that all the poles of the dynamics $\ddot{\xi}_1 = \xi_3, \ddot{\xi}_4 = \xi_6$ are in the left half real plane. Then we solve θ as a function of x_1, x_2, z_1, z_2 via $\xi_3 = -(T\sin\theta)/m$ and $\xi_6 = -(T\cos\theta)/m + g$. Moreover, by (5.22), the original controls R and D are functions of x_1, z_1, θ (since $\tilde{u}$ is zero). So we can always express stabilizing feedback controls R and D as functions of x_1, x_2, z_1, z_2.

Example 5.5.3. (2-D crane with dynamics of actuators) We consider the model of a 2-D crane described by equation (5.20), together with its actuator dynamics [68]:

$$\begin{cases} M\ddot{D} = \mathscr{F} - \lambda\dot{D} + T\sin\theta \\ (J/\rho^2)\ddot{R} = \mathscr{C} - (\mu/\rho)\dot{R} - T\rho, \end{cases} \tag{5.25}$$

where $(\mathscr{F}, \mathscr{C})$ are the new input variables, representing the external force applied to the trolley and the hoisting torque respectively, where M, J, ρ, λ, μ are constant parameters representing the characteristics of the actuator.

If we write the whole system in the form of DAECS (5.1), we get

$$\Xi^u : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J/(\rho)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{D}_1 \\ \dot{D}_2 \\ \dot{R}_1 \\ \dot{R}_2 \\ \dot{\theta} \\ \dot{T} \end{bmatrix} = \begin{bmatrix} x_2 \\ -T\sin\theta \\ z_2 \\ -T\cos\theta + mg \\ D_2 \\ -\lambda D_1 + T\sin\theta \\ R_2 \\ -(\mu/\rho)R_1 - T\rho \\ -x_1 + R_1\sin\theta + D_1 \\ -z_1 + R_1\cos\theta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathscr{F} \\ \mathscr{C} \end{bmatrix}, \tag{5.26}$$

where $D_1 = D$, $D_2 = \dot{D}$ and $R_1 = R$, $R_2 = \dot{R}$. Thus the control variables R and D of (5.21) become variables of the "generalized" state x of Ξ^u. We consider Ξ^u around an admissible point $x_a = (x_{1a}, x_{2a}, z_{1a}, z_{2a}, D_{1a}, D_{2a}, R_{1a}, R_{2a}, \theta_a, T_a)$, where

$$x_{1a} = 1, \ \ x_{2a} = 0, \ \ z_{1a} = 0, \ \ z_{2a} = 0, \ \ D_{1a} = 0,$$
$$D_{2a} = 0, \ \ R_{1a} = 1, \ \ R_{2a} = 0, \ \ \theta_a = \pi/2, \ \ T_a = 0.$$

The above admissible point represents a configuration that the load is at the same horizontal level as the trolley and notice that this point is not an equilibrium. Then, we can see that DAECS Ξ^u does not satisfy condition (EFL2) of Theorem 5.4.6. Thus Ξ^u is not completely external feedback linearizable around that admissible point.

We now search for the maximal controlled invariant submanifold M^* for Ξ^u and then we will transform Ξ^u into its **MCISF** via Theorem 5.3.10. First, calculate M^* by the algorithm given in Theorem 5.3.10, to get

$$\begin{aligned} M_0 &= \{x \in \mathbb{R}^9 \times S^1 : R_1\sin\theta + D_1 - x_1 = 0, \ \ R_1\cos\theta - z_1 = 0\}, \\ M_1 &= \{x \in M_0 : (D_2 - x_2)\sin\theta + R_2 - z_2\cos\theta = 0\}, \\ M^* &= M_2 = M_1. \end{aligned}$$

We can see that (A1) of Theorem 5.3.10 is satisfied ($E(x)$ and $G(x)$ are constant matrices in the present example). Subsequently, use the procedure given in the proof of Theorem 5.3.10, to transform Ξ^u into its **MCISF** step by step:

Step 1: choose new coordinates

$$\xi_2 = \begin{bmatrix} \tilde{x}_1 \\ \tilde{z}_1 \\ \tilde{R}_2 \end{bmatrix} = \begin{bmatrix} R_1\sin\theta + D_1 - x_1 \\ R_1\cos\theta - z_1 \\ (D_2 - x_2)\sin\theta + R_2 - z_2\cos\theta \end{bmatrix}, \quad \xi_1 = \begin{bmatrix} \tilde{x}_2 \\ \tilde{z}_2 \\ \tilde{d}_1 \\ \tilde{d}_2 \\ \tilde{R}_1 \\ \tilde{\theta} \\ \tilde{T} \end{bmatrix} = \begin{bmatrix} x_2 \\ z_2 \\ D_1 \\ D_2 \\ R_1 \\ \theta \\ T \end{bmatrix}.$$

In the $\xi = (\xi_1, \xi_2)$-coordinate system, (5.26) becomes

$$\begin{bmatrix} E_1(\xi) & E_2(\xi) \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = F(\xi) + G(\xi)u,$$

where

$$\begin{bmatrix} E_1(\xi) & E_2(\xi) \end{bmatrix} = \left[\begin{array}{ccccccc|ccc} 0 & 1 & 0 & \sin\theta & 0 & R_1\cos\theta\sin\theta & 0 & 1 & 0 & 0 \\ -m\cos\theta & 0 & m & 0 & m & m\cdot a(\xi) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & 0 & -R_1(\sin\theta)^2 & 0 & 0 & 1 & 0 \\ m\sin\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & J/(\rho)^2 \\ 0 & 0 & 0 & 0 & (J\sin\theta)/(\rho)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$F(\xi) = \begin{bmatrix} x_2 \\ -T\sin\theta \\ z_2 \\ -T\cos\theta + mg \\ D_2 \\ -\lambda D_1 + T\sin\theta \\ R_2 \\ -(\mu/\rho)R_1 - T\rho \\ -x_1 + R_1\sin\theta + D_1 \\ -z_1 + R_1\cos\theta \end{bmatrix}, \quad G(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

with $a(\xi) = z_2\sin\theta + \cos\theta(D_2 - x_2)$

Step 2: Left-multiply the above DAECS by the following invertible matrix

$$Q_1(\xi) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{m\sin\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{-a(\xi)\cos\theta}{R_1\sin\theta} & 1/m & a(\xi)/R_1 & \frac{\cos\theta}{m\sin\theta} & \frac{a(\xi)\cos\theta}{R_1\sin\theta} & -1/M & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{R_1\sin^2\theta} & 0 & 0 & 0 & \frac{\cos\theta}{R_1\sin^2\theta} & 0 & 0 & 0 \\ -\sin\theta & 0 & -\cos\theta & 0 & \sin\theta & 0 & 1 & 0 & 0 & 0 \\ a(\xi)\frac{J\cos\theta}{R_1\rho^2} & -\frac{J\sin\theta}{m\rho^2} & -a(\xi)\frac{J\sin\theta}{R_1\rho^2} & -\frac{J\cos\theta}{m\rho^2} & -\frac{J\cos\theta}{R_1\rho^2} & \frac{J\sin\theta}{M\rho^2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We get the following equation after the multiplication by $Q_1(\xi)$:

$$Q_1(\xi)\begin{bmatrix} E_1(\xi) & E_2(\xi) \end{bmatrix}\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = Q_1(\xi)F(\xi) + Q_1(\xi)G(\xi)u, \tag{5.27}$$

where

$$Q_1(\xi)\begin{bmatrix} E_1(\xi) & E_2(\xi) \end{bmatrix} = \left[\begin{array}{ccccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & J/\rho^2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{-a(\xi\cos\theta)}{R_1\sin\theta} & a(\xi)/R_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{-1}{R_1\sin^2\theta} & \frac{J\cos\theta}{R_1\rho^2\sin^2\theta} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin\theta & -\cos\theta & J/\rho^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\xi)\frac{J\cos\theta}{R_1\rho^2} & -a(\xi)\frac{J\sin\theta}{R_1\rho^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$Q_1(\xi)F(\xi) = \begin{bmatrix} \frac{mg - T\cos\theta}{m\sin\theta} \\ D_2 \\ -(D_1\lambda - T\sin\theta)/M \\ R_2 \\ F_5(\xi) \\ \frac{R_2\cos\theta - z_2}{R_1\sin^2\theta} \\ \tilde{R}_2 \\ F_8(\xi) \\ \tilde{x}_1 \\ \tilde{z}_1 \end{bmatrix}, \quad Q_1(\xi)G(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/M & 0 \\ 0 & 0 \\ -1/M & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{J\sin\theta}{M\rho^2} & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

for some function $F_5(\xi)$ and $F_8(\xi)$. Thus we have

$$Q_1(\xi)\begin{bmatrix} E_1(\xi) & E_2(\xi) & G(\xi) \end{bmatrix} = \begin{bmatrix} \tilde{E}_1(\xi) & \tilde{E}_2(\xi) & G_1(\xi) \\ 0 & 0 & G_2(\xi) \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$\tilde{E}_1(\xi) = \left[\begin{array}{c|c} I_{r_1} & E_1^2 \\ \hline 0 & E_1^4 \end{array}\right] = \left[\begin{array}{cccccc|c} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ \hline 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \end{array}\right], \quad G_1(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/M & 0 \\ 0 & 0 \\ -1/M & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{J\sin\theta}{M\rho^2} & 1 \end{bmatrix},$$

$$\tilde{E}_2(\xi) = \left[\begin{array}{c|c} E_2^1 & E_2^2 \\ \hline E_2^3 & E_2^4 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J/\rho^2 \\ \frac{-a(\xi\cos\theta)}{R_1\sin\theta} & a(\xi)/R_1 & 0 \\ 0 & \frac{-1}{R_1\sin^2\theta} & \frac{J\cos\theta}{R_1\rho^2\sin^2\theta} \\ \hline -\sin\theta & -\cos\theta & J/\rho^2 \\ a(\xi)\frac{J\cos\theta}{R_1\rho^2} & -a(\xi)\frac{J\sin\theta}{R_1\rho^2} & 0 \end{array}\right].$$

Notice that G_2 vanishes since $\operatorname{Im} G(\xi) \subset \operatorname{Im} E(\xi)$.

Step 3: Use the feedback transformation

$$\begin{bmatrix} \mathscr{F} \\ \mathscr{C} \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ \frac{J\sin\theta}{M\rho^2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ F_8(\xi) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \frac{J\sin\theta}{M\rho^2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and pre-multiply equation (5.27) by the following invertible matrix

$$Q_2(\xi) = \begin{bmatrix} 1&0&0&0&0&0 & 0 & 0 & 0&0 \\ 0&1&0&0&0&0 & 0 & 0 & 0&0 \\ 0&0&1&0&0&0 & 0 & 0 & 0&0 \\ 0&0&0&1&0&0 & 0 & 0 & 0&0 \\ 0&0&0&0&1&0 & 0 & \frac{\rho^2}{J\sin\theta} & 0&0 \\ 0&0&0&0&0&1 & \frac{-\cos\theta}{R_1\sin^2\theta} & \frac{-\rho^2}{a(\xi)J\sin\theta} & 0&0 \\ 0&0&0&0&0&0 & -\sin\theta & \frac{R_1\rho^2\cos\theta}{a(\xi)J} & 0&0 \\ 0&0&0&0&0&0 & -\cos\theta & -\frac{R_1\rho^2\sin\theta}{a(\xi)J} & 0&0 \\ 0&0&0&0&0&0 & 0 & 0 & 1&0 \\ 0&0&0&0&0&0 & 0 & 0 & 0&1 \end{bmatrix},$$

we get the following DAECS, which is in the **MCISF**:

$$\left[\begin{array}{ccccccc|cc c} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&J/\rho^2 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&\frac{-J\sin\theta}{\rho^2} \\ 0&0&0&0&0&0&0&0&1&\frac{-J\cos\theta}{\rho^2} \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right] \begin{bmatrix} \dot{\tilde{x}}_2 \\ \dot{\tilde{z}}_2 \\ \dot{\tilde{D}}_1 \\ \dot{\tilde{D}}_2 \\ \dot{\tilde{R}}_1 \\ \dot{\tilde{\theta}} \\ \dot{\tilde{T}} \\ \hline \dot{\tilde{x}}_1 \\ \dot{\tilde{z}}_1 \\ \dot{\tilde{R}}_2 \end{bmatrix} = \begin{bmatrix} \frac{mg-T\cos\theta}{m\sin\theta} \\ D_2 \\ -(D_1\lambda - T\sin\theta)/M \\ R_2 \\ \frac{R_2\cos\theta - z_2}{R_1\sin^2\theta} \\ F_6(\xi) \\ -\tilde{R}_2\sin\theta \\ -\tilde{R}_2\cos\theta \\ \tilde{x}_1 \\ \tilde{z}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{M} & 0 \\ 0 & 0 \\ \frac{-1}{M} & \frac{\rho^2}{J\sin\theta} \\ 0 & \frac{-\rho^2}{a(\xi)J\sin\theta} \\ 0 & \frac{R_1\rho^2\cos\theta}{a(\xi)J} \\ 0 & \frac{-R_1\rho^2\sin\theta}{a(\xi)J} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad (5.28)$$

for some $F_6 : U \to \mathbb{R}$.

Remark 5.5.4. (i) The admissible point we considered in this example is a singular point in the discussion of flatness of [68] and for control law design of [72]. However, we show in this example that, around this singular point Ξ_a, the system still can be simplified by bringing it to the normal form **MCISF** .

(ii) We do not give precise formula for $F_6(\xi)$ in order to save space. But it is easy to see from the above **MCISF** that Ξ^u is in-fb-equivalent to

$$\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\end{bmatrix}\begin{bmatrix}\dot{\tilde{x}}_2\\\dot{\tilde{z}}_2\\\dot{\tilde{D}}_1\\\dot{\tilde{D}}_2\\\dot{\tilde{R}}_1\\\dot{\tilde{\theta}}\\\dot{\tilde{T}}\end{bmatrix}=\begin{bmatrix}\frac{mg-\tilde{T}\cos\tilde{\theta}}{m\sin\tilde{\theta}}\\\tilde{D}_2\\-(\tilde{D}_1\lambda-\tilde{T}\sin\tilde{\theta})/M\\\tilde{z}_2\cos\tilde{\theta}-(\tilde{D}_2-\tilde{x}_2)\sin\tilde{\theta}\\\frac{(\tilde{x}_2-\tilde{D}_2)\cot\tilde{\theta}-\tilde{z}_2}{\tilde{R}_1}\\F_6(\xi_1,0)\end{bmatrix}+\begin{bmatrix}0\\0\\\frac{1}{M}\\0\\-\frac{1}{M}\\0\end{bmatrix}u_1.$$

It follows that T is a free variable of the "generalized" states (and $\dot{T}$ is a driving variable) and $\mathscr{C}$ is an input constrained by an algebraic constraint.

Example 5.5.5. Consider the following academic example borrowed from [13]:

$$\Xi^u:\begin{bmatrix}x_2&x_1&0\\0&0&0\\1&0&1\end{bmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0\\0\\x_2^2-x_1^3+x_3\end{bmatrix}+\begin{bmatrix}1&-1\\1&1\\0&0\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}. \tag{5.29}$$

Note that in [13], some outputs are considered for the above DAECS $\Xi^u=(E,F,G)$. In the present example, however, we are only interested in the system without outputs. Moreover, we consider an admissible point $x_a=(x_{1a},x_{2a},x_{3a})$, where

$$x_{1a}=1,\ \ x_{2a}=1,\ \ x_{3a}=0.$$

Clearly, there exists a neighborhood U ($x_1\neq 0$ for all $x\in U$) of x_a such that both

$$\operatorname{Im}E(x)=\operatorname{Im}\begin{bmatrix}x_2&x_1&0\\0&0&0\\1&0&1\end{bmatrix}\quad\text{and}\quad\operatorname{Im}E(x)+\operatorname{Im}G(x)=\operatorname{Im}\begin{bmatrix}x_2&x_1&0&1&-1\\0&0&0&1&1\\1&0&1&0&0\end{bmatrix}$$

are of constant dimension. Moreover, for all x around x_a, we have

$$\begin{bmatrix}0\\0\\x_2^2-x_1^3+x_3\end{bmatrix}\in\operatorname{Im}\begin{bmatrix}x_2&x_1&0&1&-1\\0&0&0&1&1\\1&0&1&0&0\end{bmatrix}.$$

Thus (EFL1) and (EFL2) of Theorem 5.4.6 are satisfied. Subsequently, via

$$Q=\begin{bmatrix}1&1&0\\0&0&1\\0&1&0\end{bmatrix},\quad\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}\begin{bmatrix}\tilde{u}_1\\\tilde{u}_2\end{bmatrix},$$

Ξ^u is ex-fb-equivalent to

$$\begin{bmatrix}x_2&x_1&0\\1&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0\\x_2^2-x_1^3+x_3\\0\end{bmatrix}+\begin{bmatrix}2&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}\tilde{u}_1\\\tilde{u}_2\end{bmatrix}.$$

We drop the tilde of the above u_1 for simplicity of notation, then $M^* = U$ and

$$\Xi^u|^{red}_{M^*} : \begin{bmatrix} x_2 & x_1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2^2 - x_1^3 + x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1.$$

Now an ODECS $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u|^{red}_{M^*})$ can be taken as

$$\Sigma^{uv} : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_2^2 - x_1^3 + x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2/x_1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} x_1 \\ -x_2 \\ -x_1 \end{bmatrix} v,$$

where v is a driving variable. Now calculate the distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ for the system Σ^{uv}, which are

$$\hat{\mathcal{D}}_1 = \operatorname{span}\{g^v\}, \ \ \mathcal{D}_1 = \operatorname{span}\{g^u, g^v\}, \ \ \mathcal{D}_2 = \hat{\mathcal{D}}_2 = \operatorname{span}\{g^u, g^v, ad_f g^v\},$$

where

$$g^v = \begin{bmatrix} x_1 \\ -x_2 \\ -x_1 \end{bmatrix}, \quad g^u = \begin{bmatrix} 0 \\ 2/x_1 \\ 0 \end{bmatrix}, \quad ad_f g^v = \begin{bmatrix} 0 \\ 0 \\ 3x_1^3 + 2x_2^2 + x_1 \end{bmatrix}.$$

Clearly, the distributions above are of constant rank and $\mathcal{D}_2 = \hat{\mathcal{D}}_2 = T_x U$ for all $x \in U$. Additionally, $[g^u, g^v] = 0 \in \mathcal{D}_1$, $\hat{\mathcal{D}}_1$ is of rank one and $\hat{\mathcal{D}}_2$ is TU, so the distributions $\hat{\mathcal{D}}_1$, $\mathcal{D}_1$, $\hat{\mathcal{D}}_2$ are all involutive. Thus, condition (EFL3) of Theorem 5.4.6 is satisfied. Therefore, system Ξ^u is completely external feedback linearizable.

In fact, we can choose $\varphi_1(x)$ and $\varphi_2(x)$ such that

$$\operatorname{span}\{d\varphi_1\} = \mathcal{D}_1^{\perp}, \quad \operatorname{span}\{d\varphi_1, d\varphi_2\} = \hat{\mathcal{D}}_1^{\perp}.$$

By solving some first order partial differential equations with the constraint that

$$(\varphi_1(x_a), \varphi_2(x_a)) = (0, 0),$$

we get

$$\varphi_1(x) = x_1 + x_3 - x_{1a}, \quad \varphi_2(x) = x_1 x_2 - x_{1a} x_{2a}.$$

Moreover, setting

$$\varphi_3(x) = L_f \varphi_1(x) = -(x_1)^3 + (x_2)^2 + x_3,$$

we conclude that $\varphi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x))$ is a local diffeomorphism. Furthermore, use the following coordinates change and feedback transformation

$$\xi = \varphi_2(x), \ \ z_1 = \varphi_1(x), \ \ z_2 = \varphi_3(x),$$

$$\begin{bmatrix} u_1 \\ v \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ \frac{2}{3(x_1)^3 + x_1 + 2(x_2)^2} & -\frac{1}{3(x_1)^3 + x_1 + 2(x_2)^2} \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{(x_2)^2 - (x_1)^3 + x_3}{3(x_1)^3 + x_1 + 2(x_2)^2} \end{bmatrix},$$

the system Σ^{uv} becomes

$$\Lambda^{\tilde{u}\tilde{v}}:\begin{cases}\dot{\xi} &= \tilde{u}_1\\ \dot{z}_1 &= z_2\\ \dot{z}_2 &= \tilde{v}.\end{cases}$$

Note that the above feedback transformation has a triangular form as we indicate in (5.12) and Definition 5.2.7.

Now by Theorem 5.2.9, $\Xi^u|_{M^*}^{red}$ is ex-fb-equivalent to the following linear DAECS Δ^u, since $\Sigma^{uv}\in\mathbf{Expl}(\Xi^u|_{M^*}^{red})$, $\Lambda^{\tilde{u}\tilde{v}}\in\mathbf{Expl}(\Delta^u)$, and $\Sigma^{uv}\overset{sys-fb}{\sim}\Lambda^{\tilde{u}\tilde{v}}$,

$$\Delta^u:\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}\dot{\xi}\\ \dot{z}_1\\ \dot{z}_2\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}\xi\\ z_1\\ z_2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}u_1.$$

Therefore, the original DAECS Ξ^u is ex-fb-equivalent to the following completely controllable linear DAECS:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{\xi}\\ \dot{z}_1\\ \dot{z}_2\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}\xi\\ z_1\\ z_2\end{bmatrix}+\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}\tilde{u}_1\\ \tilde{u}_2\end{bmatrix}$$

via

$$Q(x)=\begin{bmatrix}1&1&0\\0&0&1\\0&1&0\end{bmatrix},\quad \begin{bmatrix}\xi\\ z_1\\ z_2\end{bmatrix}=\begin{bmatrix}x_1x_2-x_{1a}x_{2a}\\ x_1+x_3-x_{1a}\\ -(x_1)^3+(x_2)^2+x_3\end{bmatrix},\quad \begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}1/2&0\\-1&1\end{bmatrix}\begin{bmatrix}\tilde{u}_1\\ \tilde{u}_2\end{bmatrix}.$$

5.6 Proofs of the results

Proof of Proposition 5.2.5. If. Suppose that Σ^{uv} and $\tilde{\Sigma}^{u,\tilde{v}}$ are equivalent via the transformations given in (5.10). First, $\operatorname{Im}\tilde{g}^{\tilde{v}}(x)\overset{(5.10)}{=}\operatorname{Im}\tilde{g}^{\tilde{v}}(x)\beta(x)=\ker E_1(x)=\ker E(x)$ proves that $\tilde{g}^{\tilde{v}}(x)$ is another choice such that $\operatorname{Im}\tilde{g}^{\tilde{v}}(x)=\ker E(x)$. Then we have

$$\tilde{\Sigma}^{u,\tilde{v}}:\begin{cases}\dot{x} &= \tilde{f}+\tilde{g}^u u+\tilde{g}^{\tilde{v}}\tilde{v}\overset{(5.10)}{=}f+\gamma h+g^v\alpha^v+(g^u+\gamma l^u+g^v\lambda)u+g^v\beta^v\tilde{v}\\ \tilde{y} &= \tilde{h}+\tilde{l}^u u\overset{(5.10)}{=}\eta h+\eta l^u u,\end{cases}$$

Now pre-multiply the differential part of $\tilde{\Sigma}$ by $E_1(x)$, to get (notice that $f=E_1^{\dagger}F_1$, $g^u=E_1^{\dagger}G_1$, $h=F_2$, $l^u=G_2$ and $\operatorname{Im}g^v=\ker E_1$)

$$\begin{cases}E_1\dot{x}=F_1+E_1\gamma F_2+(G_1+E_1\gamma G_2)u\\ \tilde{y}=\eta F_2+\eta G_2 u.\end{cases}$$

Thus $\tilde{\Sigma}^{u,\tilde{v}}$ is an $(I_l,\tilde{v})$-explicitation of the following DAECS:

$$\begin{bmatrix}E_1\\0\end{bmatrix}\dot{x}=\begin{bmatrix}F_1+E_1\gamma F_2\\ \eta F_2\end{bmatrix}+\begin{bmatrix}G_1+E_1\gamma G_2\\ \eta G_2\end{bmatrix}u.$$

Since the above DAECS can be transformed from Ξ^u via $\tilde{Q}(x) = Q'Q(x)$, where $Q'(x) = \begin{bmatrix} I_q & E_1(x)\gamma(x) \\ 0 & \eta(x) \end{bmatrix}$, it proves that $\tilde{\Sigma}^{u,\tilde{v}}$ is a $(\tilde{Q}, \tilde{v})$-explicitation of Ξ^u corresponding to the choice of invertible matrix $\tilde{Q}(x)$. Finally, by $E_1(x)\tilde{f}(x) = F_1(x) + E_1(x)\gamma(x)F_2(x)$, $E_1(x)\tilde{g}^u(x) = G_1(x) + E_1(x)\gamma(x)G_2(x)$, we get $\tilde{f}(x) = \tilde{E}_1^\dagger(x)(F_1(x) + \gamma(x)F_2(x))$ and $\tilde{g}^u(x) = \tilde{E}_1^\dagger(x)(G_1(x) + \gamma(x)G_2(x))$ for another choice of right inverse $\tilde{E}_1^\dagger(x)$ of $E_1(x)$.

Only if. Suppose that $\tilde{\Sigma}^{u,\tilde{v}} \in \mathbf{Expl}(\Xi^u)$ via $\tilde{Q}(x)$, $\tilde{E}_1^\dagger(x)$ and $\tilde{g}^{\tilde{v}}(x)$. First by $\operatorname{Im}\tilde{g}^{\tilde{v}}(x) = \ker E(x) = \operatorname{Im} g^v(x)$, there exists an invertible matrix $\beta^v(x)$ such that $\tilde{g}(x) = g(x)\beta^v(x)$. Moreover, since $E_1^\dagger(x)$ is a right inverse of $E_1(x)$ if and only if any solution $\dot{x}$ of $E_1(x)\dot{x} = w$ is given by $E_1^\dagger(x)w$, we have $E_1(x)E_1^\dagger(x)(F_1(x) + G(x)u) = F_1(x) + G(x)u$ and $E_1(x)\tilde{E}_1^\dagger(x)F_1(x) = F_1(x) + G(x)u$. It follows that $E_1(\tilde{E}_1^\dagger - E_1^\dagger)(x)(F_1(x) + G(x)u) = 0$, so $(\tilde{E}_1^\dagger - E_1^\dagger)(x)F_1(x) \in \ker E_1(x)$, $(\tilde{E}_1^\dagger - E_1^\dagger)(x)G_1(x) \in \ker E_1(x)$. Since $\ker E_1(x) = \operatorname{Im} g^v(x)$, it follows that $(\tilde{E}_1^\dagger - E_1^\dagger)(x)F_1(x) = g(x)\alpha^v(x)$ and $(\tilde{E}_1^\dagger - E_1^\dagger)(x)G_1(x) = g(x)\lambda(x)$ for suitable $\alpha^v(x)$ and $\lambda(x)$. Furthermore, since $Q(x)$ is such that $E_1(x)$ of $Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$ is of full row rank, it follows that for any other $\tilde{Q}(x)$, such that $\tilde{E}_1(x)$ of $\tilde{Q}(x)E(x) = \begin{bmatrix} \tilde{E}_1(x) \\ 0 \end{bmatrix}$ is full row rank, must be of the form $\tilde{Q}(x) = Q'(x)Q(x)$, where $Q' = \begin{bmatrix} Q_1(x) & Q_2(x) \\ 0 & Q_4(x) \end{bmatrix}$. Thus via $\tilde{Q}(x)$, Ξ^u is ex-equivalent to

$$\begin{aligned} & Q'\begin{bmatrix} E_1 \\ 0 \end{bmatrix}\dot{x} = Q'\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + Q'\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}u \\ \Rightarrow & \begin{bmatrix} Q_1E_1 \\ 0 \end{bmatrix}\dot{x} = \begin{bmatrix} Q_1F_1 + Q_2F_2 \\ Q_4F_2 \end{bmatrix} + \begin{bmatrix} Q_1G_1 + Q_2G_2 \\ Q_4G_2 \end{bmatrix}u. \end{aligned}$$

The bottom equation of the above can be expressed, using $\tilde{E}_1^\dagger(x)$ and $\tilde{g}^{\tilde{v}}(x)$, as:

$$\begin{cases} \dot{x} & = \tilde{E}_1^\dagger F_1 + \tilde{E}_1^\dagger Q_1^{-1}Q_2F_2 + (\tilde{E}_1^\dagger G_1 + \tilde{E}_1^\dagger Q_1^{-1}Q_2G_2)u + \tilde{g}^{\tilde{v}}v \\ & = E_1^\dagger F_1 + g^v\alpha^v + E_1^\dagger Q_1^{-1}Q_2h + (E_1^\dagger F_1 + g^v\lambda + E_1^\dagger Q_1^{-1}Q_2l^u)u + g^v\beta^v\tilde{v} \\ 0 & = Q_4F_2 + Q_4G_2 = Q_4h + Q_4l^u. \end{cases}$$

Thus the explicitation of Ξ via $\tilde{Q}(x)$, $\tilde{E}_1^\dagger(x)$ and $\tilde{g}(x)$ is

$$\tilde{\Sigma} : \begin{cases} \dot{x} = f + \gamma(h + l^u u) + g^v(\alpha^v + \lambda u + \beta^v\tilde{v}) = \tilde{f} + \tilde{g}^u u + \tilde{g}^{\tilde{v}}\tilde{v} \\ \tilde{y} = \eta h + \eta l^u u = \tilde{h} + \tilde{l}^u u. \end{cases}$$

where $\gamma(x) = E_1^\dagger Q_1^{-1}Q_2(x)$, $\eta(x) = Q_4(x)$. Now we can see that Σ^{uv} and $\tilde{\Sigma}^{u,\tilde{v}}$ are equivalent via transformations listed in (5.10). □

Proof of Theorem 5.2.9. By the assumptions that $\operatorname{rank} E(x)$ and $\operatorname{rank}\tilde{E}(x)$ are constant and equal to r around x^0 and $\tilde{x}^0$, respectively, there exist invertible matrix-valued functions $Q : U \to Gl(l, \mathbb{R})$ and $\tilde{Q} : \tilde{U} \to Gl(l, \mathbb{R})$, defined on neighborhoods U of x^0 and $\tilde{U}$ of $\tilde{x}^0$, such that $E'(x) = Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$ and $\tilde{E}'(\tilde{x}) = \tilde{Q}(\tilde{x})\tilde{E}(\tilde{x}) = \begin{bmatrix} \tilde{E}_1(\tilde{x}) \\ 0 \end{bmatrix}$, where E_1 :

$U \to R^{r\times n}$ and $\tilde{E}_1 : \tilde{U} \to R^{r\times n}$ are of full row rank. We have $\Xi^u \overset{ex-fb}{\sim} \Xi^{u'} = (E', F', G')$ and $\tilde{\Xi}^{\tilde{u}} \overset{ex-fb}{\sim} \tilde{\Xi}^{u'} = (\tilde{E}', \tilde{F}', \tilde{G}')$ via $Q(x)$ and $\tilde{Q}(\tilde{x})$, respectively, where

$$F'(x) = Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}, \quad G'(x) = Q(x)G(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix},$$
$$\tilde{F}'(\tilde{x}) = \tilde{Q}(\tilde{x})\tilde{F}(\tilde{x}) = \begin{bmatrix} \tilde{F}_1(\tilde{x}) \\ \tilde{F}_2(\tilde{x}) \end{bmatrix}, \quad \tilde{G}'(\tilde{x}) = \tilde{Q}(\tilde{x})\tilde{G}(\tilde{x}) = \begin{bmatrix} \tilde{G}_1(\tilde{x}) \\ \tilde{G}_2(\tilde{x}) \end{bmatrix}.$$

In this proof, without loss of generality, we will assume that $\Xi^u = \Xi^{u'}$ and $\tilde{\Xi}^{\tilde{u}} = \tilde{\Xi}^{u'}$, since $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ if and only if $\Xi^{u'} \overset{ex-fb}{\sim} \tilde{\Xi}^{u'}$.

Moreover, set

$$\begin{array}{llll} f(x) &= E_1^\dagger(x)F_1(x), \quad g^u(x) &= E_1^\dagger(x)G_1(x), \quad \operatorname{Im} g^v(x) = \ker E_1(x), \\ h(x) &= F_2(x), \quad l^u(x) &= G_2(x), \quad \tilde{f}(\tilde{x}) &= \tilde{E}_1^\dagger(\tilde{x})\tilde{F}_1(\tilde{x}), \\ \tilde{g}^{\tilde{u}}(\tilde{x}) &= \tilde{E}_1^\dagger(\tilde{x})\tilde{g}(\tilde{x}), \quad \operatorname{Im} \tilde{g}^{\tilde{v}}(\tilde{x}) = \ker \tilde{E}_1(\tilde{x}), \quad \tilde{h}(\tilde{x}) &= \tilde{F}_2(\tilde{x}), \\ \tilde{l}^{\tilde{u}}(\tilde{x}) &= \tilde{G}_2(\tilde{x}), \end{array} \tag{5.30}$$

where $E_1^\dagger(x)$ and $\tilde{E}_1^\dagger(x)$ are right inverses of $E_1(x)$ and $\tilde{E}_1(\tilde{x})$, respectively. Then by Definition 5.2.2, $\Sigma^{uv} = (f, g^u, g^v, h, l^u) \in \mathbf{Expl}(\Xi^u)$ and $\tilde{\Sigma}^{\tilde{u}\tilde{v}} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}}) \in \mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$. By Proposition 5.2.5, any control system in $\mathbf{Expl}(\Xi^u)$ is sys-fb-equivalent to Σ^{uv} and any control system in $\mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$ is sys-fb-equivalent to $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$. Without loss of generality, in the remaining part of the proof, we use Σ^{uv} and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$ with system matrices given by (5.30) to represent the two ODECSs in $\mathbf{Expl}(\Xi^u)$ and $\mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$, respectively.

If. Suppose that locally $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$. Then there exist a local diffeomorphism $\tilde{x} = \psi(x)$ and matrix-valued functions $\alpha^u(x)$, $\alpha^v(x)$, $\lambda(x)$, $\gamma(x)$, $\beta^u(x)$, $\beta^v(x)$, $\eta(x)$ such that the system matrices satisfy relations (5.11) of Definition 5.2.7.

First, consider $\tilde{g}^{\tilde{v}}(\psi(x)) = \frac{\partial\psi(x)}{\partial x} g^v(x)\beta^v(x)$. By $\operatorname{Im} g^v(x) = \ker E_1(x)$, $\operatorname{Im} \tilde{g}^{\tilde{v}}(x) = \ker \tilde{E}_1(x)$, we have $\ker \tilde{E}_1(\psi(x)) = \frac{\partial\psi(x)}{\partial x} \ker E_1(x)$. Thus there exists $Q_1(x) \in Gl(r, \mathbb{R})$ such that

$$\tilde{E}_1 \circ \psi = Q_1 E_1 \left(\frac{\partial\psi}{\partial x}\right)^{-1}. \tag{5.31}$$

Then, by (5.11), the following relation holds:

$$\begin{bmatrix} \tilde{f} \circ \psi & \tilde{g}^{\tilde{u}} \circ \psi \\ \tilde{h} \circ \psi & \tilde{l}^{\tilde{u}} \circ \psi \end{bmatrix} = \begin{bmatrix} \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial x}\gamma\eta \\ 0 & \eta \end{bmatrix} \begin{bmatrix} f & g^u & g^v \\ h & l^u & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \alpha^u & \beta^u \\ \alpha^v + \lambda\alpha^u & \lambda\beta^u \end{bmatrix}.$$

By substituting (5.30) into the above equation, we get

$$\begin{bmatrix} \tilde{E}_1^\dagger \circ \psi \cdot F_1 \circ \psi & \tilde{E}_1^\dagger \circ \psi \cdot G_1 \circ \psi \\ \tilde{F}_2 \circ \psi & \tilde{G}_2 \circ \psi \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial x}\gamma\eta \\ 0 & \eta \end{bmatrix} \begin{bmatrix} E_1^\dagger F_1 & E_1^\dagger G_1 & g^v \\ F_2 & G_2 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \alpha^u & \beta^u \\ \alpha^v + \lambda\alpha^u & \lambda\beta^u \end{bmatrix}.$$

Premultiply the above equation by

$$\begin{bmatrix}\tilde{E}_1\circ\psi & 0\\ 0 & I_p\end{bmatrix} = \begin{bmatrix}Q_1E_1\left(\frac{\partial\psi}{\partial x}\right)^{-1} & 0\\ 0 & I_p\end{bmatrix},$$

to get

$$\begin{bmatrix}\tilde{F}_1\circ\psi & \tilde{G}_1\circ\psi\\ \tilde{F}_2\circ\psi & \tilde{G}_2\circ\psi\end{bmatrix} = \begin{bmatrix}Q_1 & Q_1E_1\gamma\eta\\ 0 & \eta\end{bmatrix}\begin{bmatrix}F_1 & G_1\\ F_2 & G_2\end{bmatrix}\begin{bmatrix}I_n & 0\\ \alpha^u & \beta^u\end{bmatrix}. \tag{5.32}$$

Now from equations (5.31), (5.32) and Definition 5.2.2, it can be seen that $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ via $\tilde{x}=\psi(x)$, $Q(x)=\begin{bmatrix}Q_1 & Q_1E_1\gamma\eta\\ 0 & \eta\end{bmatrix}(x)$, $\alpha^u(x)$ and $\beta^u(x)$.

Only if. Suppose that $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ (locally in a neighborhood U of x^0). Assume that Ξ^u and $\tilde{\Xi}^u$ are ex-fb-equivalent via an invertible matirx $Q(x)=\begin{bmatrix}Q_1(x) & Q_2(x)\\ Q_3(x) & Q_4(x)\end{bmatrix}$, $\tilde{x}=\psi(x)$, $\alpha^u(x)$, $\beta^u(x)$, where $Q_1: U\to\mathbb{R}^{r\times r}$ and $Q_2(x)$, $Q_3(x)$,$Q_4(x)$ are matrix-valued functions of appropriate sizes.

Then by

$$Q(x)E(x)=\tilde{E}(\psi(x))\frac{\partial\psi(x)}{\partial x}\Rightarrow\begin{bmatrix}Q_1(x) & Q_2(x)\\ Q_3(x) & Q_4(x)\end{bmatrix}\begin{bmatrix}E_1(x)\\ 0\end{bmatrix}=\begin{bmatrix}\tilde{E}_1(\psi(x))\\ 0\end{bmatrix}\frac{\partial\psi(x)}{\partial x},$$

we can deduce that

$$\tilde{E}_1\circ\psi = Q_1E_1\left(\frac{\partial\psi}{\partial x}\right)^{-1}. \tag{5.33}$$

Moreover, $Q_3(x)=0$ and $Q_1(x)$ is invertible (since both $E_1(x)$ and $\tilde{E}_1(x)$ are of full row rank), which implies that $Q_4(x)$ is invertible as well (since $Q(x)$ is invertible). Subsequently, by

$$\tilde{F}\circ\psi = Q(F+G\alpha^u)\Rightarrow\begin{bmatrix}\tilde{F}_1(\psi)\\ \tilde{F}_2(\psi)\end{bmatrix}=\begin{bmatrix}Q_1 & Q_2\\ 0 & Q_4\end{bmatrix}\left(\begin{bmatrix}F_1\\ F_2\end{bmatrix}+\begin{bmatrix}G_1\\ G_2\end{bmatrix}\alpha^u\right),$$

we have

$$\tilde{F}_1(\psi(x)) = Q_1(x)(F_1(x)+G_1(x)\alpha^u(x))+Q_2(x)(F_2(x)+G_2(x)\alpha^u(x)) \tag{5.34}$$

and

$$\tilde{F}_2\circ\psi = Q_4(F_2+G_2\alpha^u). \tag{5.35}$$

Moreover, by

$$\tilde{G}(\psi(x)) = Q(x)G(x)\beta^u(x)\Rightarrow\begin{bmatrix}\tilde{G}_1(\psi(x))\\ \tilde{G}_2(\psi(x))\end{bmatrix}=\begin{bmatrix}Q_1(x) & Q_2(x)\\ 0 & Q_4(x)\end{bmatrix}\begin{bmatrix}G_1(x)\\ G_2(x)\end{bmatrix}\beta^u(x),$$

we have

$$\tilde{G}_1 \circ \psi = Q_1 G_1 \beta^u + Q_2 G_2 \beta^u \tag{5.36}$$

and

$$\tilde{G}_2 \circ \psi = Q_4 G_2 \beta^u. \tag{5.37}$$

Recall the system matrices given in equation (5.30). First, from $\operatorname{Im} g^v(x) = \ker E_1(x)$, $\operatorname{Im} \tilde{g}^{\tilde{v}}(x) = \ker \tilde{E}_1(\tilde{x})$, and equation (5.33), it is seen that there exists $\beta^v : U \to Gl(s, \mathbb{R})$ such that

$$\tilde{g}^{\tilde{v}} \circ \psi = \frac{\partial \psi}{\partial x} g^v \beta^v. \tag{5.38}$$

Secondly, by equations (5.33) and (5.34), we have

$$\begin{aligned} \tilde{f} \circ \psi &= \tilde{E}_1^\dagger \circ \psi \tilde{F}_1 \circ \psi \\ &= \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} F_1 + G_1 \alpha^u \\ F_2 + G_2 \alpha^u \end{bmatrix} \\ &= \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} F_1 + G_1 \alpha^u + E_1 g^v \left(\lambda \alpha^u + \alpha^v \right) \\ F_2 + G_2 \alpha^u \end{bmatrix} \\ &= \frac{\partial \psi}{\partial x} \Big(f + g^u \alpha^u + g^v \left(\lambda \alpha^u + \alpha^v \right) + E_1^\dagger Q_1^{-1} Q_2 \left(h + l^u \alpha^u \right) \Big), \end{aligned} \tag{5.39}$$

where $\alpha^v(x)$ and $\lambda(x)$ are matrix-valued functions of appropriate sizes. Thirdly, by equation (5.36), we have

$$\begin{aligned} \tilde{g}^{\tilde{u}} \circ \psi &= \tilde{E}_1^\dagger \circ \psi \tilde{G}_1 \circ \psi \\ &= \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} G_1 \beta^u \\ G_2 \beta^u \end{bmatrix} \\ &= \frac{\partial \psi}{\partial x} E_1^\dagger Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} G_1 \beta^u + E_1 g^v \lambda \\ G_2 \beta^u \end{bmatrix} \\ &= \frac{\partial \psi}{\partial x} \Big(g^u \beta^u + g^v \lambda + E_1^\dagger Q_1^{-1} Q_2 l^u \beta^u \Big). \end{aligned} \tag{5.40}$$

Note that we use the equations $E_1 g^v \left(\lambda \alpha^u + \alpha^v \right) = 0$ and $E_1 g^v \lambda = 0$ to deduce (5.39) and (5.40). At last, by equations (5.35) and (5.37) we have

$$\tilde{h} \circ \psi = \tilde{F}_2 \circ \psi = Q_4 (F_2 + G_2 \alpha^u) = Q_4 \left(h + l^u \alpha^u \right) \tag{5.41}$$

and

$$\tilde{l}^{\tilde{u}} \circ \psi = \tilde{G}_2 \circ \psi = Q_4 G_2 \beta^u = Q_4 l^u \beta^u. \tag{5.42}$$

Finally, it can be seen from (5.39), (5.40), (5.41) and (5.42), that $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$ via $\tilde{x} = \psi(x)$, $\alpha^v(x)$, $\beta^v(x)$, $\alpha^u(x)$, $\beta^u(x)$, $\lambda(x)$, $\gamma(x) = E_1^\dagger Q_1^{-1} Q_2(x)$ and $\eta(x) = Q_4(x)$. □

Proof of Proposition 5.3.7. Since M is a smooth submanifold of dimension n_1, there exist a neighborhood U_0 of x_a and $n-n_1$ independent scalar functions $\varphi_1, \cdots, \varphi_{n-n_1} : U_0 \to \mathbb{R}$ such that

$$M \cap U_0 = \{x \in U_0 : \varphi_1(x) = \ldots = \varphi_{n-n_1}(x) = 0\}.$$

Let $z_2 = [\varphi_1(x), \cdots, \varphi_{n-n_1}(x)]^T$ and choose local coordinates $z = \psi(x) = (z_1(x), z_2(x))$, where $z_1 : U_0 \to \mathbb{R}^{n_1}$ such that $dz_1 \wedge dz_2 \neq 0$. Denote $E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1}$ by $\begin{bmatrix} E_1(x) & E_2(x) \end{bmatrix}$, where $E_1 : U_0 \to \mathbb{R}^{l\times n_1}$ and $E_2 : U_0 \to \mathbb{R}^{l\times(n-n_1)}$. It follows that

$$\begin{aligned} &E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1}\left(\frac{\partial\psi(x)}{\partial x}\right)\dot{x} = F(x) + G(x)u \\ &\Rightarrow \begin{bmatrix} E_1(z_1, z_2) & E_2(z_1, z_2) \end{bmatrix}\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = F(z_1, z_2) + G(z_1, z_2)u. \end{aligned}$$

For all $z \in M \cap U_0$, we have $z_2 = 0$ and for $(z, \dot{z}) \in T_zM$, the DAE Ξ^u has the following form:

$$\begin{aligned} &\begin{bmatrix} E_1(z_1, 0) & E_2(z_1, 0) \end{bmatrix}\begin{bmatrix} \dot{z}_1 \\ 0 \end{bmatrix} = F(z_1, 0) + G(z_1, 0)u \Rightarrow \\ &E_1(z_1, 0)\dot{z}_1 = F(z_1, 0) + G(z_1, 0)u. \end{aligned}$$

Now from the assumptions

$$\dim\left(E(z)T_zM\right) = const. = r, \quad \dim\left(E(z)T_zM + \operatorname{Im} G(z)\right) = const. = r + m_2,$$

locally for $z \in M$, it follows that there exists a neighborhood $U_1 \subseteq U_0$ and an invertible matrix $Q(z_1) : M \cap U_1 \to Gl(l, \mathbb{R})$ such that:

$$Q(z_1)\begin{bmatrix} E_1(z_1, 0) & G(z_1, 0) \end{bmatrix} = \begin{bmatrix} E_1^1(z_1) & G_1(z_1) \\ 0 & G_2(z_1) \\ 0 & 0 \end{bmatrix},$$

where $E_1(z_1, 0)$, $G_1(z_1)$ and $G_2(z_1)$ are smooth functions defined on $M \cap U_1$ with values in $\mathbb{R}^{r\times n_1}$, $\mathbb{R}^{r\times m}$ and $\mathbb{R}^{m_2\times m}$, respectively and, moreover, $E_1^1(z_1)$ and $G_2(z_1)$ are of full row rank.

Denote $Q(z_1)F(z_1) = \operatorname{col}[F_1(z_1), F_2(z_1), F_3(z_1)]$, where F_1, F_2, F_3 are matrix-valued functions of appropriate sizes. Then for all $z \in M \cap U_1$, Ξ^u has the following form:

$$\begin{bmatrix} E_1^1(z_1) \\ 0 \\ 0 \end{bmatrix}\dot{z}_1 = \begin{bmatrix} F_1(z_1) \\ F_2(z_1) \\ F_3(z_1) \end{bmatrix} + \begin{bmatrix} G_1(z_1) \\ G_2(z_1) \\ 0 \end{bmatrix}u. \tag{5.43}$$

Now by $F(z) \in E(z)T_zM + \operatorname{Im} G(z)$ locally for $z \in M$ (since M is locally controlled invariant), we have

$$\begin{bmatrix} F_1(z_1) \\ F_2(z_1) \\ F_3(z_1) \end{bmatrix} \in \operatorname{Im}\begin{bmatrix} E_1^1(z_1) & G_1(z_1) \\ 0 & G_2(z_1) \\ 0 & 0 \end{bmatrix},$$

which implies $F_3(z_1) = 0$.

Subsequently, since $G_2(z_1)$ is of full row rank, we can always assume $\begin{bmatrix} G_1(z_1) \\ G_2(z_1) \end{bmatrix} = \begin{bmatrix} G_1^1(z_1) & G_1^2(z_1) \\ G_2^1(z_1) & G_2^2(z_1) \end{bmatrix}$ such that $G_2^2 : M \cap U \to Gl(m_2, \mathbb{R})$. Since if not, we can always permute the components of u such that $G_2^2(z_1)$ is invertible. Then, applying the following feedback transformation to DAECS (5.43)

$$\tilde{u} = \begin{bmatrix} 0 \\ F_2(z_1) \end{bmatrix} + \begin{bmatrix} I_{n-m_2} & 0 \\ G_2^1(z_1) & G_2^2(z_1) \end{bmatrix} u,$$

we get

$$\begin{bmatrix} E_1^1(z_1) \\ 0 \\ 0 \end{bmatrix} \dot{z}_1 = \begin{bmatrix} \tilde{F}_1(z_1) \\ 0 \\ F_3(z_1) \end{bmatrix} + \begin{bmatrix} \tilde{G}_1^1(z_1) & \tilde{G}_1^2(z_1) \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $(u_1, u_2) = \tilde{u}$ and $\dim u_1 = n - m_2 = m_1$, $\tilde{F}_1 = F_1 - G_1^2(G_2^2)^{-1}F_2$, $\tilde{G}_1^1 = G_1^1 - G_1^2(G_2^2)^{-1}G_2^1$ and $\tilde{G}_2^2 = G_1^2(G_2^2)^{-1}$.

Premultiply the above equation by

$$Q(z_1) = \begin{bmatrix} I_r & -\tilde{G}_1^2(z_1) & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & I_{l-r-m_2} \end{bmatrix},$$

then it follows, for all $z \in M \cap U_1$, that Ξ^u has the following form (notice that $F_3(z_1) = 0$)

$$\begin{bmatrix} E_1^1(z_1) \\ 0 \\ 0 \end{bmatrix} \dot{z}_1 = \begin{bmatrix} \tilde{F}_1(z_1) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1^1(z_1) & 0 \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Finally, by Definition 5.3.4 and 5.3.6, we have

$$\Xi^u|_M^{red} : E_1^1(z_1)\dot{z}_1 = \tilde{F}_1(z_1) + \tilde{G}_1^1(z_1)u_1.$$

Obviously, $\Xi^u|_M^{red} = (E_1^1, \tilde{F}_1, \tilde{G}_1^1)$ is a DAECS of form (5.1) of dimensions r, n_1 and $m_1 = m - m_2$, i.e., $\Xi^u|_M^{red} = \Xi^{u'}_{r,n_1,m_1}$, where the notation u' indicates that the dimension of the control u has been changed. Moreover, since $E_1^1(z_1)$ is of full row rank, from the procedure of explicitation, it is seen that $\mathbf{Expl}(\Xi^u|_M^{red})$ is not empty and any ODECS in $\mathbf{Expl}(\Xi^u|_M^{red})$ has no outputs. □

Proof of Theorem 5.3.10. First, we prove that $M^* = M_{k^*}^c$ is a locally maximal controlled invariant submanifold. Under the assumption that $M_k \cap U_k$, $k > 0$ are smooth submanifolds, by $M_{k+1} \subseteq M_k$, there exists an index k^* such that $M_{k^*+1} = M_{k^*}^c$. Note that k^* is strictly smaller than n, since $\dim M_0 \le n$ and $\dim M_k - \dim M_{k+1} \ge 1$ for all $k < k^*$ (notice that $\dim M_k \ne 0$).

For $k = k^*$, we have

$$F(x) \in E(x)T_xM^c_{k^*} + \operatorname{Im} G(x), \quad x \in M^c_{k^*} \tag{5.44}$$

Now consider the assumption that $M^* = M^c_{k^*}$ satisfies the regularity condition **(Reg)**, by Proposition 5.3.3, $M^* = M^c_{k^*}$ is a locally controlled invariant submanifold. Then we prove M^* is locally maximal by induction. Consider any other controlled invariant submanifold $\hat{M}$, we can see that $\hat{M} \subseteq M_0$ (since $F(x) \in E(x)T_x\hat{M} + \operatorname{Im} G(x)$ for all $x \in \hat{M}$). Suppose that $\hat{M} \subseteq M^c_k$, which implies $F(x) \in E(x)T_xM^c_k + \operatorname{Im} G(x)$ for all $x \in \hat{M}$. By equation (5.14), it can be deduced that $x \in M_{k+1}$ for all $x \in \hat{M}$. Thus $\hat{M} \subseteq M_k$ for all $k < n$, which implies $\hat{M} \subseteq M_{k^*}$. Therefore, $M^* = M^c_{k^*}$ is locally maximal.

Next we prove that under the additional assumption (A1): Ξ^u is locally ex-fb-equivalent to the **MCISF** given by (5.15). Denote $\dim M^* = n_1$, there exist a neighborhood U_1 of x^0 and two vector-valued functions $\xi_1 : U_1 \to \mathbb{R}^{n_1}$ and $\xi_2 : U_1 \to \mathbb{R}^{n_2}$ such that $M^* \cap U_1 = \{x : \xi_2(x) = 0\}$ and $d\xi_1 \wedge d\xi_2 \neq 0$. In local (ξ_1, ξ_2)-coordinates, defined by the local diffeomorphism $\xi(x) = \operatorname{col}[\xi_1(x), \xi_2(x)]$, the system Ξ^u is expressed as

$$\begin{bmatrix} E_1(\xi) & E_2(\xi) \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = F(\xi) + G(\xi)u,$$

where $E_1 : U_1 \to \mathbb{R}^{l \times n_1}$ and $E_2 : U_1 \to \mathbb{R}^{l \times n_2}$.

Then, by assumption (A1), there exists a neighborhood $U_2 = U_1 \cap U_{k^*}$ of x^0 such that for $\xi \in U_2$,

$$\operatorname{rank} \begin{bmatrix} E_1(\xi) & E_2(\xi) \end{bmatrix} = const. = r, \;\; \operatorname{rank} \begin{bmatrix} E_1(\xi) & E_2(\xi) & G(\xi) \end{bmatrix} = const. = r + m_2.$$

Choose $Q_1 : U_2 \to Gl(l, \mathbb{R})$ such that the matrices $[\tilde{E}^1(\xi), \tilde{E}^2(\xi)]$ and $G_2(\xi)$ are of full row rank:

$$Q_1(\xi) \begin{bmatrix} E_1(\xi) & E_2(\xi) & G(\xi) \end{bmatrix} = \begin{bmatrix} \tilde{E}_1(\xi) & \tilde{E}_2(\xi) & G_1(\xi) \\ 0 & 0 & G_2(\xi) \\ 0 & 0 & 0 \end{bmatrix},$$

where $\tilde{E}_1 : U_2 \to \mathbb{R}^{r \times n_1}$ and $G_2 : U_2 \to \mathbb{R}^{m_2 \times m}$.

By assumption **(Reg)**, we have $\dim E(x)T_xM^* = const.$ for $x \in M^*$, and denote this dimension by r_1. Then it is immediate to see that $\operatorname{rank} \tilde{E}_1(\xi) = r_1$ for $\xi \in M^*$. By the smoothness of $E(x)$, there exist r_1 columns of $\tilde{E}_1(\xi)$ are independent for $\xi \in U_2$. Now consider the matrix

$$\begin{bmatrix} \tilde{E}_1(\xi) & | & \tilde{E}_2(\xi) \end{bmatrix} = \left[\begin{array}{cc|cc} E_1^1(\xi) & E_1^2(\xi) & E_2^1(\xi) & E_2^2(\xi) \\ E_1^3(\xi) & E_1^4(\xi) & E_2^3(\xi) & E_2^4(\xi) \end{array}\right],$$

where $E_1^1 : U_2 \to \mathbb{R}^{r_1 \times r_1}$ and $E_2^3 : U_2 \to \mathbb{R}^{r_2 \times r_2}$ and where $r_2 = r - r_1$. We can always permute the rows (by a constant Q-transformation) and the columns (by permuting the components of ξ_1) of the above matrix such that $E_1^1(\xi)$ is invertible. Then by a suitable Q-transformation, $[\tilde{E}_1, \tilde{E}_2]$ admits the form

$$\begin{bmatrix} \tilde{E}_1(\xi) & | & \tilde{E}_2(\xi) \end{bmatrix} = \left[\begin{array}{cc|cc} I_{r_1} & E_1^2(\xi) & E_2^1(\xi) & E_2^2(\xi) \\ 0 & E_1^4(\xi) & E_2^3(\xi) & E_2^4(\xi) \end{array}\right].$$

Since $\operatorname{rank} E(x) = r$, we have $[E_1^4, E_2^3, E_4^2]$ is of full row rank $r_2 = r - r_1$. Notice that $E_1^4(\xi) = 0$ for $\xi \in M^*$ (since $\operatorname{rank} \tilde{E}_1(\xi) = r_1$ for $\xi \in M^*$). By the smoothness of $E(x)$, we have $[E_2^3(\xi), E_4^2(\xi)]$ is of full row rank r_2 for $x \in U_2$. Now we can always permute the columns (by permuting the components of ξ_2) of $\tilde{E}_2$ such that E_2^3 is invertible. On the other hand, let

$$\left[\begin{array}{c} G_1(\xi) \\ \hline G_2(\xi) \end{array}\right] = \left[\begin{array}{cc} G_1^1(\xi) & G_1^2(\xi) \\ G_1^3(\xi) & G_1^4(\xi) \\ \hline G_2^1(\xi) & G_2^2(\xi) \end{array}\right],$$

where $G_2^2(\xi)$ is a $m_2 \times m_2$ matrix. Since G_2 is of full row rank m_2, we can permute the components of u (by a feedback transformation) such that $G_2^2(\xi)$ is invertible. Since $E_2^3(\xi)$ and $G_2^2(\xi)$ are invertible, set

$$Q_2(\xi) = \begin{bmatrix} I_{r_1} & Q_1^2(\xi) & Q_1^3(\xi) & 0 \\ 0 & Q_2^2(\xi) & Q_2^3(\xi) & 0 \\ 0 & 0 & Q_3^3(\xi) & 0 \\ 0 & 0 & 0 & I_{l-m-r} \end{bmatrix},$$

where $Q_1^2 = -E_2^1(E_2^3)^{-1}$, $Q_1^3 = -(G_1^2 - E_2^1(E_2^3)^{-1}G_1^4)(G_2^2)^{-1}$, $Q_2^2 = (E_2^3)^{-1}$, $Q_2^3 = -(E_2^3)^{-1}G_1^4(G_2^2)^{-1}$, $Q_3^3 = (G_2^2)^{-1}$, and we have

$$Q_2(\xi)Q_1(\xi)\left[\begin{array}{cc|c} E_1(\xi) & E_2(\xi) & G(\xi) \end{array}\right] = \left[\begin{array}{cccc|cc} I_{r_1} & \tilde{E}_1^2(\xi) & 0 & \tilde{E}_2^2(\xi) & \tilde{G}_1^1(\xi) & 0 \\ 0 & \tilde{E}_1^4(\xi) & I_{r_1} & \tilde{E}_2^4(\xi) & \tilde{G}_1^3(\xi) & 0 \\ 0 & 0 & 0 & 0 & \tilde{G}_2^1(\xi) & I_{m_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

where $\tilde{E}_1^2 = E_1^2 + Q_1^2E_1^4$, $\tilde{E}_2^2 = E_2^2 + Q_1^2E_2^4$, $\tilde{G}_1^1 = G_1^1 + Q_1^2E_1^4 + Q_1^3G_2^1$, $\tilde{E}_1^4 = Q_2^2E_1^4$, $\tilde{E}_2^4 = Q_2^2E_2^4$, $\tilde{G}_1^3 = Q_2^2G_1^3 + Q_2^3G_2^1$, $\tilde{G}_2^1 = Q_3^3G_2^1$.

Denote $Q_2Q_1F = \operatorname{col}[F_1, F_2, F_3, F_4]$. Then by the feedback transformation

$$\begin{bmatrix} 0 \\ F_3(\xi) \end{bmatrix} + \begin{bmatrix} I_{m_1} & 0 \\ \tilde{G}_2^1(\xi) & I_{m_2} \end{bmatrix} u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

both $\tilde{G}_2^1$ and F_3 become zero. Rewrite $z = \xi$, $(z_1, z_2) = \xi_1$, $(z_3, z_4) = \xi_2$, $E_1^2 = \tilde{E}_1^2$, $E_2^2 = \tilde{E}_1^4$, $E_1^4 = \tilde{E}_2^2$, $E_2^4 = \tilde{E}_2^4$, $G_1 = \tilde{G}_1^1$, $G_2 = \tilde{G}_1^3$, then it is not hard to see Ξ^u is transformed into the normal form given by (5.15). Since $\dim\, E(x)T_xM^* = \operatorname{rank} \begin{bmatrix} I_{r_1} & E_1^2(z) \\ 0 & E_2^2(z) \end{bmatrix} = r_1$ for $z \in M^*$, we have $E_2^2(z) = 0$ for $z \in M^*$. Since $F(z) \in E(z)T_zM^* + \operatorname{Im} G(z)$ (because M^* is a controlled invariant submanifold), we have $F_4(z) = 0$ for $z \in M^*$.

Now we prove that under additional conditions (A1) and (A2), Ξ^u is locally ex-fb-equivalent the **SMCISF**, given by (5.16). The construction of the **SMCISF** is similar to the above construction of the normal form (5.15), but we choose new coordinates differently as shown in the following. By (A2), there exist a neighborhood U_1 of x^0 and two vector-valued functions $\xi_1 : U_1 \to \mathbb{R}^{n_1}$ and $\xi_2 : U_1 \to \mathbb{R}^{n_2}$ such that $\operatorname{span}\{d\xi_1\} = \operatorname{span}\{d\xi_1^1, \ldots, d\xi_1^{n_1}\} = \mathcal{D}^\perp$ (recall that $\mathcal{D}$ is involutive) and $d\xi_1 \wedge d\xi_2 \neq 0$. Since $\mathcal{D}(x) = T_xM^*$ for $x \in M^*$, we still have $M^* \cap U_1 = \{x : \xi_2(x) = 0\}$. Thus by

(A1) and the above construction of (5.15), we can always transform Ξ^u into the normal form of (5.15). However, by (A2), in (5.15),

$$\dim E(z)\mathcal{D}(z) = \operatorname{rank}\begin{bmatrix} I_{r_1} & E_1^2(z) \\ 0 & E_2^2(z) \end{bmatrix} = r_1,$$

$$\dim\left(E(z)\mathcal{D}(z) + \operatorname{Im} G(z)\right) = \operatorname{rank}\begin{bmatrix} I_{r_1} & E_1^2(z) & G_1(z) & 0 \\ 0 & E_2^2(z) & G_2(z) & 0 \\ 0 & 0 & 0 & I_{m_2} \end{bmatrix} = r_1 + m_2,$$

locally for $z \in U_2$. Thus both E_2^2 and G_2 become zero. Then by $F(z) \in E(z)T_zM^* + \operatorname{Im} G(z)$ (since M^* is a controlled invariant submanifold), we have $F_2(z) = 0$, $F_4(z) = 0$ for $z \in M^*$. Therefore, under assumptions (A1) and (A2), Ξ^u is always locally ex-fb-equivalent to the **SMCISF**, given by (5.16). □

Proof of Theorem 5.4.5. Denote $\dim\, M^* = n^*$, $\dim\left(E(x)T_xM^*\right) = r^*$ and

$$\dim\left(E(x)T_xM^* + \operatorname{Im} G(x)\right) = r^* + (m - m^*)$$

(the dimensions being constant by assumption (A1)). Then by Proposition 5.3.7, a DAE $\Xi^u|^{red}_{M^*}$ is of the form

$$\Xi^u|^{red}_{M^*} : E^*(x)\dot{z} = F^*(x) + G^*(x)u^*, \tag{5.45}$$

where $E^*(x)$ is of full row rank r^*, $G^*(x)$ is of full column rank m^*, and $\Xi^u|^{red}_{M^*}$ is thus $\Xi^u|^{red}_{M^*} = \Xi^u_{r^*,n^*,m^*}$. An ODECS $\Sigma^*_{n^*,m^*,s^*} = (f, g^u, g^v) \in \mathit{Expl}(\Xi^u|^{red}_{M^*})$ is a system without outputs and of the form

$$\Sigma^* : \dot{x} = f(x) + g^u(x)u + g^v(x)v,$$

where $s^* = n^* - r^*$.

Only if. Suppose that Ξ^u is locally completely internal feedback linearizable (see Definition 5.4.3), which means that $\Xi^u|^{red}_{M^*}$ is locally ex-fb-equivalent to a completely controllable linear DAECS

$$\Delta^{u*} : E^*\dot{z} = H^*z + L^*u^*,$$

denoted by $\Delta^{u*} = (E^*, H^*, L^*)$, where E^*, H^*, L^* are constant matrices of appropriate sizes. A linear ODECS $\Lambda^* = (A^*, B^{u*}, B^{v*}) \in \mathbf{Expl}(\Delta^{u*})$ is of the form

$$\Lambda^* : \dot{x} = A^*x + B^{u*}u + B^{v*}v.$$

Then by Lemma 5.4.2, the complete controllability of Δ^{u*} implies $\hat{\mathscr{W}}_{n^*}(\Delta^{u*}) = \mathscr{W}_{n^*}(\Delta^{u*}) = \mathbb{R}^{n^*}$. By Proposition 3.2.9 of Chapter 3 (see also Remark 5.4.4(ii)), we get $\hat{\mathcal{W}}_{n^*}(\Lambda^*) = \mathcal{W}_{n^*}(\Lambda^*) = \hat{\mathscr{W}}_{n^*}(\Delta^{u*}) = \mathscr{W}_{n^*}(\Delta^{u*}) = \mathbb{R}^{n^*}$. Since Λ^* is a linear control system without outputs, we have $\hat{\mathcal{D}}_{n^*}(\Lambda^*) = \hat{\mathcal{W}}_{n^*}(\Lambda^*)$ and $\mathcal{D}_{n^*}(\Lambda^*) = \mathcal{W}_{n^*}(\Lambda^*)$. Hence, $\hat{\mathcal{D}}_{n^*}(\Lambda^*) = \mathcal{D}_{n^*}(\Lambda^*) = \mathbb{R}^{n^*}$. Thus Σ^* satisfies condition (FL2). Moreover, since Λ^* is a linear control system, it satisfies conditions (FL1) and (FL3) in an obvious way. Conditions (FL1)-(FL3) are invariant under sys-fb-equivalence, so Σ^* (which is sys-fb-equivalent to Λ^*) also satisfies them.

After having proved that Σ^* satisfies (FL1)-(FL3) of Theorem 5.4.5, we will now prove that any ODECS belonging to $\mathbf{Expl}(\Xi^u|_{M^*}^{red})$ satisfies these conditions as well. Consider two ODECSs $\Sigma^* \in \mathbf{Expl}(\Xi^u|_{M^*}^{red})$ and $\hat{\Sigma}^* \in \mathbf{Expl}(\Xi^u|_{M^*}^{red})$, we have $\Sigma^* \overset{sys-fb}{\sim} \hat{\Sigma}^*$ by Theorem 5.2.9. Since Σ^* and $\hat{\Sigma}^*$ are control systems without outputs, sys-fb-equivalence reduces to feedback equivalence. Thus Σ^* and $\hat{\Sigma}^*$ are feedback equivalent (via two kinds of feedback transformations, see Remark 5.2.8). It is easy to verify that if $\hat{\mathcal{D}}_i$ and $\mathcal{D}_i$ are involutive, then the two distribution sequences are invariant under the two kinds of feedback transformations. Hence any ODECS belonging to the class $\mathbf{Expl}(\Xi^u|_{M^*}^{red})$ satisfies conditions (FL1)-(FL3) of Theorem 5.4.5.

If. Suppose that an ODECS $\Sigma^* \in \mathbf{Expl}(\Xi^u|_{M^*}^{red})$ satisfies conditions (FL1)-(FL3). We claim that $\Sigma_{n^*,m^*,s^*} = (f, g^u, g^v)$ is locally feedback equivalent to the Brunovský canonical form (see [31] for standard linear ODECSs and Chapter 3 for linear ODECSs with two kinds of inputs), via two kinds of feedback transformations (see equation (5.46)). We now describe a procedure to construct new coordinates $z = T_x(x)$ and feedback transformations:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} F_u(x) \\ F_v(x) \end{bmatrix} + \begin{bmatrix} T_u(x) & 0 \\ R(x) & T_u(x) \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \tag{5.46}$$

to transform Σ^* into the Brunovský canonical form. Note that $T_u(x)$ and $T_v(x)$ have to be invertible around x_a.

Step 1: Denote the smallest i such that $\mathcal{D}_i = \mathcal{D}_{n^*} = TM^*$ by k^* (meaning $\mathcal{D}_{k^*} = \mathcal{D}_{k^*+1} = TM^*$) and define

$$m_1 = \dim\, \hat{\mathcal{D}}_{k^*}^{\perp} - \dim\, \mathcal{D}_{k^*}^{\perp}, \quad s_1 = \dim\, \mathcal{D}_{k^*-1}^{\perp} - \dim\, \hat{\mathcal{D}}_{k^*}^{\perp}.$$

Now by involutivity of $\mathcal{D}_{k^*-1}$ and $\hat{\mathcal{D}}_{k^*}$ (condition (FL1)), we can choose scalar functions $h_i^u(x)$, $1 \le i \le m_1$ and $h_i^v(x)$, $1 \le i \le s_1$ such that

$$\begin{aligned} &\text{span}\,\{dh_i^u, 1 \le i \le m_1\} = \hat{\mathcal{D}}_{k^*}^{\perp}, \\ &\text{span}\,\{dh_i^u, 1 \le i \le m_1\} + \text{span}\,\{dh_i^v, 1 \le i \le s_1\} = \mathcal{D}_{k^*-1}^{\perp}, \end{aligned}$$

which implies that

$$\begin{aligned} &\left\langle dh_i^u(x), ad_f^k g_j^u(x)\right\rangle = 0, \;\; 1 \le i \le m_1, \;\; 1 \le j \le m, \;\; 0 \le k \le k^* - 2; \\ &\left\langle dh_i^u(x), ad_f^k g_j^v(x)\right\rangle = 0, \;\; 1 \le i \le m_1, \;\; 1 \le j \le s, \;\; 0 \le k \le k^* - 1; \\ &\left\langle dh_i^v(x), ad_f^k g_j^u(x)\right\rangle = 0, \;\; 1 \le i \le s_1, \;\; 1 \le j \le m, \;\; 0 \le k \le k^* - 2; \\ &\left\langle dh_i^v(x), ad_f^k g_j^v(x)\right\rangle = 0, \;\; 1 \le i \le s_1, \;\; 1 \le j \le s, \;\; 0 \le k \le k^* - 2. \end{aligned} \tag{5.47}$$

where the vector fields $g_j^u(x)$, $1 \le j \le m$, are the columns of $g^u(x)$ and the vector fields $g_j^v(x)$, $1 \le j \le s$ are the columns of $g^v(x)$.

Recall the following result [92][151]:

$$\begin{aligned} &\left\langle dh(x), ad_f^k g(x)\right\rangle = 0, \;\; 0 \le k \le l-2 \;\;\text{ then} \\ &\left\langle dh(x), ad_f^{l-1} g(x)\right\rangle = (-1)^k \left\langle dL_f^k h, ad_f^{l-1-k} g\right\rangle, \;\; 0 \le k \le l-1, \end{aligned} \tag{5.48}$$

where $h(x)$ is a scalar function, $f(x)$ and $g(x)$ are vector fields.

Then by (5.47) and (5.48), we have:

$$\begin{aligned}
&L_{g_j^u} L_f^k h_i^u(x) = 0, \ \ 1 \le i \le m_1, \ \ 1 \le j \le m, \ \ 0 \le k \le k^* - 2;\\
&L_{g_j^v} L_f^k h_i^u(x) = 0, \ \ 1 \le i \le m_1, \ \ 1 \le j \le s, \ \ 0 \le k \le k^* - 1;\\
&L_{g_j^u} L_f^k h_i^v(x) = 0, \ \ 1 \le i \le s_1, \ \ 1 \le j \le m, \ \ 0 \le k \le k^* - 2;\\
&L_{g_j^v} L_f^k h_i^v(x) = 0, \ \ 1 \le i \le s_1, \ \ 1 \le j \le s, \ \ 0 \le k \le k^* - 2.
\end{aligned} \tag{5.49}$$

Let $H_{m_1}^u : M^* \to R^{m_1}$ be a vector-valued map with entries $h_i^u(x)$, $1 \le i \le m_1$ and $H_{s_1}^v : M^* \to R^{s_1}$ be a vector-valued map with entries h_i^v, $1 \le i \le s_1$. By construction, $dH_{m_1}^u(x)$ and $dH_{s_1}^v(x)$ are matrices of full row rank around x_a. Then, we claim:

(a) The following $m_1 \times m$ matrix is of full row rank at x_a:

$$T_u^{m_1}(x) = L_{g^u} L_f^{k^*-1} H_{m_1}^u(x) = \left\langle dL_f^{k^*-1} H_{m_1}^u(x), g^u(x) \right\rangle$$

and the following $s_1 \times s$ matrix is of full row rank at x_a:

$$T_v^{s_1}(x) = L_{g^v} L_f^{k^*-1} H_{s_1}^v(x) = \left\langle dL_f^{k^*-1} H_{s_1}^v(x), g^v(x) \right\rangle .$$

First suppose $\operatorname{rank} T_u^{m_1}(x_a) < m_1$. Then, by (5.47) and (5.48), there exists a nonzero vector $c_{m_1}^u \in \mathbb{R}^{m_1}$ such that

$$c_{m_1}^u \left\langle dL_f^{k^*-1} H_{m_1}^u(x_a), g_j^u(x_a) \right\rangle = (-1)^{k^*-1} c_{m_1}^u \left\langle dH_{m_1}^u(x_a), ad_f^{k^*-1} g_j^u(x_a) \right\rangle = 0, \ \ 1 \le j \le m.$$

Note that $\operatorname{span}\{dh_i^u, 1 \le i \le m_1\} \in \hat{\mathcal{D}}_{k^*}^{\perp}$ by construction. Thus the above equation implies that $c_{m_1}^u dH_{m_1}^u(x_a) \in \mathcal{D}_{k^*}^{\perp}(x_a)$. The matrix $dH_{m_1}^u(x)$ is of full row rank and $\dim \mathcal{D}_{k^*} = n^*$, so $c_{m_1}^u$ has to be zero, this contradiction implies that $T_u^{m_1}(x_a)$ is of full row rank. Then, suppose $\operatorname{rank} T_v^{s_1}(x_a) < s_1$. Also by (5.47) and (5.48), there exists a nonzero row vector $c_{s_1}^v \in \mathbb{R}^{s_1}$ such that

$$c_{s_1}^v \left\langle dL_f^{k^*-1} H_{s_1}^v(x_a), g_j^v(x_a) \right\rangle = c_{s_1}^v \left\langle dH_{s_1}^v(x_a), ad_f^{k^*-1} g_j^v(x_a) \right\rangle = 0, \ \ 1 \le j \le s.$$

Note that $\operatorname{span}\{dh_i^v, 1 \le i \le s_1\} \in \mathcal{D}_{k^*-1}^{\perp}$ by construction. Thus the above equation implies that $c_{s_1}^v dH_{s_1}^v(x_a) \in \hat{\mathcal{D}}_{k^*}^{\perp}(x_a)$. It follows that, there exists a nonzero vector $b_{m_1}^u \in \mathbb{R}^{m_1}$ such that

$$c_{s_1}^v dH_{s_1}^v(x_a) = b_{m_1}^u dH_{m_1}^u(x_a),$$

which is a contradiction since $dh_i^u, 1 \le i \le m_1$ and $dh_i^v, 1 \le i \le s_1$ are independent. Hence, the whole claim is true.

Subsequently, if $m_1 + s_1 = m + s$, then $T_u^{m_1}(x)$ and $T_v^{s_1}(x)$ are invertible around x_a. Set

$$\begin{aligned}
&T_x(x) = \operatorname{col}\left[H_{m_1}^u, H_{s_1}^v, L_f H_{m_1}^u, L_f H_{s_1}^v, \ldots, L_f^{k^*-1} H_{m_1}^u, L_f^{k^*-1} H_{s_1}^v\right](x),\\
&T_u(x) = T_u^{m_1}(x), \quad T_v(x) = T_v^{s_1}(x),\\
&F_u(x) = L_f^{k^*} H_{m_1}^u(x), \quad F_v = L_f^{k^*} H_{s_1}^v(x), \quad R = \left\langle dL_f^{k^*-1} H_{s_1}^v(x), g^u(x) \right\rangle .
\end{aligned}$$

It can be seen that the system, mapped via the transformations constructed above, is in the Brunovský canonical form with indices $\rho_1 = \rho_2 = \cdots = \rho_m = \bar{\rho}_1 = \bar{\rho}_2 = \cdots = \bar{\rho}_s = k^*$. If $m_1 + s_1 < m + s$, go to the next step.

Step 2: We claim that, arpund x_a,

(b) $\mathcal{D}^{\perp}_{k^*-1} \cap \operatorname{span}\{dL_f h^u_i, 1 \le i \le m_1\} = 0$, and

$$\mathcal{D}^{\perp}_{k^*-1} + \operatorname{span}\{dL_f h^u_i, 1 \le i \le m_1\} \subset \hat{\mathcal{D}}^{\perp}_{k^*-1}. \tag{5.50}$$

(c) $\hat{\mathcal{D}}^{\perp}_{k^*-1} \cap \operatorname{span}\{dL_f h^v_i(x), 1 \le i \le s_1\} = 0$, and

$$\hat{\mathcal{D}}^{\perp}_{k^*-1} + \operatorname{span}\{dL_f h^v_i(x), 1 \le i \le s_1\} \subset \mathcal{D}^{\perp}_{k^*-2}. \tag{5.51}$$

First, we prove that claim (b) is true. Consider (5.50), then $\mathcal{D}^{\perp}_{k^*-1} \subset \hat{\mathcal{D}}^{\perp}_{k^*-1}$ is obvious (since $\hat{\mathcal{D}}_{k^*-1} \subset \mathcal{D}_{k^*-1}$). By (5.47) and (5.48), we have

$$\begin{aligned}
\left\langle dL_f h^u_i(x), ad^k_f g^u_j(x)\right\rangle = 0, \;\; 1 \le i \le s_1, \;\; 1 \le j \le m, \;\; 0 \le k \le k^* - 3,\\
\left\langle dL_f h^u_i(x), ad^k_f g^v_j(x)\right\rangle = 0, \;\; 1 \le i \le s_1, \;\; 1 \le j \le s, \;\; 0 \le k \le k^* - 2,
\end{aligned}$$

which implies that $\{dL_f h^u_i, 1 \le i \le m_1\} \in \hat{\mathcal{D}}^{\perp}_{k^*-1}$, hence (5.50) is true.

Now suppose $\mathcal{D}^{\perp}_{k^*-1}(x_a) \cap \{dL_f h^u_i(x_a), 1 \le i \le m_1\} \ne 0$, which implies that there exists a nonzero row vector $c^u_{m_1} \in \mathbb{R}^{m_1}$ such that $c^u_{m_1} dL_f H^u_{m_1}(x_a) \in \mathcal{D}^{\perp}_{k^*-1}(x_a)$. This would mean

$$\begin{aligned}
c^u_{m_1}\left\langle dL_f H^u_{m_1}(x_a), ad^k_f g^u_j(x_a)\right\rangle = 0, \quad 1 \le j \le m, \;\; 0 \le k \le k^* - 2,\\
c^u_{m_1}\left\langle dL_f H^u_{m_1}(x_a), ad^k_f g^v_j(x_a)\right\rangle = 0, \quad 1 \le j \le s, \;\; 0 \le k \le k^* - 2,
\end{aligned}$$

which in turn, implies by (5.48)

$$\begin{aligned}
c^u_{m_1}\left\langle dH^u_{m_1}(x_a), ad^k_f g^u_j(x_a)\right\rangle = 0, \quad 1 \le j \le m, \;\; 0 \le k \le k^* - 1,\\
c^u_{m_1}\left\langle dH^u_{m_1}(x_a), ad^k_f g^v_j(x_a)\right\rangle = 0, \quad 1 \le j \le s, \;\; 0 \le k \le k^* - 1.
\end{aligned}$$

Thus by the above equations, we have $c^u_{m_1} dH^u_{m_1}(x_a) \in \mathcal{D}^{\perp}_{k^*}$. Then, by $\dim \mathcal{D}_{k^*} = n^*$ and $dH^u_{m_1}(x_a)$ being of full row rank, $c^u_{m_1}$ has to be zero. This contradiction completes the proof of claim (b).

Then we prove claim (c) is true. Consider (5.51), then $D^{\perp}_{k^*-1} \subset \mathcal{D}^{\perp}_{k^*-2}$ is obvious (since $\mathcal{D}_{k^*-2} \subset \hat{\mathcal{D}}_{k^*-1}$). By (5.47) and (5.48), we have

$$\begin{aligned}
\left\langle dL_f h^v_i(x), ad^k_f g^u_j(x)\right\rangle = 0, \;\; 1 \le i \le m_1, \;\; 1 \le j \le m, \;\; 0 \le k \le k^* - 3,\\
\left\langle dL_f h^v_i(x), ad^k_f g^v_j(x)\right\rangle = 0, \;\; 1 \le i \le m_1, \;\; 1 \le j \le s, \;\; 0 \le k \le k^* - 3,
\end{aligned}$$

which implies $\{dL_f h^v_i, 1 \le i \le s_1\} \in \mathcal{D}^{\perp}_{k^*-2}$, hence (5.51) holds.

Now suppose $\hat{\mathcal{D}}^{\perp}_{k^*-1}(x_a) \cap \{dL_f h^v_i(x_a), 1 \le i \le s_1\} \ne 0$, which implies that there exists a nonzero row vector $c^v_{s_1} \in \mathbb{R}^{s_1}$ such that $c^v_{s_1} dL_f H^v_{s_1}(x_a) \in \hat{\mathcal{D}}^{\perp}_{k^*-1}(x_a)$. This would mean

$$c^v_{s_1}\left\langle dL_f H^v_{s_1}(x_a), ad^k_f g^u_j(x_a)\right\rangle = 0, \quad 1 \le j \le m, \;\; 0 \le k \le k^* - 3,$$

$$c^v_{s_1}\left\langle dL_fH^v_{s_1}(x_a), ad^k_f g^v_j(x_a)\right\rangle = 0, \quad 1 \le j \le s, \;\; 0 \le k \le k^* - 2,$$

which in turn, implies by (5.48)

$$\begin{aligned} c^v_{s_1}\left\langle dH^v_{s_1}(x_a), ad^k_f g^u_j(x_a)\right\rangle = 0, \quad 1 \le j \le m, \;\; 0 \le k \le k^* - 2, \\ c^v_{s_1}\left\langle dH^v_{s_1}(x_a), ad^k_f g^v_j(x_a)\right\rangle = 0, \quad 1 \le j \le s, \;\; 0 \le k \le k^* - 1. \end{aligned}$$

Thus, by the above equations, we have $c^v_{s_1}dH^v_{s_1}(x_a) \in \hat{\mathcal{D}}^\perp_{k^*}(x_a)$. It follows that there exists a row vector $b^u_{m_1} \in \mathbb{R}^{m_1}$ such that $c^v_{s_1}dH^v_{s_1}(x_a) = b^u_{m_1}dH^u_{m_1}(x_a)$, which is a contradiction since $dh^u_i, 1 \le i \le m_1$ and $dh^v_i, 1 \le i \le s_1$ are independent by construction. This contradiction completes the proof of claim (c).

Based on claims (b) and (c), define

$$m_2 = (\dim \hat{\mathcal{D}}^\perp_{k^*-1}) - (\dim \mathcal{D}^\perp_{k^*-1}) - m_1, \quad s_2 = (\dim \mathcal{D}^\perp_{k^*-2}) - (\dim \hat{\mathcal{D}}^\perp_{k^*-1}) - s_1.$$

Then choose m_2 scalar functions $h^u_i(x)$, $m_1 + 1 \le i \le m_1 + m_2$, such that

$$\mathcal{D}^\perp_{k^*-1} + \text{span}\left\{dL_fh^u_i, 1 \le i \le m_1\right\} + \text{span}\left\{dh^u_i, m_1+1 \le i \le m_1+m_2\right\} = \hat{\mathcal{D}}^\perp_{k^*-1}, \tag{5.52}$$

and choose s_2 scalar functions $h^v_i(x)$, $s_1 + 1 \le i \le s_1 + s_2$, such that

$$\hat{\mathcal{D}}^\perp_{k^*-1} + \text{span}\left\{dL_fh^v_i, 1 \le i \le s_1\right\} + \text{span}\left\{dh^v_i, s_1+1 \le i \le s_1+s_2\right\} = \mathcal{D}^\perp_{k^*-2}. \tag{5.53}$$

Moreover, set

$$T^{m_2}_u(x) = L_{g^u}L^{k^*-2}_f H^u_{m_2}(x), \quad T^{s_2}_v(x) = L_{g^u}L^{k^*-2}_f H^v_{s_2}(x).$$

where $H^u_{m_2}(x)$ and $H^v_{s_2}$ denote vector-valued functions whose rows consist of $h^u_i(x)$, $m_1 + 1 \le i \le m_1 + m_2$, and $h^v_i(x)$, $s_1 + 1 \le i \le s_1 + s_2$, respectively. We claim that

(d) The $(m_1 + m_2) \times m$ matrix

$$T^{m_1+m_2}_u(x) = \begin{bmatrix} T^{m_1}_u(x) \\ T^{m_2}_u(x) \end{bmatrix}$$

is of full row rank at x_a and the $(s_1 + s_2) \times s$ matrix

$$T^{s_1+s_2}_v(x) = \begin{bmatrix} T^{s_1}_v(x) \\ T^{s_2}_v(x) \end{bmatrix}$$

is of full row rank at x_a. First suppose that $\text{rank}\, T^{m_1+m_2}_u(x_a) < m_1 + m_2$, which implies that there exist some row vectors $c^u_{m_1} \in \mathbb{R}^{m_1}$ and $c^u_{m_2} \in \mathbb{R}^{m_2}$ such that

$$c^u_{m_1}\left\langle dL^{k^*-1}_f H^u_{m_1}(x_a), g^u_j(x_a)\right\rangle + c^u_{m_2}\left\langle dL^{k^*-2}_f H^u_{m_2}(x_a), g^u_j(x_a)\right\rangle = 0, \;\; 1 \le j \le m, \tag{5.54}$$

which implies, by (5.48), that

$$c^u_{m_1}\left\langle dL_fH^u_{m_1}(x_a), ad^{k^*-2}_f g^u_j(x_a)\right\rangle + c^u_{m_2}\left\langle dH^u_{m_2}(x_a), ad^{k^*-2}_f g^u_j(x_a)\right\rangle = 0, \;\; 1 \le j \le m.$$

Thus

$$\left\langle c^u_{m_1} dL_f H^u_{m_1} + c^u_{m_2} dH^u_{m_2}, ad_f^{k^*-2}(x_a) g^u_j(x_a) \right\rangle = 0, \ \ 1 \le j \le m. \tag{5.55}$$

Notice that by (5.52), we have

$$c^u_{m_1} dL_f H^u_{m_1}(x_a) + c^u_{m_2} dH^u_{m_2}(x_a) \in \hat{\mathcal{D}}^{\perp}_{k^*-1}(x_a). \tag{5.56}$$

As a consequence of (5.55) and (5.56), we get

$$c^u_{m_1} dL_f H^u_{m_1}(x_a) + c^u_{m_2} dH^u_{m_2}(x_a) \in \mathcal{D}^{\perp}_{k-1}(x_a),$$

which contradicts the independence of the differentials used in (5.52). Thus $T^{m_1+m_2}_u(x_a)$ is of full row rank. Then suppose $\operatorname{rank} T^{s_1+s_2}_v(x_a) < s_1 + s_2$, which implies that there exist some row vectors $c^v_{s_1} \in \mathbb{R}^{s_1}$, $c^v_{s_1} \in \mathbb{R}^{s_2}$ such that

$$c^v_{s_1} \left\langle dL_f^{k^*-1} H^u_{s_1}(x_a), g^v_j(x_a) \right\rangle + c^v_{s_2} \left\langle dL_f^{k^*-2} H^v_{s_2}(x_a), g^v_j(x_a) \right\rangle = 0, \ \ 1 \le j \le s.$$

By (5.48), the above equation gives

$$c^v_{s_1} \left\langle dL_f H^v_{s_1}(x_a), ad_f^{k^*-2} g^v_j(x_a) \right\rangle + c^v_{s_2} \left\langle dH^v_{s_2}(x_a), ad_f^{k^*-2} g^v_j(x_a) \right\rangle = 0, \ \ 1 \le j \le m.$$

Thus

$$\left\langle c^v_{s_1} dL_f H^v_{s_1}(x_a) + c^v_{s_2} dH^v_{s_2}(x_a), ad_f^{k^*-2} g^v_j \right\rangle = 0 \ \ 1 \le j \le s.$$

Notice that by (5.53), we have $c^v_{s_1} dL_f H^v_{s_1} + c^v_{s_2} dH^v_{s_2} \in \mathcal{D}^{\perp}_{k^*-2}$. As a consequence,

$$c^v_{s_1} dL_f H^v_{s_1}(x_a) + c^v_{s_2} dH^v_{s_2}(x_a) \in \hat{\mathcal{D}}^{\perp}_{k^*-1},$$

which contradicts the independence of the differentials used in (5.53). Therefore, $T^{s_1+s_2}_v(x_a)$ is of full row rank.

Now if $m_1 + m_2 + s_1 + s_2 = m + s$, we have $T^{m_1+m_2}_u(x)$, $T^{s_1+s_2}_v(x)$ are invertible around x_a. Set

$$\begin{aligned}
T_x(x) &= \operatorname{col}\left[L_f^j H^u_{m_i}(x), \ \ L_f^j H^v_{s_i}(x), \ 1 \le i \le 2, \ \ 0 \le j \le k^* - i\right], \\
T_u(x) = T^{m_1+m_2}_u(x), \quad T_v(x) &= T^{s_1+s_2}_v(x), \quad R(x) = \operatorname{col}[L_f^{k^*-1} H^v_{s_1} B_u, L_f^{k^*-2} H^v_{s_2} B_u](x), \\
F_u(x) = \operatorname{col}\left[L_f^{k^*} H^u_{m_1}, L_f^{k^*-1} H^u_{m_2}\right](x), &\quad F_v(x) = \operatorname{col}\left[L_f^{k^*} H^v_{s_1}, L_f^{k^*-1} H^v_{s_2}\right](x).
\end{aligned}$$

Then the transformations given by the above matrices bring $\Sigma^* = (f, g^u, g^v)$ into its Brunovský canonical form with indices

$$\begin{aligned}
\rho_1 = \rho_2 = \cdots = \rho_{m_1} = \bar{\rho}_1 = \bar{\rho}_2 = \cdots = \bar{\rho}_{s_1} = k^*, \\
\rho_{m_1+1} = \rho_{m_1+2} = \cdots = \rho_m = \bar{\rho}_{s_1+1} = \bar{\rho}_{s_2+2} = \cdots = \bar{\rho}_s = k^* - 1.
\end{aligned}$$

If $m_1 + s_1 < m + s$, go to next step.

Step k: After $k-1$ iterations of the above procedure, we can find $(k-1)m_1 + (k-2)m_2 + \cdots + m_{k-1} + (k-1)s_1 + (k-2)s_2 + \cdots + s_{k-1}$ independent row vectors

$$\begin{cases} dh_i^u(x), dL_f h_i^u(x), \ldots, dL_f^{k-2} h_i^u(x), & 1 \le i \le m_1 \\ \ldots & \\ dh_i^u(x), dL_f h_i^u(x), & m_1 + \ldots + m_{k-3} + 1 \le i \le m_1 + \ldots + m_{k-2} \\ dh_i^u(x), & m_1 + \ldots + m_{k-2} + 1 \le i \le m_1 + \ldots + m_{k-1} \\ \hline dh_i^v(x), dL_f h_i^v(x), \ldots, dL_f^{k-2} h_i^v(x), & 1 \le i \le s_1 \\ \ldots & \\ dh_i^v(x), dL_f h_i^v(x), & s_1 + \ldots + s_{k-3} + 1 \le i \le s_1 + \ldots + s_{k-2} \\ dh_i^v(x), & s_1 + \ldots + s_{k-2} + 1 \le i \le s_1 + \ldots + s_{k-1} \end{cases}$$

that span $\mathcal{D}^{\perp}_{k^*-(k-1)}$. Then, define

$$\mathcal{C} = \text{span}\left\{dL_f^j h_i^u(x), m_1 + \cdots + m_{k-j-1} + 1 \le i \le m_1 + \cdots + m_{k-j}, 1 \le j \le k-1\right\}$$

and

$$\hat{\mathcal{C}} = \text{span}\left\{dL_f^j h_i^v(x), \;\; s_1 + \ldots + s_{k-j-1} + 1 \le i \le s_1 + \ldots + s_{k-j}, \;\; 1 \le j \le k-1\right\}.$$

By similar arguments as those used to prove claims (b) and (c) above, we can show that around x_a,

$$\mathcal{D}^{\perp}_{k^*-(k-1)} + \mathcal{C} \in \hat{\mathcal{D}}^{\perp}_{k^*-(k-1)},$$

and that the intersection of the $\mathcal{D}^{\perp}_{k^*-(k-1)}$ and $\mathcal{C}$ is zero,

$$\hat{\mathcal{D}}^{\perp}_{k^*-(k-1)} + \hat{\mathcal{C}} \in \mathcal{D}^{\perp}_{k^*-k}.$$

and that the intersection of the co-distributions $\hat{\mathcal{D}}^{\perp}_{k^*-(k-1)}$ and $\hat{\mathcal{C}}$ is zero. Then, define

$$\begin{aligned} m_k &= (\dim\, \hat{\mathcal{D}}^{\perp}_{k^*-(k-1)}) - (\dim\, \mathcal{D}^{\perp}_{k^*-(k-1)}) - (m_1 + \ldots + m_{k-1}), \\ s_k &= (\dim\, \mathcal{D}^{\perp}_{k^*-k}) - (\dim\, \hat{\mathcal{D}}^{\perp}_{k^*-(k-1)}) - (s_1 + \ldots + s_{k-1}). \end{aligned}$$

We can choose m_k independent row vectors $dh_i^u(x)$, $m_1 + \ldots + m_{k-1} + 1 \le i \le m_1 + \ldots + m_k$, such that

$$\hat{\mathcal{D}}^{\perp}_{k^*-(k-1)} - \mathcal{D}^{\perp}_{k^*-(k-1)} = \mathcal{C} + \text{span}\left\{h_i^u, m_1 + \ldots + m_{k-1} + 1 \le i \le m_1 + \ldots + m_k\right\}.$$

Subsequently, we can choose s_k independent row vectors $h_i^v(x), s_1 + \ldots + s_{k-1} + 1 \le i \le s_1 + \ldots + s_k$, such that

$$\mathcal{D}^{\perp}_{k^*-k} - \hat{\mathcal{D}}^{\perp}_{k^*-(k-1)} = \hat{\mathcal{C}} + \text{span}\left\{h_i^v, s_1 + \ldots + s_{k-1} + 1 \le i \le s_1 + \ldots + s_k\right\}.$$

Then, by a similar argument that used to prove claim (d), it is possible to show that the $(m_1 + m_2 + \cdots + m_k) \times m$ matrix

$$T_u^{m_1+m_2+\cdots+m_k}(x) = \begin{bmatrix} T_u^{m_1}(x) \\ \vdots \\ T_u^{m_k}(x) \end{bmatrix}$$

is of full row rank at x_a, where $T_u^{m_i}(x) = L_{g^u} L_f^{k^*-i} H_{m_i}^u(x)$, $1 \leq i \leq k$. The $(s_1 + s_2 + \cdots + s_k) \times s$ matrix

$$T_v^{s_1+s_2+\cdots+s_k}(x) = \begin{bmatrix} T_v^{s_1}(x) \\ \vdots \\ T_v^{s_k}(x) \end{bmatrix}$$

is of full row rank at x_a, where $T_v^{s_i}(x) = L_{g^v} L_f^{k^*-i} H_{s_i}^v(x)$, $1 \leq i \leq k$.

Now if $m_1 + \cdots + m_k + s_1 + \cdots + s_k = m + s$, we have $T_u^{m_1+m_2+\cdots+m_k}(x)$ and $T_v^{s_1+s_2+\cdots+s_k}(x)$ invertible at x_a. With a similar construction to that given in Step 2, we can construct suitable $T_x(x)$, $T_u(x)$, $T_v(x)$, $F_u(x)$, $F_v(x)$, $R(x)$ such that the transformation given by these matrices brings Σ^* into its Brunovský form for systems with two kinds of inputs (see Corollary 3.4.3 of Chapter 3) with indices:

$$\begin{gathered} \rho_1 = \rho_2 = \cdots = \rho_{m_1} = \bar{\rho}_1 = \bar{\rho}_2 = \cdots = \bar{\rho}_{s_1} = k^*, \\ \rho_{m_1+1} = \rho_{m_1+2} = \cdots = \rho_{m_2} = \bar{\rho}_{s_1+1} = \bar{\rho}_{s_2+2} = \cdots = \bar{\rho}_{s_2} = k^* - 1, \\ \cdots \\ \rho_{m_{k-1}+1} = \rho_2 = \cdots = \rho_{m_k} = \bar{\rho}_{s_{k-1}+1} = \bar{\rho}_{s_{k-1}+2} = \cdots = \bar{\rho}_{s_k} = k^* - k + 1. \end{gathered}$$

If $m_1 + s_1 < m + s$, go to the next step.

Finally, we conclude that one can always find suitable transformation matrices to bring $\Sigma^* = (f, g^u, g^v)$ into its Brunovský canonical form by implementing at most $k = k^*$ steps of the above procedure.

Now we will prove that $\Xi^u|_{M^*}^{red}$, given by (5.45), is locally ex-fb-equivalent to:

$$\Delta^{uc} : \begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar{\rho}} \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\bar{\rho}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho \\ 0 \end{bmatrix} \tilde{u}^*. \tag{5.57}$$

Note that by Lemma 5.4.2, the linear DAECS Δ^{uc}, given by (5.57), is completely controllable. Denote the Brunovský form (for systems with two kinds of inputs) of Σ^* by Σ^*_{Br}. Then by the construction above, Σ^* is locally sys-fb-equivalent to Σ^*_{Br}. Moreover, it is not hard to see that $\Sigma^*_{Br} \in \mathbf{Expl}(\Delta^{uc})$. Furthermore, recall that $\Sigma^* \in \mathbf{Expl}(\Xi^u|_{M^*}^{red})$. Therefore, by Theorem 5.2.9, $\Xi^u|_{M^*}^{red}$ is locally ex-fb-equivalent to Δ^{uc}. Hence Ξ^u is locally completely internal feedback linearizable. □

Proof of Theorem 5.4.6. Only if. Suppose that Ξ^u is locally completely external feedback linearizable. Then Ξ^u is locally ex-fb-equivalent, via $z = \psi(x)$, $Q(x)$ and $u = \alpha^u(x) + \beta^u(x)\tilde{u}$, to a linear completely controllable DAECS:

$$\Delta^{\tilde{u}} : \tilde{E}\dot{z} = \tilde{H}z + \tilde{L}\tilde{u}. \tag{5.58}$$

Thus by Definition 5.2.2, we have

$$\begin{aligned} Q(x)E(x) &= \tilde{E} \cdot \tfrac{\partial \psi(x)}{\partial x}, \\ Q(x)(F(x) + G(x)\alpha^u(x)) &= \tilde{H} \cdot \psi(x), \\ Q(x)G(x)\beta^u(x) &= \tilde{L}. \end{aligned} \tag{5.59}$$

Obviously, $\Delta^{\tilde{u}}$ satisfies condition (EFL1). The system Ξ^u satisfies condition (EFL1) as well because the ranks of $E(x)$ and $[E(x), G(x)]$ are invariant under ex-fb-equivalence. Moreover, the complete controllability of $\Delta^{\tilde{u}}$ implies $\tilde{H}z \in \operatorname{Im}\tilde{E} + \operatorname{Im}L$ (see Lemma 5.4.2). By substituting (5.59), we get

$$Q(x)(F(x) + G(x)\alpha^u(x)) \in \operatorname{Im} Q(x)E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1} + \operatorname{Im} Q(x)G(x)\beta^u(x)$$
$$\Rightarrow F(x) + G(x)\alpha^u(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x) \Rightarrow F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x).$$

Thus, Ξ^u satisfies condition (EFL2). Furthermore, by conditions (EFL1) and (EFL2), and Theorem 5.3.10, it is seen that the locally maximal controlled invariant submanifold $M^* = U$. Now consider the restricted and reduced system $\Delta^{\tilde{u}}|^{red}_{M^*} = \Delta^{\tilde{u}}|^{red}_{U}$, which is a linear completely controllable DAECS without outputs. This means that Ξ^u is locally internally feedback linearizable. Thus by Theorem 5.4.5, Ξ^u satisfies condition (EFL3) on $M^* = U$.

If. Suppose that in a neighborhood U of x^0, Ξ^u satisfies conditions (EFL1)-(EFL3). Let $\operatorname{rank} E(x) = r$ and $\operatorname{rank}[E(x), G(x)] = r + m_2$ and $m_1 = m - m_2$. Then, by (EFL1), there exist an invertible $Q(x)$ defined on U and a partition of $u = (u_1, u_2)$ such that

$$Q(x)E(x)\dot{x} = Q(x)F(x) + Q(x)G(x)u \Rightarrow$$
$$\begin{bmatrix} E^*(x) \\ 0 \\ 0 \end{bmatrix}\dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{bmatrix} + \begin{bmatrix} G_1^1(x) & G_1^2(x) \\ G_2^1(x) & G_2^2(x) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $E^*(x)$ is of full row rank r and $G_2^2(x)$ is a $m_2 \times m_2$ invertible matrix-valued function defined on U. Moreover, by condition (EFL2), we have $F_3(x) = 0$ for $x \in U$. Now use the feedback transformation

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} I_{m_1} & 0 \\ G_2^1(z_1) & G_2^2(z_1) \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ F_2(z_1) \end{bmatrix} + \begin{bmatrix} I_{m_1} & 0 \\ G_2^1(z_1) & G_2^2(z_1) \end{bmatrix}^{-1}\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix},$$

and the system becomes

$$\begin{bmatrix} E^*(x) \\ 0 \\ 0 \end{bmatrix}\dot{x} = \begin{bmatrix} F_1(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} G_1^1(x) & G_1^2(x) \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix},$$

Premultiply the above equation by $\begin{bmatrix} I_r & -G_1^2(x) \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix}$ to get

$$\begin{bmatrix} E^*(x) \\ 0 \\ 0 \end{bmatrix}\dot{x} = \begin{bmatrix} F^*(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} G^*(x) & 0 \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u^* \\ \bar{u}^* \end{bmatrix}, \tag{5.60}$$

where $F^* = F_1$, $G^* = G_1^1$, $u_1^* = \tilde{u}_1$ and $u_2^* = \tilde{u}_2$.

By Definition 5.3.4 and 5.3.6, it is seen that $\Xi^u|^{red}_{M^*} = \Xi^u|^{red}_{U}$ is the following system:

$$E^*(x)\dot{x} = F^*(x) + G^*(x)u^*.$$

By condition (EFL3) and Theorem 5.4.5, $\Xi^u|_{M^*}^{red}$ is locally ex-fb-equivalent to a linear DAECS Δ^{cu} of (5.57). It can bee seen from (5.60) that Ξ^u is locally ex-fb-equivalent to

$$\begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} E_\rho & 0 \\ 0 & 0 \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix},$$

which is completely controllable by Lemma 5.4.2. Therefore, Ξ^u is locally completely external feedback linearizable. □

5.7 Conclusions and perspectives

In this chapter, we propose a maximal controlled invariant submanifold form (a normal form) for nonlinear DAECSs, which is our first main result. This form requires only the existence of a maximal controlled invariant submanifold and some constant rank assumptions of system matrices. Moreover, we give necessary and sufficient conditions to the problem for a nonlinear DAECS to be locally internally (second main result) or externally (third main result) feedback equivalent to a completely controllable linear one. The conditions are based on an ODECS given by the explicitation with driving variables procedure. Some examples are given to illustrate the construction of the maximal controlled invariant submanifold form, and how to externally or internally feedback linearize a nonlinear DAECS.

A natural problem for future works is that of when a nonlinear DAE system is ex-fb-equivalent to a linear one which is not necessarily completely controllable. Actually, this problem is more involved than the problem of complete external feedback linearization. Indeed, since in Theorem 5.4.6, the maximal controlled invariant submanifold M^* on U is $M^* = U$, it follows that the algebraic constraints are directly governed by some variables of u. Thus the in-fb-equivalence is very close to the ex-fb-equivalence. However, if $M^* \neq U$, then the algebraic constraints may affect the generalized state. Moreover, since the explicitation is defined up to a generalized output injection, it may happen that one system of the explicitation is feedback linearizable but another is not. The general feedback linearizability problem remains open and, in view of the above points, is challenging.

Chapter 6

Internal and External Linearization of Semi-Explicit Differential Algebraic Equations

Abstract: In this chapter, we study two kinds of linearization (internal and external) of nonlinear differential-algebraic equations DAEs of semi-explicit SE form. The difference of external and internal linearization is illustrated by an example of a mechanical system. Moreover, we define different levels of external equivalence for two SE DAEs. A proposed explicitation procedure allows us to treat a given SE DAE as a control system defined up to feedback transformation (a class of control systems). Then sufficient and necessary conditions, expressed via explicitation procedure, are given to describe when a given SE DAE is level-3 externally equivalent to a linear SE DAE of some specific forms. At last, we show by an example that level-2 external linearization can be achieved if its explicitation is level-2 input-output linearizable.

6.1 Introduction

We study differential-algebraic equations DAEs of semi-explicit SE form

$$\Xi^{se}: \begin{cases} \mathcal{R}(x)\dot{x} &= a(x) \\ 0 &= c(x), \end{cases} \tag{6.1}$$

where $\mathcal{R}(x)$, $a(x)$, and $c(x)$ are smooth maps with values in $\mathbb{R}^{r\times n}$, $\mathbb{R}^r$, and $\mathbb{R}^p$, respectively, and the word smooth will mean throughout $\mathcal{C}^\infty$-smooth, and where $x \in X$ is called the generalized state and X is an open subset of $\mathbb{R}^n$. A SE DAE of form (6.1) will be denoted by $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ or, simply, Ξ^{se}. A solution of Ξ^{se} is a curve $x(t) \in \mathcal{C}^1(I; X)$ with an open interval I such that for all $t \in I$, $x(t)$ solves (6.1). An admissible point of (6.1) is a point $x_0 \in X$ such that through x_0, there passes at least one solution. The motivation of studying SE DAEs is their presence in modeling of electrical circuits [165], chemical processes [120] and constrained mechanical systems [35],[141], etc.

Definition 6.1.1. (External equivalence). Consider two SE DAEs $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ and $\tilde{\Xi}^{se}_{n,r,p} = (\tilde{\mathcal{R}}, \tilde{a}, \tilde{c})$. If there exists a diffeomorphism $\psi : X \to \tilde{X}$ and a smooth invertible

$r \times r$-matrix $Q^a(x)$ such that

$$\tilde{\mathcal{R}}(\psi(x)) = Q^a(x)\mathcal{R}(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1},$$
$$\tilde{a}(\psi(x)) = Q^a(x)a(x),$$

and if, additionally,

(i) there exists a smooth invertible $p \times p$-matrix $Q^c(x)$ such that $\tilde{c}(\psi(x)) = Q^c(x)c(x)$, we call Ξ^{se} and $\tilde{\Xi}^{se}$ externally equivalent, or shortly ex-equivalent, of level-1;

(ii) there exists a smooth invertible $p \times p$-matrix $Q^c(x)$ such that $\tilde{c}(\psi(x)) = Q^c(x)c(x)$ and $Q^c(x) = S(c(x))$ for some invertible $S(x)$, we call Ξ^{se} and $\tilde{\Xi}^{se}$ ex-equivalent of level-2;

(iii) there exists a constant invertible $p \times p$-matrix T such that $\tilde{c}(\psi(x)) = Tc(x)$, we call Ξ^{se} and $\tilde{\Xi}^{se}$ ex-equivalent of level-3.

The level i ($i = 1, 2, 3$) ex-equivalence of two SE DAEs will be denoted by $\Xi^{se} \overset{ex-i}{\sim} \tilde{\Xi}^{se}$. If $\psi : X_0 \to \tilde{X}_0$ is a local diffeomorphism between neighborhoods X_0 of x_0 and $\tilde{X}_0$ of $\tilde{x}_0$, and $Q^a(x)$, $Q^c(x)$ are defined locally on X_0, we will speak about local ex-equivalence.

Remark 6.1.2. For SE DAEs, we introduce three kinds of output multiplication which correspond to three levels of external equivalence. The interpretation of the three level ex-equivalence is as follows.

(i) Two constraints $0 = c(x)$ and $0 = \tilde{c}(x)$ are level-1 ex-equivalent if and only if $M_0 = \tilde{M}_0$, where $M_0 = \{x \,|\, c(x) = 0\}$ and $\tilde{M}_0 = \{x \,|\, \tilde{c}(x) = 0\}$;

(ii) Two constraints are level-2 ex-equivalent means that the foliations M_d and $\tilde{M}_{\tilde{d}}$ coincide, where $d, \tilde{d} \in \mathbb{R}^p$, $M_d = \{x \,|\, c(x) = d\}$ and $\tilde{M}_{\tilde{d}} = \left\{x \,|\, \tilde{c}(x) = \tilde{d}\right\}$, i.e., there exists a diffeomorphism ϕ such that $\tilde{M}_{\tilde{d}} = M_{\phi(d)}$. It also implies that the set of motions $x(t)$ respecting the constraint $c(x) = d$ (equivalently, $dc(x(t)) \cdot \dot{x}(t) = 0$) coincides with that respecting $\tilde{c}(x) = \tilde{d}$;

(iii) Two constraints are level-3 ex-equivalent means the foliations M_d and $\tilde{M}_{\tilde{d}}$ coincide via a linear parameter transformation, i.e., $\tilde{M}_{\tilde{d}} = M_{Td}$.

There are two kinds of equivalence relations for DAEs, namely, external equivalence and internal equivalence (for the details of internal equivalence, we refer Chapter 2 and Chapter 3). We will show the differences of these two equivalent relations for SE DAEs in Section 6.3 by examples. Roughly speaking, the word "internal" means that we consider the DAE on its constrained submanifold [162] only (also called invariant submanifold in Chapter 3 and 5, see also [48] and [45], or configuration subspace [177]), i.e., where the solutions of the DAE exist. Correspondingly, the word "external" means that we consider the DAE in a whole neighborhood and for some points in that neighborhood there may not exist solutions. More precisely, solutions of $\mathcal{R}(x)\dot{x} = a(x)$ pass through each point of the neighborhood but may not respect the algebraic constraint $c(x) = 0$. Therefore, external equivalence is interesting for all problems, where the nominal point does not respect the constraints but we want to steer the solution towards the constraint (in finite time or asymptotically). So the form of the DAE matters not only on the constraint set but in a neighborhood as well.

The purpose of this chapter is to discuss when a SE DAE, given by (6.1), is locally equivalent to a linear SE DAE. Some results for linearization of DAEs can be found in [111],[101], however, the concepts of external and internal equivalence are not mentioned in those papers. In the present chapter, we will use a new tool named *explicitation* (see Definition 6.3.1) to represent DAEs as explicit control systems. As shown in the examples of Section 6.3, the internal linearizability has direct relations with the feedback linearizability of the explicit control system on its maximal output zeroing submanifold. For the external linearizability, we only consider level-3 and level-2 external equivalence, level-1 will be discussed in future. The level-3 external linearizability of SE DAEs is closely related to the involutivity of some distributions of an explicit control system (obtained via *explicitation*), as is shown in Section 6.4. Moreover, in Section 6.5 we provide an example of a system that is level-2 externally linearizable but not level-3 externally linearizable.

6.2 Some results for the linear case

In this section, we introduce some concepts of linear semi-explicit DAEs of form

$$\Delta^{se}: \begin{cases} R\dot{x} = Ax \\ \ 0 \ = Cx, \end{cases} \tag{6.2}$$

where $R \in \mathbb{R}^{r\times n}$, $A \in \mathbb{R}^{r\times n}$, $C \in \mathbb{R}^{p\times n}$. We assume R to be of full row rank. A DAE of form (6.2) will be denoted by $\Delta^{se}_{r,n,p} = (R, A, C)$ or, simply, Δ^{se}. From the Kronecker canonical form KCF, see e.g. [117] or [11], for matrix pencils $sE - H$ (or equivalently, for linear DAEs $E\dot{x} = Hx$), the following canonical form SCF can be deduced for linear SE DAEs. Definition 6.1.1 applied to linear systems says that two linear SE DAEs $\Delta^{se} = (R, A, C)$ and $\tilde{\Delta}^{se} = (\tilde{R}, \tilde{A}, \tilde{C})$ are ex-equivalent if there exists constant invertible matrices P, Q^a, Q^c such that $\tilde{R} = Q^aRP^{-1}$, $\tilde{A} = Q^aAP^{-1}$, $\tilde{C} = Q^cCP^{-1}$.

Proposition 6.2.1. *Any linear SE DAE $\Delta^{se}_{r,n,p} = (R, A, C)$ is ex-equivalent to the following semi-explicit canonical form:*

$$SCF: \begin{cases} \dot{z}^1 = A^1z^1 + B^1w^1 & +K^1y \\ \dot{z}^2 = A^2z^2 & +K^2y \\ \dot{z}^3 = A^3z^3 + B^3w^3 & +K^3y \\ \dot{z}^4 = A^4z^4 & +K^4y \\ \ 0 \ = C^3z^3 + D^3w^3 & \\ \ 0 \ = C^4z^4, & \end{cases}$$

where $y = (y^3, y^4)$, $y^3 = C^3z^3 + D^3w^3$ and $y^4 = C^4z^4$. Up to injection terms

$$(K^1y, K^2y, K^3y, K^4y),$$

for each $k = 1, 2, 3, 4$, the equation for $\dot{z}^k$ consists of, respectively, a,b,c,d equations of the form

$$\begin{aligned} \dot{z}_i &= A_i^2 z_i, \quad z_i \in z^2, \\ z_i^{(\mu_i)} &= \begin{cases} w_i, & z_i \in z^1, z^3, \\ 0, & z_i \in z^4, \end{cases} \\ 0 &= \begin{cases} z_i, & z_i \in z^3, z^4, \ \ \mu_i \neq 0, \\ w_i, & z_i \in z^3, \ \ \mu_i = 0. \end{cases} \end{aligned}$$

where A_i^2 are constant matrices in the Jordan canonical form of real matrices.

Proof. From the theory of the Kronecker canonical form, there always exist invertible Q, P such that the DAE $E\dot{x} = Hx$ given by

$$E = Q \begin{bmatrix} R \\ 0 \end{bmatrix} P^{-1} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad H = Q \begin{bmatrix} A \\ C \end{bmatrix} P^{-1} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix},$$

is in the Kronecker canonical form, then by row permutations, we put all the algebraic constrains "$0 = *$" at the bottom of the system (thus the matrix E keeps the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$). Since E is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ and R is full row rank, it is not hard to see that Q has to be of a triangular form (as Q has to preserve the zeros in the lower part of $\begin{bmatrix} R \\ 0 \end{bmatrix}$), i.e., $Q = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix}$, where $Q_1 \in \mathbb{R}^{r \times r}$ and Q_2, Q_4 are of appropriate sizes. Now in view of Definition 6.1.1, in the semi-explicit DAE case, we put $Q_2 = 0$. Thus, using Q_1, Q_4, P, we can transform Δ^{se} into

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} z + \begin{bmatrix} -Q_2(Q_4)^{-1}y \\ 0 \end{bmatrix},$$

which is the desired form SCF. □ □

Remark 6.2.2. The indices μ_i, $i = 1, 2, 3, 4$ together with the numbers a, b, c, d of equations are the Kronecker indices of the matrix pencil $\left(s \begin{bmatrix} R \\ 0 \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} \right)$. The above canonical form differs from the KCF (after some row permutations) only by the drift injection terms (K^1y, K^2y, K^3y, K^4y).

Remark 6.2.3. If we regard the algebraic constraint as the zero output of the control system, the above canonical form coincides with the Morse canonical form MCF [146] (under coordinates change and feedback transformation only, without output injection) for linear control systems.

Now let $\mathscr{M}^*$ be the largest subspace $\mathscr{M}$ such that $\begin{bmatrix} A \\ C \end{bmatrix} \mathscr{M} \subseteq \begin{bmatrix} R \\ 0 \end{bmatrix} \mathscr{M}$. The Wong sequences $\mathscr{V}_i$ and $\mathscr{W}_i$ (see [191] and [11]) of Δ^{se} are defined as:

$$\mathscr{V}_0 := \mathbb{R}^n, \quad \mathscr{V}_{i+1} := \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \begin{bmatrix} R \\ 0 \end{bmatrix} \mathscr{V}_i, \quad i \in \mathbb{N},$$

$$\mathscr{W}_0 := \{0\}, \quad \mathscr{W}_{i+1} := \begin{bmatrix} R \\ 0 \end{bmatrix}^{-1} \begin{bmatrix} A \\ C \end{bmatrix} \mathscr{W}_i, \quad i \in \mathbb{N}.$$

The limits of $\mathscr{V}_i$ and $\mathscr{W}_i$ are denoted by $\mathscr{V}^*$ and $\mathscr{W}^*$, respectively. Notice that the solutions of Δ^{se} exist on $\mathscr{M}^*$ only and, moreover, $\mathscr{M}^* = \mathscr{V}^*$ (see, e.g. Chapter 2). Now we introduce the following regularity and reachability concepts (compare [17]).

Definition 6.2.4. $\Delta^{se}_{r,n,p} = (R, A, C)$ is called

- ***internally regular***, if $\forall\, x^0 \in \mathscr{M}^*$, $\exists$ only one solution $x(t)$ such that $x(0) = x^0$,
- ***regular***, if it is *internally regular* and $r + p = n$,
- ***internally reachable***, if $\forall\, x^0, x^e \in \mathscr{M}^*$, $\exists\, t_e > 0$ and a solution $x(t)$ of Δ^{se} such that $x(0) = x^0$ and $x(t_e) = x^e$,
- ***constraint-freely reachable***, if $\forall x^0, x^e \in \mathbb{R}^n$, $\exists\, t_e > 0$ and a solution $x(t)$ of $R\dot{x} = Ax$ such that $x(0) = x^0$ and $x(t_e) = x^e$.

Lemma 6.2.5. $\Delta^{se}_{r,n,p} = (R, A, C)$ *is*

(i) internally regular $\Leftrightarrow \dim \mathscr{V}^* = \dim(R\mathscr{V}^*) \Leftrightarrow \mathscr{V}^* \cap \mathscr{W}^* = 0 \Leftrightarrow$ *the subsystem* z^1 *in SCF is absent,*

(ii) regular $\Leftrightarrow \mathscr{V}^* \cap \mathscr{W}^* = 0$ *and* $\mathscr{V}^* \oplus \mathscr{W}^* = \mathbb{R}^n \Leftrightarrow$ *the subsystems* z^1 *and* z^4 *in SCF are absent,*

(iii) internally reachable $\Leftrightarrow \mathscr{V}^* \subseteq \mathscr{W}^* \Leftrightarrow$ *the subsystems* z^2 *in SCF is absent,*

(iv) constraint-freely reachable $\Leftrightarrow R\dot{x} = Ax$ *is internally reachable.*

The above lemma can be easily proved using the SCF described in Proposition 6.2.1. The purpose of this lemma is to show how the concepts of Definition 6.2.4 correspond to certain forms of linear SE DAEs and that they are closely related to the Wong sequences.

6.3 Explicitation and internal linearization

We start this section by the definition of *explicitation* for SE DAEs. Throughout the chapter, we will assume that $\mathcal{R}(x)$ is of full row rank equal to r in a neighborhood X_0 of the nominal point x_0.

Definition 6.3.1. (Explicitation) For $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$, set $m = n - r$. Then the *explicitation* of Ξ^{se}, denoted by **Expl**(Ξ^{se}), is a class of control systems of the following form:

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)v \\ y = h(x), \end{cases} \tag{6.3}$$

where $v \in \mathbb{R}^m$ is called the driving variable, $h(x)$ is a smooth $\mathbb{R}^p$-valued function on X_0, and where $f, g_1, \ldots, g_m$ are smooth vector fields on X_0 satisfying

$$f(x) = R^\dagger(x)a(x), \ \ \mathrm{Im} g(x) = \ker \mathcal{R}(x), \ \ h(x) = c(x).$$

Above $R^\dagger(x)$ is a right inverse of $\mathcal{R}(x)$, i.e., $R(x)R^\dagger(x) = I_r$ and $g = (g_1, \ldots, g_m)$. We will denote control system (6.3) by $\Sigma_{n,m,p} = (f, g, h)$ or, simply, Σ.

Notice that $\mathbf{Expl}(\Xi^{se})$ is a class of control systems. Indeed, f is given up to $\ker \mathcal{R}(x)$ and the distribution spanned by $g_1, \ldots, g_m$ is given uniquely but not the vector fields $g_1, \ldots, g_m$ themselves. We will use the notation $\Sigma \in \mathbf{Expl}(\Xi^{se})$ to indicate that control system (6.3) belongs to the class of *explicitation* of Ξ^{se}. By setting $y = 0$ for system (6.3), we get a SE DAE parametrized by the driving variable v. The definition of f and g implies that $\dot{x} = f(x) + g(x)v$ and $\mathcal{R}(x)\dot{x} = a(x)$ have the same solutions. Thus, via *explicitation*, we can study the solutions of Σ yielding a zero output instead of studying the solutions of Ξ^{se} directly. Since the *explicitation* allows to treat a SE DAE as a class of control systems, we give the definition of equivalence for control systems.

Definition 6.3.2. (System equivalence) Consider two control systems $\Sigma_{n,m,p} = (f, g, h)$ and $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$ defined on X and $\tilde{X}$, respectively. If there exists a diffeomorphism $\psi : X \to \tilde{X}$, an $\mathbb{R}^m$-valued function $\alpha(x)$ and an invertible $m \times m$-matrix-valued function $\beta(x)$ satisfying

$$\begin{aligned} \tilde{f}(\psi(x)) &= \tfrac{\partial \psi(x)}{\partial x} \left(f + g\alpha \right)(x), \\ \tilde{g}(\psi(x)) &= \tfrac{\partial \psi(x)}{\partial x} (g\beta)(x), \end{aligned}$$

and, additionally,

(i) there exists a smooth invertible $p \times p$-matrix $T(x)$ such that $\tilde{h}(\psi(x)) = T(x)h(x)$, we call Σ and $\tilde{\Sigma}$ system equivalent, shortly sys-equivalent, of level-1;

(ii) there exists a diffeomorphism $\varphi : \mathbb{R}^p \to \mathbb{R}^p$ such that $\tilde{h}(\psi(x)) = \varphi(h(x))$, we call the two control systems sys-equivalent of level-2;

(iii) there exists a constant invertible matrix T such that $\tilde{h}(\psi(x)) = Th(x)$, we call the two control systems sys-equivalent of level-3.

The sys-equivalence of level-i ($i = 1, 2, 3$) of two control systems will be denoted by $\Sigma \overset{sys-i}{\sim} \tilde{\Sigma}$. If $\psi : X_0 \to \tilde{X}_0$ is a local diffeomorphism between neighborhoods X_0 of x_0 and $\tilde{X}_0$ of x_0, φ is a local diffeomorphism around $h(x_0)$, and $\alpha(x)$, $\beta(x)$ are defined locally on X_0, we will speak about local sys-equivalence.

Actually the above defined system equivalence for two nonlinear control systems of the form (6.3) is widely considered in nonlinear control theory, e.g., [139, 96, 92, 151]. The following result is essential since it connects control systems with SE DAEs.

Proposition 6.3.3. *(i) Consider two control systems* $\Sigma_{n,m,p} = (f, g, h)$ *and* $\Sigma_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$*, that are belong to* explicitation *class of* $\Xi^{se}_{n,r,p}$*, i.e.* $\Sigma, \tilde{\Sigma} \in$ ***Expl***(Ξ^{se})*. Then there exist* $\alpha(x)$*,* $\beta(x)$ *such that*

$$\tilde{f}(x) = f(x) + g(x)\alpha(x), \ \tilde{g}(x) = g(x)\beta(x).$$

*(ii) Two SE DAEs $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ and $\tilde{\Xi}^{se}_{n,r,p} = (\tilde{\mathcal{R}}, \tilde{a}, \tilde{c})$ are ex-equivalent of level-2 (respectively, level-3) if and only if two control systems $(f, g, h) = \Sigma \in$ **Expl**(Ξ^{se}) and $(\tilde{f}, \tilde{g}, \tilde{h}) = \tilde{\Sigma} \in$ **Expl**$(\tilde{\Xi}^{se})$ are sys-equivalent of level-2 (respectively, level-3).*

Remark 6.3.4. If Σ and $\tilde{\Sigma}$ are as in Proposition 6.3.3(i), then obviously $h(x) = c(x) = \tilde{h}(x)$.

Proof. (i) By Definition 6.3.1, $\mathrm{Im} g(x) = \mathrm{Im} \tilde{g}(x) = \ker \mathcal{R}(x)$. Thus there exists an invertible $\beta(x)$ such that $\tilde{g}(x) = g(x)\beta(x)$ (since $\beta(x)$ preserves images). Then from Definition 6.3.1, both $f(x)$ and $\tilde{f}(x)$ are given as $R^{\dagger}a(x)$ but for different choices of the right inverse $\mathcal{R}^{\dagger}$. Actually, at each x, $f(x)$ and $\tilde{f}(x)$ are two solutions Z of the equation $\mathcal{R}(x)Z = a(x)$ and thus their difference $\tilde{f}(x) - f(x)$ is in $\ker \mathcal{R}(x)$ implying there exists $\alpha(x)$ such that $\tilde{f}(x) = f(x) + g(x)\alpha(x)$.

(ii): We will only prove the more general case, which is the level-2 case:
If. Suppose that $\Sigma \overset{sys-2}{\sim} \tilde{\Sigma}$. By $\tilde{g}(\psi(x)) = \frac{\partial \psi(x)}{\partial x}(g\beta)(x)$ from Definition 6.3.2 and $\mathrm{Im} g(x) = \ker \mathcal{R}(x)$, $\mathrm{Im}\tilde{g}(x) = \ker \tilde{\mathcal{R}}(x)$ from Definition 6.3.1, it can be deduced that there exists an invertible matrix $Q^a(x)$ such that $\tilde{\mathcal{R}}(x) = Q^a(x)\mathcal{R}(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}$. Moreover, we have $\tilde{f}(\psi(x)) = \frac{\partial \psi(x)}{\partial x}\left(f + g\alpha\right)(x)$ which implies that

$$(\tilde{\mathcal{R}}^{\dagger}\tilde{a})(x) = \frac{\partial \psi(x)}{\partial x}\left(\mathcal{R}^{\dagger}a + g\alpha\right)(x).$$

Left-multiply the above equation by $\tilde{\mathcal{R}}(x)$, we get $\tilde{a}(x) = Q^a(x)a(x)$. Now by $\tilde{h}(x) = \varphi(h(x))$ from Definition 6.3.2, we have $\tilde{c}(x) = \varphi(c(x))$. Then choose coordinates $(y, z) = (y_1, \ldots, y_p, z_1, \ldots, z_{n-p})$, where $y_i = c^i(x)$. We have $\tilde{c}^i(x) = \varphi^i(y)$. Denote $\varphi^i_1 = \varphi^i$, $\varphi^i_2(y_2, \ldots, y_p) = \varphi^i_1(0, y_2, \ldots, y_p)$ and $\phi^i_1 = \varphi^i_1 - \varphi^i_2$. Now $\phi^i_1(0, y_2, \ldots, y_p) \equiv 0$ and by the Taylor series expansion with respect to y_1, we have $\phi^i_1 = y_1 Q^i_1(y)$ and thus $\varphi^i = y^1 Q^i_1(y) + \varphi^i_2$. Repeat the above procedure replacing φ^i_1 by φ^i_2, i.e., set $\varphi^i_3(y_3, \ldots, y_p) = \varphi^i_2(0, y_3, \ldots, y_p)$ and $\phi^i_2 = \varphi^i_2 - \varphi^i_3$. Since $\phi^i_2(0, y_3, \ldots, y_p) \equiv 0$, by the Taylor series expansion, we have $\phi^i_2 = y_2 Q^i_2(y)$. By an induction argument, we get $\tilde{c}^i = \varphi^i(y) = y^j Q^i_j(y)$, that is, $\tilde{c}^i = c^j Q^i_j(y)$, where $Q^i_j = Q^i_j(c(x))$. Therefore, by Definition 6.1.1, we have $\Xi^{se} \overset{ex-2}{\sim} \tilde{\Xi}^{se}$.

Only if. Suppose that $\Xi^{se} \overset{ex-2}{\sim} \tilde{\Xi}^{se}$. From $\tilde{\mathcal{R}}(x) = Q^a(x)\mathcal{R}(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}$ of Definition 6.1.1 and $\mathrm{Im} g(x) = \ker \mathcal{R}(x)$, $\mathrm{Im}\tilde{g}(x) = \ker \tilde{\mathcal{R}}(x)$ of Definition 6.3.1, it can be deduced that there exists an invertible $\beta(x)$ such that $\tilde{g}(\psi(x)) = \frac{\partial \psi(x)}{\partial x}(g\beta)(x)$. Moreover, we have

$$f(x) = (\mathcal{R}^{\dagger}a)(x) \ \text{ and } \ \tilde{f} = \tilde{R}^{\dagger}\tilde{a} = \frac{\partial \psi(x)}{\partial x}(R^{-1}(Q^a)^{-1}Q^a a)(x) = \frac{\partial \psi(x)}{\partial x}(\mathcal{R}^{\dagger}a)(x).$$

It follows that $f(x)$ and $\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}\tilde{f}(x)$ are two solutions Z of the equation $\mathcal{R}(x)Z = a(x)$. Thus their difference $\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}\tilde{f}(x) - f(x)$ is in $\ker \mathcal{R}(x)$, implying there exists $\alpha(x)$ such that $\tilde{f}(x) = \frac{\partial \psi(x)}{\partial x}(f + g\alpha)(x)$. Furthermore, $\tilde{c}(\psi(x)) = Q^c(x)c(x)$ and $Q^c(x) = Q(c(x))$

of Definition 6.1.1 implies that there exists an invertible matrix $P(x)$ such that $d\tilde{h} = Pdh$ (with $P = \frac{\partial(Q(y)y)}{\partial y}$, where $y = c(x)$). Finally, it can be seen from $d\tilde{h} = Pdh$ that there exists a diffeomorphism φ such that $\tilde{h}(x) = \varphi(h(x))$. Therefore, by Definition 6.3.2, we have $\Sigma \overset{sys-2}{\sim} \tilde{\Sigma}$. □

Now we apply the above defined *explicitation* to the internal analysis of SE DAEs. For a SE DAE Ξ^{se}, a submanifold M^* is called a *maximal invariant submanifold* (for details, see Chapter 3) if M^* is the largest submanifold of X such that $\forall\, x_0 \in M^*$, $\exists\, x(t)$ such that $x(0) = x_0$. M^* can be seen as a nonlinear generalization of the invariant space $\mathscr{M}^*$ for linear DAEs. But note that $\mathscr{M}^*$ always exists while M^* may not exist. Denote by $\Xi^{se}|_{M^*}$ a semi-explicit DAE Ξ^{se} restricted to its maximal invariant submanifold M^*.

Definition 6.3.5. (Internal equivalence) Consider two SE DAEs $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ and $\tilde{\Xi}^{se}_{n,r,p} = (\tilde{\mathcal{R}}, \tilde{a}, \tilde{c})$. Let M^* and $\tilde{M}^*$ be their maximal invariant submanifolds. We call Ξ^{se} and $\tilde{\Xi}^{se}$ internally equivalent, shortly in-equivalent, if $\Xi^{se}|_{M^*}$ and $\tilde{\Xi}^{se}|_{\tilde{M}^*}$ are ex-equivalent.

Theorem 6.3.6. *For $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$, the followings are equivalent:*

(i) Ξ^{se} is in-equivalent to a linear SE DAE Δ^{se}, given by equation (6.2);

*(ii) A (and then any) control system $(f^*, g^*) = \Sigma^* \in$ **Expl**$(\Xi^{se}|_{M^*})$ is feedback linearizable;*

(iii) The linearizability distributions $G_i(\Sigma^*)$, given by (6.13) below, are involutive and of constant rank and $G^*(\Sigma^*) = TM^*$.

The following example illustrates the above theorem. Note that in Chapter 3, it is proved the maximal invariant submanifold M^* of DAEs coincide with the output zeroing submanifold of any control system in its *explicitation* class.

Example 6.3.7. (The Kapitsa pendulum with auxiliary controls). Consider the following equation of the Kapitsa pendulum taken from [68].

$$\begin{cases} \dot{\alpha} = p + \frac{u_1}{l}\sin\alpha \\ \dot{p} = \left(\frac{g}{l} - \frac{(u_1)^2}{l^2}\cos\alpha - \frac{(u_2)^2}{2l^2}\cos\alpha\right)\sin\alpha - \frac{u_1}{l}p\cos\alpha \\ \dot{z} = u_1. \end{cases} \tag{6.4}$$

We subject the system to two different holonomic constraints and analyze the modified system from the DAE point of view.

Case 1: Consider the following holonomic constraint:

$$z + l\cos\alpha = c_{10}, \tag{6.5}$$

where c_{10} denotes a fixed constant. This holonomic constraint assures that the end joint of the pendulum keeps the same vertical position as its initial point. Now combine equations (6.4) and (6.5), and denote $x = (x_1, \ldots, x_5)$, where

$$x_1 = \alpha, \;\; x_2 = p, \;\; x_3 = z, \;\; x_4 = u_1, \;\; x_5 = u_2. \tag{6.6}$$

We get the following SE DAE:

$$\Xi_1^{se}:\left\{\begin{array}{rcl}\mathcal{R}_1(x)\dot{x} &=& a_1(x)\\ 0 &=& c_1(x),\end{array}\right. \tag{6.7}$$

where

$$\mathcal{R}_1(x)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\end{bmatrix},\quad c_1(x)=x_3+l\cos x_1-c_{10},$$
$$a_1(x)=\begin{bmatrix}x_2+\frac{x_4}{l}\sin x_1\\ \left(\frac{g}{l}-\frac{(x_4)^2}{l^2}\cos x_1-\frac{(x_5)^2}{2l^2}\cos x_1\right)\sin x_1-\frac{x_4}{l}x_2\cos x_1\\ x_4\end{bmatrix}.$$

Consider the above DAE around an admissible point $x_0=(x_{10},\dots,x_{50})$ such that

$$x_{50}\cos x_{10}\sin x_{10}\neq 0.$$

The *explicitation* of DAE (6.7) contains the following control system, see Definition 6.3.1, denoted by $\Sigma_1=(f_1,g_1,h_1)$, with driving variables $v_1=\dot{x}_4$, $v_2=\dot{x}_5$,

$$\Sigma_1:\left\{\begin{array}{l}\dot{x}=\begin{bmatrix}a_1(x)\\0\\0\end{bmatrix}+\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}\\ y=x_3+l\cos x_1-c_{10}.\end{array}\right. \tag{6.8}$$

Recall that the *explicitation* of our DAE is the above control system defined up to feedback transformation. By the zero dynamics algorithm (see [92]), the maximal output zeroing submanifold of Σ_1, denoted by M_1^*, can be expressed as:

$$M_1^*=\left\{x\,|\,x_3+l\cos x_1-c_{10}=x_4\cos^2 x_1-lx_2\sin x_1=0\right\}.$$

Notice that $x_0\in M_1^*$. Then system (6.8) restricted on M_1^* is

$$\left\{\begin{array}{l}\dot{x}_1=\frac{x_2}{\cos^2 x_1}\\ \dot{x}_2=\left(\frac{g}{l}-\frac{(x_2)^2}{\cos^3 x_1}-\frac{(x_5)^2}{2l^2}\cos x_1\right)\sin x_1\\ \dot{x}_5=v_2.\end{array}\right. \tag{6.9}$$

System (6.9) is locally static feedback equivalent to the following chained form around x_0:

$$\dot{\tilde{x}}_1=\tilde{x}_2,\quad \dot{\tilde{x}}_2=\tilde{x}_5,\quad \dot{\tilde{x}}_5=\tilde{v}_2,$$

where $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_5)$ are new coordinates and $\tilde{v}_2$ is a new control. It follows by Theorem 6.3.6 that Ξ_1^{se} is internally equivalent to the following linear DAE:

$$\left\{\begin{array}{l}\dot{\tilde{x}}_1=\tilde{x}_2\\ \dot{\tilde{x}}_2=\tilde{x}_5.\end{array}\right.$$

Case 2: Consider again system (6.4) but now under the following dummy holonomic constraints

$$\begin{cases} 0 = z \\ 0 = \ln|\tan\frac{\alpha}{2}| + (k-1)z, \end{cases}$$

where $k \in \mathbb{R}$. Following the notations of Case 1, we get

$$\Xi_2^{se} : \begin{cases} \mathcal{R}_2(x)\dot{x} &= a_2(x) \\ 0 &= c_2(x), \end{cases} \tag{6.10}$$

where $\mathcal{R}_2(x) = \mathcal{R}_1(x)$, $a_2(x) = a_1(x)$ and

$$c_2(x) = \begin{bmatrix} x_3 \\ \ln|\tan\frac{x_1}{2}| + (k-1)x_3 \end{bmatrix}.$$

Consider Ξ_2^{se} around an admissible point x_0. Then the explicitation of Ξ_2^{se} gives a control system $\Sigma_2 \in$ **Expl**(Ξ_2^{se}), where $\Sigma_2 = (f_2, g_2, h_2)$ is given by

$$\Sigma_2 : \begin{cases} \dot{x} = \begin{bmatrix} a_2(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_4 \\ \ln|\tan\frac{x_1}{2}| + (k-1)x_3 \end{bmatrix}. \end{cases} \tag{6.11}$$

The maximal output zeroing submanifold M_2^*, given by the zero dynamics algorithm applied to Σ_2, is:

$$M_2^* = \left\{ x \;\middle|\; \begin{array}{l} \ln|\tan\frac{x_1}{2}| + (k-1)x_3 = x_2 = x_4 = \\ 2lg - (x_5)^2\cos x_1 = 0 \end{array} \right\}.$$

The zero dynamics of Σ_2 is

$$\dot{x}_3 = 0.$$

Since x_3 does not depend on time, Σ_2 is just the point $(x_{10}, 0, x_{30}, 0, x_{50})$ on its output zeroing submanifold (note that $x_0 \in M_2^*$). It implies that internally Ξ_2^{se} consists of the fixed admissible point x_0 only.

6.4 Level-3 external linearization

We start by reviewing the results of the linearization of input-output map for control systems, firstly given in [96]. Denote by $r(A(x))$ the point-wise rank of the matrix $A(x)$ and denote by $r_{\mathbb{R}}(A(x))$ the dimension of the vector space spanned over $\mathbb{R}$ by the rows of $A(x)$.

Theorem 6.4.1. ([96],[50]) *For a control system $\Sigma_{n,m,p} = (f, g, h)$, the following conditions are equivalent.*

(i) System Σ is level-3 input-output linearizable;

(ii) The Toeplitz matrices

$$M_k = \begin{bmatrix} T_0(x) & T_1(x) & \cdots & T_k(x) \\ 0 & T_0(x) & \cdots & T_{k-1}(x) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & T_0(x) \end{bmatrix}$$

satisfy $r(M_k(x)) = r_{\mathbb{R}}(M_k(x))$ *for all* $k \leq 2n-1$, *where* $T_k(x) = L_g L_f^k h(x)$*;*

(iii) System Σ *is level-3 sys-equivalent to*

$$\begin{cases} \dot{\xi}^1 = f^1(\xi) + g^1(\xi)v^1 + g^3(\xi)v^3 \\ \dot{\xi}^3 = \hat{A}^3\xi^3 + \hat{B}^3v^3 \qquad\qquad +\hat{K}^3 y \\ \dot{\xi}^4 = f^4(\xi^4) \qquad\qquad\qquad +\hat{K}^4 y \\ y^3 = \hat{C}^3\xi^3 \\ y^4 = \hat{C}^4\xi^4, \end{cases} \tag{6.12}$$

where $y = (y^3, y^4)$ *and* $(\hat{A}^3, \hat{B}^3, \hat{C}^3)$ *is prime (see Definition 2.9.1 in Chapter 2, or* [146] *for the definition of prime form).*

Note that in [96] and [50], the implication $(i) \Rightarrow (ii)$ is proved by the *structural algorithm*, from which a linearizing feedback can be constructed via a $r_{2n-1} \times m$ full row rank decoupling matrix $L_g\Gamma(x)$. Due to the reason of saving space, here we will not re-implement the structural algorithm but emphasize that this rank r_{2n-1} will be used for the external linearization problem below.

For a nonlinear control system $\Sigma_{n,m,p} = (f, g, h)$, define sequences of distributions G_i, S_i and codistributions P_i by

$$\begin{aligned} G_1 &:= G := \operatorname{span}\{g_1, \ldots, g_m\}, \\ G_{i+1} &:= G_i + [f, G_i], \\ G^* &:= \sum_{i\geq 1} G_i \end{aligned} \tag{6.13}$$

$$\begin{aligned} S_1 &:= G, \\ S_{i+1} &:= S_i + [f, S_i \cap \ker dh] + \textstyle\sum_{j=1}^m [g_j, S_i \cap \ker dh], \\ S^* &:= \sum_{i\geq 1} S_i \\ P_1 &:= \operatorname{span}\{dh_1, \ldots, dh_p\}, \\ P_{i+1} &:= P_i + L_f(P_i \cap G^\perp) + \textstyle\sum_{j=1}^m L_{g_j}(P_i \cap G^\perp), \\ P^* &:= \sum_{i\geq 1} P_i \end{aligned}$$

The above distributions and co-distributions, together with $V_i := P_i^\perp$, $V^* := (P^*)^\perp$, play an important role in the problems of linearization and decoupling of nonlinear control systems, see e.g. [92],[151].

Theorem 6.4.2. *Consider* $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ *around a point* x_0*. Then in a neighborhood* X_0 *of* x_0*,* Ξ^{se} *is level-3 ex-equivalent to a linear SE DAE* Δ^{se} *with* internal regularity *and* constraint-free reachability *if and only if a (and then any) control system* $\Sigma = (f, g, h) \in$ ***Expl***(Ξ^{se})*, satisfies the following conditions in* X_0*:*

(i) Σ *is level-3 input-output linearizable;*

(ii) $G^* = TX_0$*;*

(iii) $[ad^k_{\tilde{f}}\tilde{g}_i, ad^l_{\tilde{f}}\tilde{g}_j] = 0$ *for* $1 \leq i, j \leq m$, $0 \leq l + k \leq 2n - 1$, *where* $\tilde{f}$ *and* $\tilde{g}_i$ *are vector fields modified by a feedback transformation resulting from the structure algorithm;*

(iv) $V^* \cap S^* = 0$.

Moreover, Δ^{se} *is* regular *if and only if* Ξ^{se} *satisfies (i)-(iv) and, additionally, condition*

(v) $V^* \oplus S^* = TX_0$.

Remark 6.4.3. (i) The distributions V^* and S^* are, obviously, the nonlinear generalizations of the limits of Wong sequences $\mathscr{V}^*$ and $\mathscr{W}^*$, respectively.

(ii) Condition (iv) above can be replaced by (iv)': The rank r_{2n-1} of the decoupling matrix $L_g\Gamma(x)$ in the structural algorithm equals m. Condition (v) can be replaced by (v)': $r + p = n$.

Observe that if the rank $r_{2n-1} = m$, which implies that the feedback transformation of *the structure algorithm* is unique, then condition (iii) of Theorem 6.4.2 is verifiable. However, if $r_{2n-1} < m$, which implies some inputs are not used for the purpose of input-output linearization, then condition (iii) may be difficult to check. We give the following theorem, in which the "unused" inputs serve to linearize the remaining part (contained in V^*) of the system and all conditions become checkable.

Theorem 6.4.4. *Consider* $\Xi^{se}_{n,r,p} = (\mathfrak{R}, a, c)$ *around a point* x_0*. Then in a neighborhood* X_0 *of* x_0*,* Ξ^{se} *is level-3 ex-equivalent a linear SE DAE* Δ^{se} *of the form*

$$\begin{cases} \dot{z}^1 = A^1 z^1 + B^1 w^1, \\ \dot{z}^3 = A^3 z^3 + B^3 w^3 + K^3 y, \\ 0 \ = C^3 z^3 + D^3 w^3, \end{cases} \tag{6.14}$$

where all matrices are in the SCF (z^2*- and* z^4*-subsystems are absent), if and only if a (and then any) control system* $\Sigma \in$ ***Expl***(Ξ^{se}) *satisfies the following conditions in* X_0*:*

(i) Σ *is level-3 input-output linearizable;*

(ii) S_i *and* G_i *are involutive and of constant rank;*

(iii) $S^* = TX_0$*;*

(iv) $S_i \cap V^* = G_i \cap V^*$.

Example 6.4.5. (Continuation of Example 6.3.7) Case 1: Consider Ξ^{se} around a point x_0 (not necessarily admissible). Assume $x_{50} \cos x_{10} \sin x_{10} \neq 0$. Then the control system Σ_1

satisfies conditions (i)-(iv) of Theorem 6.4.4 around x_0. In particular, via the change of coordinates

$$\begin{cases} \tilde{x}_3 = x_3 + l\cos x_1 - c_0, \;\; \tilde{x}_4 = x_4\cos^2 x_1 - lx_2\sin x_1, \\ \tilde{x}_1 = l\ln|\tan\frac{x_1}{2}| - x_3, \;\; \tilde{x}_2 = \frac{lx_2}{\sin x_1}, \\ \tilde{x}_5 = g - \frac{\cos x_1(lx_2 + x_4\sin x_1)^2}{l\sin^2 x_1} - \frac{(x_5)^2\cos x_1}{2l} \end{cases}$$

and the static feedback transformation

$$\begin{cases} \tilde{v}_1 = \tilde{\alpha}_1(x) + \cos^2 x_1 v_1, \\ \tilde{v}_2 = \tilde{\alpha}_2(x) - \frac{2(x_4\sin x_1 + lx_2)\cos x_1}{l\sin x_1}v_1 - \frac{x_5\cos x_1}{l}v_2, \end{cases}$$

where $\tilde{\alpha}_1(x) = L_f\tilde{x}_4(x)$ and $\tilde{\alpha}_2(x) = L_f\tilde{x}_5(x)$, Σ_1 is level-3 sys-equivalent to $\tilde{\Sigma}_1$ below. It follows from Proposition 6.3.3 that Ξ_1^{se} is level-3 ex-equivalent to the following Δ_1^{se} (since $\Sigma_1 \in$ **Expl**(Ξ_1^{se}) and $\tilde{\Sigma}_1 \in$ **Expl**(Δ_1^{se})).

$$\tilde{\Sigma}_1 : \begin{cases} \dot{\tilde{x}}_3 = \tilde{x}_4, \, y = \tilde{x}_3 \\ \dot{\tilde{x}}_4 = \tilde{v}_1 \\ \dot{\tilde{x}}_1 = \tilde{x}_2 \\ \dot{\tilde{x}}_2 = \tilde{x}_5 \\ \dot{\tilde{x}}_5 = \tilde{v}_2 \end{cases} \Rightarrow \Delta_1^{se} : \begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 \\ \dot{\tilde{x}}_2 = \tilde{x}_5 \\ \dot{\tilde{x}}_3 = \tilde{x}_4 \\ 0 = \tilde{x}_3. \end{cases}$$

Note that the above transformation bringing $\tilde{\Sigma}_1$ into the linear DAE, given by Δ_1^{se}, is a dual procedure to that of *explicitation* and it is called implicitation of a control system (for details, see Chapter 2,Chapter 3).

Case 2: We show that although internally Ξ_2^{se} is a trivial system whose generalized state consists of one point only (which is admissible), it is ex-equivalent to a linear SE DAE. Consider Ξ_2^{se} around x_0, which is not necessarily admissible. Assume $x_{50}\cos x_{10}\sin x_{10} \neq 0$. Since Σ_2 satisfies conditions (i)-(v) of Theorem 6.4.2 around x_0, it can be seen that Σ_2 is level-3 sys-equivalent to the following $\tilde{\Sigma}_2$ via the coordinates change

$$\begin{cases} \tilde{x}_3 = x_3, \;\; \tilde{x}_4 = x_4, \;\; \tilde{x}_1 = l\ln|\tan\frac{x_1}{2}| - x_3, \\ \tilde{x}_2 = \frac{lx_2}{\sin x_1}, \;\; \tilde{x}_5 = g - \frac{\cos x_1(lx_2 + x_4\sin x_1)^2}{l\sin^2 x_1} - \frac{(x_5)^2\cos x_1}{2l} \end{cases}$$

and the static feedback transformation

$$\begin{cases} \tilde{v}_1 = v_1, \\ \tilde{v}_2 = \tilde{\alpha}_2(x) - \frac{2(x_4\sin x_1 + lx_2)\cos x_1}{l\sin x_1}v_1 - \frac{x_5\cos x_1}{l}v_2, \end{cases}$$

where $\tilde{\alpha}_2(x) = L_f\tilde{x}_5(x)$. Moreover, since $\Sigma_2 \in$ **Expl**(Ξ_2^{se}) and obviously $\tilde{\Sigma}_2 \in$ **Expl**(Δ_2^{se}), by Proposition 6.3.3, Ξ_2^{se} is level-3 ex-equivalent to the following Δ_2^{se}, which is *regular* and *constraint-freely reachable*.

$$\tilde{\Sigma}_2 : \begin{cases} \dot{\tilde{x}}_3 = \tilde{x}_4 \\ \dot{\tilde{x}}_4 = \tilde{v}_1, \quad\quad y_1 = \tilde{x}_4 \\ \dot{\tilde{x}}_1 = \tilde{x}_2 + ky_1, \;\; y_2 = \tilde{x}_1 \\ \dot{\tilde{x}}_2 = \tilde{x}_5 \\ \dot{\tilde{x}}_5 = \tilde{v}_2 \end{cases} \Rightarrow \Delta_2^{se} : \begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 + k\tilde{x}_4 \\ \dot{\tilde{x}}_2 = \tilde{x}_5 \\ \dot{\tilde{x}}_3 = \tilde{x}_4 \\ 0 = \tilde{x}_4 \\ 0 = \tilde{x}_1. \end{cases}$$

6.5 An example which is not level-3 externally linearizable but so is level-2

Example 6.5.1. Consider a SE DAE $\Xi_3^{se} = (\mathcal{R}_3, a_3, c_3)$, described by

$$\mathcal{R}_3(x) = \begin{bmatrix} 1 & 0 & -x_1 & 0 & 0 & 0 \\ 0 & 0 & e^{3x_3} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ a_3(x) = \begin{bmatrix} 2(x_1e^{x_3})^{\frac{1}{2}}x_2 \\ -(x_5 + ke^{x_3}) \\ x_6 \end{bmatrix},$$
$$c_3(x) = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}.$$

where $k \in \mathbb{R}$. We can choose a control system $(f_3, g_3, h_3) = \Sigma_3 \in$ **Expl**(Ξ_3^{se}), given by

$$\Sigma_3 : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 2(x_1e^{x_3})^{\frac{1}{2}}x_2 \\ 0 \\ 0 \\ x_5 + ke^{x_3} \\ x_6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & x_1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{3x_3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ y_1 = x_3 \\ y_2 = x_4. \end{cases}$$

It is easy to verify that Σ_3 is not level-3 input-output linearizable (since the Toeplitz matrices $M_k(\Sigma_3)$ do not satisfy rank condition (ii) of Theorem 6.4.1). However, via a nonlinear coordinates change in the output space

$$\tilde{y}_1 = e^{y_1}, \quad \tilde{y}_2 = y_2 - \frac{1}{3}e^{3y_1},$$

the system with the new outputs $\tilde{y}_1$, $\tilde{y}_2$ is level-3 input-output linearizable. Additionally, the transformed system satisfies conditions (i)-(iv) of Theorem 6.4.4. In fact, by choosing new coordinates

$$\begin{cases} \tilde{x}_1 = (x_1e^{-x_3})^{\frac{1}{2}}, & \tilde{x}_2 = x_2, & \tilde{x}_3 = e^{x_3}, \\ \tilde{x}_4 = x_4 - \frac{1}{3}e^{3x_3}, & \tilde{x}_5 = x_5, & \tilde{x}_6 = x_6, \end{cases}$$

and the feedback transformation $v_1 = \tilde{v}_1$, $v_2 = e^{-x_3}\tilde{v}_2$, $v_3 = \tilde{v}_3$, the system Σ_3 is level-2 sys-equivalent to the linear control system $\tilde{\Sigma}_3$ below. Moreover, since $\Sigma_3 \in$ **Expl**(Ξ_3^{se}), by Proposition 6.3.3, Ξ_3^{se} is level-2 ex-equivalent to the linear DAE Δ_3^{se} below.

$$\tilde{\Sigma}_3 : \begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 \\ \dot{\tilde{x}}_2 = \tilde{v}_1 \\ \dot{\tilde{x}}_3 = \tilde{v}_2, & \tilde{y}_1 = \tilde{x}_3 \\ \dot{\tilde{x}}_4 = \tilde{x}_5 + k\tilde{y}_1, & \tilde{y}_2 = \tilde{x}_4 \\ \dot{\tilde{x}}_5 = \tilde{x}_6 \\ \dot{\tilde{x}}_6 = \tilde{v}_3 \end{cases} \Rightarrow \Delta_3^{se} : \begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 \\ \dot{\tilde{x}}_4 = \tilde{x}_5 + k\tilde{y}_1 \\ \dot{\tilde{x}}_5 = \tilde{x}_6 \\ 0 = \tilde{x}_3 \\ 0 = \tilde{x}_4. \end{cases}$$

In view of the example above, even if an explicit control system is not level-3 input-output linearizable, it may be so under level-2 sys-equivalence. Thus via further transformations, the original SE DAE is possibly level-2 externally linearizable. It suggests that the future work should be focused on level-2 input-output linearizability of control systems and corresponding SE DAEs.

6.6 Sketch of the proof of Theorem 6.4.4

Proof. Necessity. If Ξ^{se} is level-3 ex-equivalent to Δ^{se} given by (6.14), then any control system $\Sigma \in \mathbf{Expl}(\Xi^{se})$ is level-3 ex-equivalent to

$$\begin{cases} \dot{z}^1 = A^1 z^1 + B^1 w^1, \qquad \dot{w}^1 = v^1, \\ \dot{z}^3 = A^3 z^3 + B^3 w^3 + K^3 y, \, \dot{w}^3 = v^3, \\ y^3 = C^3 z^3 + D^3 w^3. \end{cases}$$

The above linear control system satisfies (i)-(iv) in an obvious way. Moreover, the invariance of S_i, G_i (clearly, G_i is involutive for the linear system), and V^*, under level-3 sys-equivalence, completes the proof of necessity.

Sufficiency. Suppose $\Sigma \in \mathbf{Expl}(\Xi^{se})$ satisfies conditions (i)-(iv), then by condition (i) and Theorem 6.4.1, Σ is level-3 sys-equivalence to a control system of the form (6.12) via the structural algorithm. Subsequently, condition (iii) implies that there is no ξ_4 in system (6.12), i.e., after input-output linearization, Σ becomes

$$\begin{cases} \dot{\xi}^1 = f^1(\xi^1, \xi^3) + g^1(\xi^1, \xi^3) v^1 + g^3(\xi^1, \xi^3) v^3 \\ \dot{\xi}^3 = \hat{A}^3 \xi^3 + \hat{B}^3 v^3 + \hat{K}^3 y^3 \\ y^3 = \hat{C}^3 \xi^3. \end{cases} \tag{6.15}$$

For ease of proof, we assume that v^1 is of dimension 1. Denote

$$f = \begin{pmatrix} f^1(\xi) \\ \hat{A}^3 \xi^3 + \hat{K}^3 y^3 \end{pmatrix}, \quad g_1 = \begin{pmatrix} g^1(\xi) \\ 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} g^3(\xi) \\ B^3 \end{pmatrix}.$$

In view of condition (iii), the key of the following proof is to find new coordinates $\tilde{\xi}^1$ and new control $\tilde{v}^1$ (we do not change ξ^3 and v^3) such that in $(\tilde{\xi}^1, \xi^3)$-coordinates and with the control $(\tilde{v}^1, v^3)$, the distributions G_i are rectified. Notice that from the involutivity of S_i in (ii), we have $S_{i+1} = S_i + [f, S_i \cap \ker dh]$. Now from $V^* = \operatorname{span}\left\{ \frac{\partial}{\partial \xi^1} \right\}$, via condition (iv) and a direct calculation of S_i, we get for (6.15),

$$G_i \cap V^* = S_i \cap V^* = \operatorname{span}\left\{ g_1, ad_f g_1, \ldots, ad_f^{i-1} g_1 \right\}.$$

Then there exists a smallest number, denoted by ρ, such that $G_\rho \cap V^* = G^* \cap V^*$ (note that $\dim(G_\rho \cap V^*) - \dim(G_{\rho-1} \cap V^*) = 1$). Thus, from the involutivity of G_i, we can choose a scalar function $\psi(\xi^1, \xi^3)$ such that $d\psi \in (G_{\rho-1})^\perp$ and $d\psi \notin (G_\rho \cap V^*)^\perp = (V^*)^\perp$. The above construction implies the dummy output $y^1 = \psi(\xi^1, \xi^3)$ has relative degree ρ and $L_{g_1} L_f^{\rho-1} \psi \neq 0$. Observe that $G_\rho \cap V^* = V^*$ and that $\operatorname{span}\left\{ d\psi, \ldots, dL_f^{\rho-1} \psi \right\} \cap (V^*)^\perp = 0$. Thus $(\psi, \ldots, L_f^{\rho-1} \psi, \xi^3)$ form a local diffeomorphism (since $d\xi^3 = (V^*)^\perp$ and

$d\psi, \ldots, dL_f^{\rho-1}\psi$ are independent). Finally, via the change of coordinates $\tilde{\xi}_1^1 = \psi, \ldots, \tilde{x}_\rho^1 = L_f^{\rho-1}\psi$ and the feedback transformation $\tilde{v}_1 = L_f^{\rho-1}\psi + v^1 L_{g_1} L_f^{\rho-1}\psi + v^3 L_{g_3} L_f^{\rho-1}\psi$, we get

$$\begin{cases} \dot{\tilde{\xi}}_1^1 = \tilde{\xi}_2^1, \ \dot{\tilde{\xi}}_2^1 = \tilde{\xi}_3^1, \ \ldots, \ \dot{\tilde{\xi}}_\rho^1 = \tilde{v}^1, \\ \dot{\xi}^3 = \hat{A}^3\xi^3 + \hat{B}^3 v^3 + \hat{K}^3 y^3, \\ y^3 = \hat{C}^3\xi^3. \end{cases}$$

□

6.7 Conclusions

In this chapter, we discuss linearization of semi-explicit differential-algebraic equations under internal and external equivalence. The difference of linearization under those two equivalence relations is illustrated by an example of a mechanical system under some holonomic constraints. Moreover, we define 3-levels of external equivalence depending on 3 kinds of output transformations and show their geometric interpretations. Then we give necessary and sufficient conditions for level-3 external linearization problem via the explicitation procedure and illustrate by an academic example the problem of level-2 external linearization.

Chapter 7

Conclusions and Perspectives

In this thesis, we study both linear and nonlinear systems described by differential-algebraic equations DAEs using geometric methods. These DAE systems are classified into different categories, which include linear DAEs Δ, linear DAE control systems DAECSs Δ^u, semi-explicit SE linear DAEs Δ^{se}, and their nonlinear counterparts Ξ, Ξ^u, Ξ^{se}, based on their system structures. The main results of the present thesis are summarized as follows.

1. Existence and uniqueness of solutions. We discuss the solutions of linear (resp. nonlinear) DAEs using the notions of invariant subspaces (resp. invariant submanifolds). We have shown that for a linear (resp. nonlinear) DAE, there passes at lest one solution through a nominal point if and only if the point belongs to its maximal invariant subspace (resp. locally maximal invariant submanifold). Moreover, if the solution is unique, we call the DAE internally regular, and thus the internal regularity of a DAE yields an ODE evolving on the maximal invariant subspace (submanifold) that has no free variables; the corresponding results are given in Proposition 2.6.12 for linear DAEs and Theorem 4.3.14 for nonlinear DAEs. The calculation of the locally maximal submanifold of a nonlinear DAE can be implemented by a reduction method commonly appeared in the nonlinear DAEs literature. We reformulate this reduction method as Algorithm 4.3.4 in Chapter 4 and show how this algorithm is related to the zero dynamics algorithm in the nonlinear control theory.

2. Internal and external (feedback) equivalence. A main difference between this thesis and the other existing results on geometric analysis of DAE systems is that we systematically distinguish the difference between the two equivalence relations. The external (feedback) equivalence of two DAEs (DAECSs) is important in every chapter of this thesis since it is the fundamental relation when considering DAE systems (locally) everywhere (not just on the subspace (submanifold) where the solutions exist). Various normal forms and canonical forms under external (feedback) equivalence are proposed in this thesis (see item 5 below) to simplify the structure of DAE systems. The internal equivalence of two DAEs is defined by the external equivalence of the two DAEs restricted to their maximal invariant subspaces (or submanifolds), i.e., where the solutions exist. We have shown that the internal equivalence is useful when we only care about where and how the solutions

evolve; the corresponding results are given in Theorem 2.6.10 for linear DAEs, and Lemma 4.2.3 and Theorem 4.3.14 for nonlinear DAEs.

3. Two kinds of explicitation procedures. In order to "explicitate" the "implicit" DAE systems and connect DAE systems with ODE systems, we propose two kinds of explicitation procedures, i.e., the explicitation with driving variables (or (Q, v)-explicitation) and without driving variables (or (Q, P)-explicitation). Through Chapters 2-6, we have shown that the explicitation procedure is a powerful tool for DAE systems since with its help we can use the knowledges from the classical linear and nonlinear control theory to analyze DAE systems. We prove that the explicitation of a DAE system is not just a system but a class of systems (or a system defined up to some transformations), as seen in Remark 2.3.3 and Proposition 3.2.3,3.2.4,4.3.18,5.2.5,6.3.3. We discuss the differences of the two explicitation procedure in Remark 3.2.5 of Chapter 3 for linear DAE systems and show in Theorem 4.3.27 that a nonlinear DAE $\Xi = (E, F)$ admits an explicitation without driving variables if and only if the distribution defined by $\ker E(x)$ is of constant rank and involutive, which also explains when Ξ is externally equivalent to a SE DAE Ξ^{se}.

4. Connections between DAE and ODE systems. The connections between the two classes of systems are built up depending on the results that the external (feedback) equivalence for DAE systems is a true counterpart of the system (feedback) equivalence (the (extended) Morse equivalence for the linear case) for ODE systems; the corresponding results are Theorem 2.3.4, 3.2.8, 4.3.21, 5.2.9 and Proposition 6.3.3. The relations of linear DAE systems and linear ODE systems are shown by connecting their geometric subspaces and canonical forms. The relations between the (augmented) Wong sequences for DAE systems and the invariant subspaces for ODE systems are given in Proposition 2.4.10 and Proposition 3.2.9. The correspondence of the Kronecker canonical form of DAEs and the Morse canonical form of ODE control systems are shown by establishing relations of their indices in Proposition 2.5.3. Similarly, the correspondence of the feedback canonical form of DAECSs and the extended Morse canonical form are shown by establishing relations of their indices in Remark 3.4.8.

5. Normal and canonical forms. In Chapter 3, we propose a Morse triangular form **MTF** (Proposition 3.3.1) to simplify the construction of the Morse normal form **MNF** (Proposition 3.3.2) of classical ODE control systems and then the **MTF** and **MNF** are generalized, respectively, to the extended Morse triangular form **EMTF** (Theorem 3.3.4) and the extended Morse normal form **EMNF** (Theorem 3.3.5) for ODE control systems with two kinds of inputs. In Theorem 3.4.2, we provide a constructive passage from the **EMNF** to the extended Morse canonical form **EMCF**. Algorithm 3.4.11 describes a way of transforming a linear DAECS into its feedback canonical form **FBCF** via the intermediate forms **EMTF**, **EMNF** and **EMCF** of its explicitation systems. In Theorem 4.3.29 of Chapter 4, a nonlinear generalization of the Weierstrass form **NWF** is proposed based on the comparison of Algorithm 4.3.4 for DAEs and the zero dynamics algorithm for the explicitation systems. In Theorem 5.3.10 of Chapter 5, two normal forms based on the notion of maximal controlled invariant submanifold are proposed to simplify the structure and to understand various types of variables of DAECSs.

6. Nonlinear generalizations of the notions in linear DAEs theory. We have shown in Chapters 4-6 that the (augmented) Wong sequences have two kinds of nonlinear generalizations, which are sequences of submanifolds and distributions, those observations are given in Remark 4.3.15(iv), 5.4.7(iv) and 6.4.3(i). The **NWF** in Theorem 4.3.29 generalizes the Weierstrass form for linear regular DAEs. The maximal controlled invariant form in Theorem 5.3.10 is the effort made to generalize the **FBCF** of linear DAECSs. The (Q, P)- and (Q, v)- explicitation for linear DAEs in Chapter 2-3 are generalized to the explicitation with and without driving variables in Chapter 4 and the correspondence of external (feedback) equivalence and system (feedback) equivalence for nonlinear systems generalizes the linear results in Theorem 3.2.8.

7. Linearization and feedback linearization. Necessary and sufficient conditions are given in Theorem 5.4.5 and 5.4.6 to describe when a nonlinear DAECS is externally and internally feedback equivalent to a completely controllable linear one, respectively. The results of linearization of semi-explicit DAEs are given in Chapter 6. We show in Theorem 6.3.6 and 6.4.2, 6.4.4, respectively, when a semi-explicit DAE is internally and externally equivalent to a linear one. All these results on linearization for nonlinear DAE systems are solved with the help of some distributions given by the explicitation systems.

We now give some perspectives for this thesis. As the explicitation procedure builds up a bridge between DAE systems and ODE systems, various results on geometric control of nonlinear ODE systems, such as disturbance decoupling, observer design by geometric methods, the zero dynamics algorithm, invertibility analysis, stabilization and tracking, dynamic feedback linearization etc, can be possibly generalized to DAE systems. The linearization problems of DAE systems need a further study since in this thesis we only give the results for some special cases, e.g. in Theorem 5.4.6, we only study when a DAECS can be linearized to a linear one with complete controllability, but a linear DAECS can have various kinds of controllability (see [17]), thus different controllability of the linearized DAE system should correspond to different conditions of linearization. Moreover, in Chapter 6, we define 3-levels of external equivalence but only give the conditions for lever-3 external linearization problem, thus the conditions for lever-1 and level-2 external linearization of semi-explicit DAEs should be further investigate. Last but not least, examples in Chapter 5 raise the interests in studying relations of the flatness of the explicitation systems and the feedback linearizablity of the DAE control system and the flatness of DAE systems is also an interesting subject to be studied in the geometric spirit of the present thesis.

Bibliography

[1] H. ALING AND J. M. SCHUMACHER, *A nine-fold canonical decomposition for linear systems*, International Journal of Control, 39 (1984), pp. 779–805.

[2] J. D. APLEVICH, *Minimal representations of implicit linear systems*, Automatica, 21 (1985), pp. 259–269.

[3] H. ARAI, K. TANIE, AND N. SHIROMA, *Nonholonomic control of a three-dof planar underactuated manipulator*, IEEE Transactions on Robotics and Automation, 14 (1998), pp. 681–695.

[4] E. ARANDA-BRICAIRE AND R. HIRSCHORN, *Equivalence of nonlinear systems to prime systems under generalized output transformations*, SIAM Journal on Control and Optimization, 37 (1998), pp. 118–130.

[5] E. ARANDA-BRICAIRE, C. MOOG, AND J.-B. POMET, *A linear algebraic framework for dynamic feedback linearization*, IEEE Transactions on Automatic Control, 40 (1995), pp. 127–132.

[6] R. BACHMANN, L. BRÜLL, T. MRZIGLOD, AND U. PALLASKE, *On methods for reducing the index of differential algebraic equations*, Computers & Chemical Engineering, 14 (1990), pp. 1271–1273.

[7] G. BASILE AND G. MARRO, *Controlled and conditioned invariant subspaces in linear system theory*, Journal of Optimization Theory and Applications, 3 (1969), pp. 306–315.

[8] ——, *On the observability of linear, time-invariant systems with unknown inputs*, Journal of Optimization Theory and Applications, 3 (1969), pp. 410–415.

[9] ——, *Controlled and Conditioned Invariants in Linear System Theory*, Prentice Hall Englewood Cliffs, 1992.

[10] T. BEELEN AND P. VAN DOOREN, *An improved algorithm for the computation of Kronecker's canonical form of a singular pencil*, Linear Algebra and its Applications, 105 (1988), pp. 9–65.

[11] T. BERGER, *On Differential-Algebraic Control Systems*, Univ.-Verlag, 2013.

[12] ——, *Controlled invariance for nonlinear differential–algebraic systems*, Automatica, 64 (2016), pp. 226–233.

[13] ——, *The zero dynamics form for nonlinear differential-algebraic systems*, IEEE Transactions on Automatic Control, PP (2016), pp. 1–1.

[14] ——, *Disturbance decoupling by behavioral feedback for linear differential–algebraic systems*, Automatica, 80 (2017), pp. 272–283.

[15] T. BERGER, A. ILCHMANN, AND T. REIS, *Zero dynamics and funnel control of linear differential-algebraic systems*, Mathematics of Control, Signals, and Systems, 24 (2012), pp. 219–263.

[16] T. BERGER, A. ILCHMANN, AND S. TRENN, *The quasi-Weierstraß form for regular matrix pencils*, Linear Algebra and its Applications, 436 (2012), pp. 4052 – 4069.

[17] T. BERGER AND T. REIS, *Controllability of linear differential-algebraic systems—a survey*, in Surveys in Differential-Algebraic Equations I, Springer, 2013, pp. 1–61.

[18] ——, *Regularization of linear time-invariant differential–algebraic systems*, Systems & Control Letters, 78 (2015), pp. 40–46.

[19] T. BERGER, T. REIS, AND S. TRENN, *Observability of linear differential-algebraic systems: a survey*, in Surveys in Differential-Algebraic Equations IV, Springer, 2017, pp. 161–219.

[20] T. BERGER AND S. TRENN, *The quasi-Kronecker form for matrix pencils*, SIAM Journal on Matrix Analysis and Applications, 33 (2012), pp. 336–368.

[21] T. BERGER AND S. TRENN, *Addition to "The quasi-Kronecker form for matrix pencils"*, SIAM Journal on Matrix Analysis and Applications, 34 (2013), pp. 94–101.

[22] P. BETSCH AND P. STEINMANN, *A DAE approach to flexible multibody dynamics*, Multibody System Dynamics, 8 (2002), pp. 365–389.

[23] W. BLAJER, *Modelling of aircraft program motion with application to circular loop simulation*, The Aeronautical Journal, 92 (1988), pp. 289–296.

[24] W. BLAJER, J. GRAFFSTEIN, AND M. KRAWCZYK, *Prediction of the dynamic characteristics and control of aircraft in prescribed trajectory flight*, Journal of Theoretical and Applied Mechanics, 39 (2001), pp. 79–103.

[25] W. BLAJER AND K. KOŁODZIEJCZYK, *A geometric approach to solving problems of control constraints: theory and a DAE framework*, Multibody System Dynamics, 11 (2004), pp. 343–364.

[26] ——, *Control of underactuated mechanical systems with servo-constraints*, Nonlinear Dynamics, 50 (2007), pp. 781–791.

[27] M. E. Bonilla and M. Malabre, *External reachability (reachability with pole assignment by PD feedback) for implicit descriptions*, Kybernetika, 29 (1993), pp. 499–510.

[28] K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, vol. 14, SIAM, 1996.

[29] R. W. Brockett, *Feedback invariants for nonlinear systems*, IFAC Proceedings Volumes, 11 (1978), pp. 1115–1120.

[30] ——, *Finite Dimensional Linear Systems*, vol. 74, SIAM, 2015.

[31] P. Brunovský, *A classification of linear controllable systems*, Kybernetika, 6 (1970), pp. 173–188.

[32] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols, *Feedback design for regularizing descriptor systems*, Linear Algebra and its Applications, 299 (1999), pp. 119–151.

[33] G. Byrne and P. Ponzi, *Differential-algebraic systems, their applications and solutions*, Computers & Chemical Engineering, 12 (1988), pp. 377–382.

[34] S. Campbell, R. Nikoukhah, and F. Delebecque, *Nonlinear descriptor systems*, in Advances in Control, Springer, 1999, pp. 247–281.

[35] S. L. Campbell, *High-index differential algebraic equations*, Journal of Structural Mechanics, 23 (1995), pp. 199–222.

[36] S. L. Campbell and S. Campbell, *Singular Systems of Differential Equations*, vol. 1, Pitman London, 1980.

[37] S. L. Campbell and C. W. Gear, *The index of general nonlinear DAEs*, Numerische Mathematik, 72 (1995), pp. 173–196.

[38] S. L. Campbell and E. Griepentrog, *Solvability of general differential algebraic equations*, SIAM Journal on Scientific Computing, 16 (1995), pp. 257–270.

[39] B. Charlet, J. Lévine, and R. Marino, *On dynamic feedback linearization*, Systems & Control Letters, 13 (1989), pp. 143–151.

[40] ——, *Sufficient conditions for dynamic state feedback linearization*, SIAM Journal on Control and Optimization, 29 (1991), pp. 38–57.

[41] C. C. Cheah and D. Wang, *Learning control for a class of nonlinear differential-algebraic systems with application to constrained robots*, Journal of Robotic Systems, 13 (1996), pp. 141–151.

[42] B. M. CHEN, X. LIU, AND Z. LIN, *Interconnection of Kronecker canonical form and special coordinate basis of multivariable linear systems*, Systems & Control Letters, 57 (2008), pp. 28–33.

[43] C.-T. CHEN, *Linear System Theory and Design*, Oxford University Press, Inc., 1998.

[44] M.-C. CHEN ET AL., *Robust and H_∞ Control*, Citeseer, 2000.

[45] Y. CHEN AND W. RESPONDEK, *Feedback linearization and controlled invariant submanifold of nonlinear differential-algebraic control systems*, (2019). Preprint.

[46] ——, *From Morse triangular form of ODE control systems to feedback canonical form of DAE control systems*, (2019). Preprint.

[47] ——, *Geometric analysis of linear differential-algebraic equations via linear control theory*, (2019). Preprint.

[48] ——, *Geometric analysis of nonlinear differential-algebraic equations via nonlinear control theory*, (2019). Preprint.

[49] ——, *Internal and external linearization of semi-explicit differential-algebraic equations*, (2019). Accepted by the 11th IFAC Symposium on Nonlinear Control Systems.

[50] D. CHENG, A. ISIDORI, W. RESPONDEK, AND T. J. TARN, *Exact linearization of nonlinear systems with outputs*, Mathematical Systems Theory, 21 (1988), pp. 63–83.

[51] L. CHUA, T. MATSUMOTO, AND S. ICHIRAKU, *Geometric properties of resistive nonlinear n-ports: transversality, structural stability, reciprocity, and anti-reciprocity*, IEEE Transactions on Circuits and Systems, 27 (1980), pp. 577–603.

[52] L. O. CHUA AND H. OKA, *Normal forms for constrained nonlinear differential equations. I. Theory*, IEEE Transactions on Circuits and Systems, 35 (1988), pp. 881–901.

[53] D. CLAUDE, *Everything you always wanted to know about linearization*, in Algebraic and Geometric Methods in Nonlinear Control Theory, Springer, 1986, pp. 181–226.

[54] D. COBB, *Feedback and pole placement in descriptor variable systems*, International Journal of Control, 33 (1981), pp. 1135–1146.

[55] ——, *Controllability, observability, and duality in singular systems*, IEEE Transactions on Automatic Control, 29 (1984), pp. 1076–1082.

[56] G. COSTANTINI, S. TRENN, AND F. VASCA, *Regularity and passivity for jump rules in linear switched systems*, in 52nd IEEE Conference on Decision and Control, IEEE, 2013, pp. 4030–4035.

[57] J. J. CRAIG, *Introduction to Robotics: Mechanics and Control*, vol. 3, Pearson/Prentice Hall Upper Saddle River, NJ, USA:, 2005.

[58] P. S. P. DA SILVA AND C. C. FILHO, *Relative flatness and flatness of implicit systems*, SIAM Journal on Control and Optimization, 39 (2001), pp. 1929–1951.

[59] L. DAI, *Singular Control Systems*, vol. 118, Springer, 1989.

[60] P. DAOUTIDIS, *DAEs in model reduction of chemical processes: an overview*, in Surveys in Differential-Algebraic Equations II, Springer, 2015, pp. 69–102.

[61] C. DE PERSIS AND A. ISIDORI, *A geometric approach to nonlinear fault detection and isolation*, IEEE Transactions on Automatic Control, 46 (2001), pp. 853–865.

[62] P. V. DOOREN, *The computation of Kronecker's canonical form of a singular pencil*, Linear Algebra and its Applications, 27 (1979), pp. 103 – 140.

[63] D. ESTÉVEZ SCHWARZ AND C. TISCHENDORF, *Structural analysis of electric circuits and consequences for mna*, International Journal of Circuit Theory and Applications, 28 (2000), pp. 131–162.

[64] P. FERREIRA, *On system equivalence*, IEEE Transactions on Automatic Control, 32 (1987), pp. 619–621.

[65] L. R. FLETCHER AND A. AASARAAI, *On disturbance decoupling in decsriptor systems*, SIAM Journal on Control and Optimization, 27 (1989), pp. 1319–1332.

[66] M. FLIESS, *A note on the invertibility of nonlinear input-output differential systems*, Systems & Control letters, 8 (1986), pp. 147–151.

[67] M. FLIESS AND M. HASLER, *Questioning the classic state-space description via circuit examples*, in Realization and Modelling in System Theory, Springer, 1990, pp. 1–12.

[68] M. FLIESS, J. LÉVINE, P. MARTIN, AND P. ROUCHON, *Flatness and defect of nonlinear systems: introductory theory and examples*, International Journal of Control, 61 (1995), pp. 1327–1361.

[69] M. FLIESS, J. LEVINE, P. MARTIN, AND P. ROUCHON, *Implicit differential equations and Lie-Backlund mappings*, in Proceedings of the 34th IEEE Conference on Decision and Control, vol. 3, IEEE, 1995, pp. 2704–2709.

[70] M. FLIESS, J. LÉVINE, P. MARTIN, AND P. ROUCHON, *Index and decomposition of nonlinear implicit differential equations*, IFAC Proceedings Volumes, 28 (1995), pp. 37–42.

[71] ——, *A Lie-Backlund approach to equivalence and flatness of nonlinear systems*, IEEE Transactions on Automatic Control, 44 (1999), pp. 922–937.

[72] M. FLIESS, J. LEVINE, AND P. ROUCHON, *A simplified approach of crane control via a generalized state-space model*, in Proceedings of the 30th IEEE Conference on Decision and Control, IEEE, 1991, pp. 736–741.

[73] M. FLIESS, P. MARTIN, N. PETIT, AND P. ROUCHON, *Active signal restoration for the telegraph equation*, in Proceedings of the 38th IEEE Conference on Decision and Control, vol. 2, IEEE, 1999, pp. 1107–1111.

[74] H. FRANKOWSKA, *On controllability and observability of implicit systems*, Systems & Control Letters, 14 (1990), pp. 219–225.

[75] F. GANTMACHER, *The Theory of Matrices*, Chelsea Publishing Co., 1959.

[76] C. W. GEAR, *Differential-algebraic equation index transformations*, SIAM Journal on Scientific and Statistical Computing, 9 (1988), pp. 39–47.

[77] ——, *Differential algebraic equations, indices, and integral algebraic equations*, SIAM Journal on Numerical Analysis, 27 (1990), pp. 1527–1534.

[78] C. W. GEAR AND L. R. PETZOLD, *ODE methods for the solution of differential/algebraic systems*, SIAM Journal on Numerical Analysis, 21 (1984), pp. 716–728.

[79] T. GEERTS, *Invariant subspaces and invertibility properties for singular systems: The general case*, Linear algebra and its applications, 183 (1993), pp. 61–88.

[80] H. GLÜSING-LÜERBEN, *Feedback canonical form for singular systems*, International Journal of Control, 52 (1990), pp. 347–376.

[81] V. GOL'DSHTEIN AND V. SOBOLEV, *Qualitative analysis of singularly perturbed systems of chemical kinetics*, Singularity Theory and Some Problems of Functional Analysis. American Mathematical Society, Translations, SG Gindikin (ed.), 153 (1992), pp. 73–92.

[82] H. GOLDSTEIN, *Classical Mechanics*, Pearson Education India, 2011.

[83] C. GRAND, F. B. AMAR, P. BIDAUD, AND G. ANDRADE, *A simulation system for behaviour evaluation of off-road mobile robots*, in Proceedings CLAWAR's of the International Conference on Climbing and Walking Robots, 2001, pp. 307–314.

[84] P. M. GRESHO, *Incompressible fluid dynamics: some fundamental formulation issues*, Annual Review of Fluid Mechanics, 23 (1991), pp. 413–453.

[85] U. HELMKE AND M. A. SHAYMAN, *A canonical form for controllable singular systems*, Systems & Control letters, 12 (1989), pp. 111–122.

[86] R. HIRSCHORN, (A, B)*-invariant distributions and disturbance decoupling of nonlinear systems*, SIAM Journal on Control and Optimization, 19 (1981), pp. 1–19.

[87] L. HUNT AND R. SU, *Linear equivalents of nonlinear time-varying systems*, in Proc . Int . Symposium on Math. Theory of Networks and Systems, Santa Monica, 1981, pp. 119–123.

[88] L. HUNT, R. SU, AND G. MEYER, *Global transformations of nonlinear systems*, IEEE Transactions on Automatic Control, 28 (1983), pp. 24–31.

[89] J.-I. IMURA, K. KOBAYASHI, AND T. YOSHIKAWA, *Nonholonomic control of 3 link planar manipulator with a free joint*, in Proceedings of the 35th IEEE Conference on Decision and Control, vol. 2, IEEE, 1996, pp. 1435–1436.

[90] M. ISHIKAWA, R. KITAYOSHI, AND T. SUGIE, *Volvot: A spherical mobile robot with eccentric twin rotors*, in IEEE International Conference on Robotics and Biomimetics (ROBIO), IEEE, 2011, pp. 1462–1467.

[91] A. ISIDORI, *Nonlinear feedback, structure at infinity and the input-output linearization problem*, in Mathematical Theory of Networks and Systems, Springer, 1984, pp. 473–493.

[92] A. ISIDORI, *Nonlinear Control Systems*, Springer-Verlag New York, Inc., Secaucus, NJ, USA, 3rd ed., 1995.

[93] A. ISIDORI, A. KRENER, C. GORI-GIORGI, AND S. MONACO, *Locally (f, g) invariant distributions*, Systems & Control letters, 1 (1981), pp. 12–15.

[94] A. ISIDORI, A. KRENER, C. GORI-GIORGI, AND S. MONACO, *Nonlinear decoupling via feedback: a differential geometric approach*, IEEE Transactions on Automatic Control, 26 (1981), pp. 331–345.

[95] A. ISIDORI AND C. MOOG, *On the nonlinear equivalent of the notion of transmission zeros*, in Modelling and Adaptive Control, Springer, 1988, pp. 146–158.

[96] A. ISIDORI AND A. RUBERTI, *On the synthesis of linear input-output responses for nonlinear systems*, Systems Control Letters, 4 (1984), pp. 17 – 22.

[97] S. JAFFE AND N. KARCANIAS, *Matrix pencil characterization of almost (A, Z)-invariant subspaces: A classification of geometric concepts*, International Journal of Control, 33 (1981), pp. 51–93.

[98] B. JAKUBCZYK AND W. RESPONDEK, *On linearization of control systems*, Bull. Acad. Polonaise Sci. Ser. Sci. Math., (1980), p. 517–522.

[99] K. JANKOWSKI, *Dynamics of controlled mechanical systems with material and program constraints—III. Illustrative examples*, Mechanism and Machine Theory, 24 (1989), pp. 175–179.

[100] K. P. JANKOWSKI AND H. VAN BRUSSEL, *Inverse dynamics task control of flexible joint robots—I: Continuous-time approach*, Mechanism and Machine Theory, 28 (1993), pp. 741–749.

[101] Z. JIANDONG AND C. ZHAOLIN, *Exact linearization for a class of nonlinear differential-algebraic systems*, in Proceedings of the 4th World Congress on Intelligent Control and Automation, vol. 1, IEEE, 2002, pp. 211–214.

[102] T. KAILATH, *Linear Systems*, vol. 156, Prentice-Hall Englewood Cliffs, NJ, 1980.

[103] R. KALMAN, *On the general theory of control systems*, IRE Transactions on Automatic Control, 4 (1959), pp. 110–110.

[104] R. KALMAN, *Kronecker invariants and feedback*, tech. rep., STANFORD UNIV CALIF DEPT OF OPERATIONS RESEARCH, 1971.

[105] R. E. KALMAN, *Canonical structure of linear dynamical systems*, Proceedings of the National Academy of Sciences, 48 (1962), pp. 596–600.

[106] R. E. KALMAN, *Mathematical description of linear dynamical systems*, Journal of the Society for Industrial and Applied Mathematics, Series A: Control, 1 (1963), pp. 152–192.

[107] S. KAPRIELIAN, K. CLEMENTS, AND J. TURI, *Feedback stabilization for an AC/DC power system model*, in Proceedings of the 29th IEEE Conference on Decision and Control, IEEE, 1990, pp. 3367–3372.

[108] O. KATO AND I. SUGIURA, *An interpretation of airplane general motion and contol as inverse problem*, Journal of Guidance, Control, and Dynamics, 9 (1986), pp. 198–204.

[109] R. KAUSAL AND S. TRENN, *Impulses in structured nonlinear switched DAEs*, in IEEE 56th Annual Conference on Decision and Control (CDC), IEEE, 2017, pp. 3181–3186.

[110] R. KAUSAR AND S. TRENN, *Water hammer modeling for water networks via hyperbolic PDEs and switched DAEs*, in XVI International Conference on Hyperbolic Problems: Theory, Numerics, Applications, Springer, 2016, pp. 123–135.

[111] S. KAWAJI AND E. Z. TAHA, *Feedback linearization of a class of nonlinear descriptor systems*, in Proceedings of the 33rd IEEE Conference on Decision and Control, vol. 4, IEEE, 1994, pp. 4035–4037.

[112] S. KOBAYASHI AND K. NOMIZU, *Foundations of differential geometry*, vol. 1, New York, 1963.

[113] J. KOVECSES, J.-C. PIEDBOEUF, AND C. LANGE, *Dynamics modeling and simulation of constrained robotic systems*, IEEE/ASME Transactions on Mechatronics, 8 (2003), pp. 165–177.

[114] A. J. KRENER, *(ad_f,g), (ad_f,g) and locally (ad_f,g) invariant and controllability distributions*, SIAM Journal on Control and Optimization, 23 (1985), pp. 523–549.

[115] ——, *Conditioned invariant and locally conditioned invariant distributions*, Systems & Control Letters, 8 (1986), pp. 69–74.

[116] H. KRISHNAN AND N. H. MCCLAMROCH, *Tracking in nonlinear differential-algebraic control systems with applications to constrained robot systems*, Automatica, 30 (1994), pp. 1885–1897.

[117] L. KRONECKER, *Algebraische Reduction der Schaaren bilinearer Formen*, 1890.

[118] M. KUIJPER AND J. M. SCHUMACHER, *Minimality of descriptor representations under external equivalence*, Automatica, 27 (1991), pp. 985–995.

[119] A. KUMAR AND P. DAOUTIDIS, *Feedback control of nonlinear differential-algebraic-equation systems*, AIChE Journal, 41 (1995), pp. 619–636.

[120] ——, *Control of nonlinear differential algebraic equation systems: an overview*, in Nonlinear Model Based Process Control, Springer, 1998, pp. 311–344.

[121] ——, *Control of Nonlinear Differential Algebraic Equation Systems with Applications to Chemical Processes*, vol. 397, CRC Press, 1999.

[122] P. KUNKEL AND V. MEHRMANN, *Index reduction for differential-algebraic equations by minimal extension*, ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics, 84 (2004), pp. 579–597.

[123] G. LE VEY, *Some remarks on solvability and various indices for implicit differential equations*, Numerical Algorithms, 19 (1998), pp. 127–145.

[124] G. LEBRET AND J. J. LOISEAU, *Proportional and proportional-derivative canonical forms for descriptor systems with outputs*, Automatica, 30 (1994), pp. 847–864.

[125] J. LÉVINE, *On necessary and sufficient conditions for differential flatness*, Applicable Algebra in Engineering, Communication and Computing, 22 (2011), pp. 47–90.

[126] F. LEWIS AND K. OZCALDIRAN, *Geometric structure and feedback in singular systems*, IEEE Transactions on Automatic Control, 34 (1989), pp. 450–455.

[127] F. L. LEWIS, *A survey of linear singular systems*, Circuits, Systems and Signal Processing, 5 (1986), pp. 3–36.

[128] ——, *A tutorial on the geometric analysis of linear time-invariant implicit systems*, Automatica, 28 (1992), pp. 119–137.

[129] X. LIU* AND D. WC HO, *Stabilization of non-linear differential-algebraic equation systems*, International Journal of Control, 77 (2004), pp. 671–684.

[130] J.-J. LOISEAU, *Some geometric considerations about the Kronecker normal form*, International Journal of Control, 42 (1985), pp. 1411–1431.

[131] J. J. LOISEAU, K. ÖZÇALDIRAN, M. MALABRE, AND N. KARCANIAS, *Feedback canonical forms of singular systems*, Kybernetika, 27 (1991), pp. 289–305.

[132] J. J. LOISEAU AND P. ZAGALAK, *On pole structure assignment in linear systems*, International Journal of Control, 82 (2009), pp. 1179–1192.

[133] D. G. LUENBERGER, *Time-invariant descriptor systems*, Automatica, 14 (1978), pp. 473–480.

[134] D. G. LUENBERGER AND A. ARBEL, *Singular dynamic Leontief systems*, Econometrica: Journal of the Econometric Society, (1977), pp. 991–995.

[135] M. MALABRE, *More geometry about singular systems*, in 26th IEEE Conference on Decision and Control, vol. 26, IEEE, 1987, pp. 1138–1139.

[136] ——, *Generalized linear systems: geometric and structural approaches*, Linear Algebra and its Applications, 122 (1989), pp. 591–621.

[137] M. MALABRE, V. KUČERA, AND P. ZAGALAK, *Reachability and controllability indices for linear descriptor systems*, Systems & Control Letters, 15 (1990), pp. 119–123.

[138] M. MANDERLA, D. SCHMITT, AND U. KONIGORSKI, *Modelling, simulation and control of a redundant parallel robotic manipulator based on invariant manifolds*, Mathematical and Computer Modelling of Dynamical Systems, 16 (2010), pp. 95–113.

[139] R. MARINO, W. RESPONDEK, AND A. VAN DER SCHAFT, *Equivalence of nonlinear systems to input-output prime forms*, SIAM Journal on Control and Optimization, 32 (1994), pp. 387–407.

[140] S. E. MATTSSON AND G. SÖDERLIND, *Index reduction in differential-algebraic equations using dummy derivatives*, SIAM Journal on Scientific Computing, 14 (1993), pp. 677–692.

[141] N. MCCLAMROCH, *Singular systems of differential equations as dynamic models for constrained robot systems*, in Proceedings of IEEE International Conference on Robotics and Automation, vol. 3, IEEE, 1986, pp. 21–28.

[142] V. MEHRMANN AND T. STYKEL, *Descriptor systems: a general mathematical framework for modelling, simulation and control*, Automatisierungstechnik, 54 (2006), pp. 405–415.

[143] S. MOBERG, *Modeling and Control of Flexible Manipulators*, PhD thesis, Linköping University Electronic Press, 2010.

[144] B. MOLINARI, *A strong controllability and observability in linear multivariable control*, IEEE Transactions on Automatic Control, 21 (1976), pp. 761–764.

[145] B. P. MOLINARI, *Structural invariants of linear multivariable systems*, International Journal of Control, 28 (1978), pp. 493–510.

[146] A. MORSE, *Structural invariants of linear multivariable systems*, SIAM Journal on Control, 11 (1973), pp. 446–465.

[147] A. MORSE, *System invariants under feedback and cascade control*, in Mathematical Systems Theory, Springer, 1976, pp. 61–74.

[148] A. MORSE AND W. WONHAM, *Status of noninteracting control*, IEEE Transactions on Automatic Control, 16 (1971), pp. 568–581.

[149] H. NIJMEIJER AND W. RESPONDEK, *Dynamic input-output decoupling of nonlinear control systems*, IEEE transactions on Automatic control, 33 (1988), pp. 1065–1070.

[150] H. NIJMEIJER AND A. VAN DER SCHAFT, *Controlled invariance for nonlinear systems*, IEEE Transactions on Automatic Control, 27 (1982), pp. 904–914.

[151] H. NIJMEIJER AND A. VAN DER SCHAFT, *Nonlinear Dynamical Control Systems*, vol. 175, Springer, 1990.

[152] K. ÖZÇALDIRAN, *A geometric characterization of the reachable and the controllable subspaces of descriptor systems*, Circuits, Systems and Signal Processing, 5 (1986), pp. 37–48.

[153] K. OZCALDIRAN, *A complete classification of controllable singular systems*, in 29th IEEE Conference on Decision and Control, IEEE, 1990, pp. 3596–3597.

[154] C. PANTELIDES, D. GRITSIS, K. MORISON, AND R. SARGENT, *The mathematical modelling of transient systems using differential-algebraic equations*, Computers & Chemical Engineering, 12 (1988), pp. 449–454.

[155] F. PASQUALETTI, F. DÖRFLER, AND F. BULLO, *Attack detection and identification in cyber-physical systems*, IEEE Transactions on Automatic Control, 58 (2013), pp. 2715–2729.

[156] P. S. PEREIRA DA SILVA AND S. BATISTA, *On state representations of nonlinear implicit systems*, International Journal of Control, 83 (2010), pp. 441–456.

[157] L. PETZOLD, *Differential/algebraic equations are not ODE's*, SIAM Journal on Scientific and Statistical Computing, 3 (1982), pp. 367–384.

[158] P. J. RABIER AND W. C. RHEINBOLDT, *A geometric treatment of implicit differential-algebraic equations*, Journal of Differential Equations, 109 (1994), pp. 110–146.

[159] ——, *Nonholonomic Motion of Rigid Mechanical Systems from a DAE Viewpoint*, vol. 68, SIAM, 2000.

[160] P. J. RABIER, W. C. RHEINBOLDT, ET AL., *A general existence and uniqueness theory for implicit differential-algebraic equations*, Differential and Integral Equations, 4 (1991), pp. 563–582.

[161] S. REICH, *On a geometrical interpretation of differential-algebraic equations*, Circuits, Systems, and Signal Processing, 9 (1990), pp. 367–382.

[162] ——, *On an existence and uniqueness theory for nonlinear differential-algebraic equations*, Circuits, Systems and Signal Processing, 10 (1991), pp. 343–359.

[163] W. RESPONDEK, *Geometric methods in linearization of control systems*, Banach Center Publications, 1 (1985), pp. 453–467.

[164] W. C. RHEINBOLDT, *Differential-algebraic systems as differential equations on manifolds*, Mathematics of Computation, 43 (1984), pp. 473–482.

[165] R. RIAZA, *Differential-Algebraic Systems: Analytical Aspects and Circuit Applications*, World Scientific, 2008.

[166] ——, *DAEs in circuit modelling: a survey*, in Surveys in Differential-Algebraic Equations I, Springer, 2013, pp. 97–136.

[167] H. H. ROSENBROCK, *State-space and Multivariable Theory*, Nelson, 1970.

[168] ——, *Structural properties of linear dynamical systems*, International Journal of Control, 20 (1974), pp. 191–202.

[169] P. ROUCHON, M. FLIESS, AND J. LÉVINE, *Kronecker's canonical forms for nonlinear implicit differential systems*, in System Structure and Control 1992, Elsevier, 1992, pp. 248–251.

[170] W. J. RUGH AND W. J. RUGH, *Linear System Theory*, vol. 2, Prentice Hall Upper Saddle River, NJ, 1996.

[171] S. SASTRY AND C. DESOER, *Jump behavior of circuits and systems*, IEEE Transactions on Circuits and Systems, 28 (1981), pp. 1109–1124.

[172] L. SILVERMAN, *Inversion of multivariable linear systems*, IEEE Transactions on Automatic Control, 14 (1969), pp. 270–276.

[173] S. N. SINGH AND W. J. RUGH, *Decoupling in a class of nonlinear systems by state variable feedback*, Journal of Dynamic Systems, Measurement, and Control, 94 (1972), pp. 323–329.

[174] P. SINHA, *State feedback decoupling of nonlinear systems*, IEEE Transactions on Automatic Control, 22 (1977), pp. 487–489.

[175] J. SJÖBERG, *Optimal Control and Model Reduction of Nonlinear DAE Models*, PhD thesis, Institutionen för systemteknik, 2008.

[176] M. S. Soto and C. Tischendorf, *Numerical analysis of DAEs from coupled circuit and semiconductor simulation*, Applied Numerical Mathematics, 53 (2005), pp. 471–488.

[177] A. Steinbrecher, *Numerical Solution of Quasi-Linear Differential-Algebraic Equations and Industrial Simulation of Multibody Systems*, PhD thesis, Technische Universität Berlin, 2006.

[178] R. Su, *On the linear equivalents of nonlinear systems*, Systems Control Letters, 2 (1982), pp. 48 – 52.

[179] F. Takens, *Constrained equations; a study of implicit differential equations and their discontinuous solutions*, in Structural Stability, the Theory of Catastrophes, and Applications in the Sciences, Springer, 1976, pp. 143–234.

[180] I. Tall and W. Respondek, *Feedback equivalence of nonlinear control systems: a survey on formal approach*, in Chaos in Automatic Control, CRC Press, 2005, pp. 156–281.

[181] S. Trenn, *A normal form for pure differential algebraic systems*, Linear Algebra and its Applications, 430 (2009), pp. 1070–1084.

[182] D. Vafiadis and N. Karcanias, *Canonical forms for descriptor systems under restricted system equivalence*, Automatica, 33 (1997), pp. 955–958.

[183] M. van Nieuwstadt, M. Rathinam, and R. M. Murray, *Differential flatness and absolute equivalence*, in Proceedings of 33rd IEEE Conference on Decision and Control, vol. 1, IEEE, 1994, pp. 326–332.

[184] A. Varga, *Computation of Kronecker-like forms of a system pencil: Applications, algorithms and software*, in Proceedings of the 1996 IEEE International Symposium on Computer-Aided Control System Design, IEEE, 1996, pp. 77–82.

[185] J. Wang and C. Chen, *Exact linearization of nonlinear differential algebraic systems*, in 2001 International Conferences on Info-Tech and Info-Net. Proceedings (Cat. No.01EX479), vol. 4, Oct 2001, pp. 284–290 vol.4.

[186] K. Weierstrass, *Zur theorie der bilinearen und quadratischen formen*, Berl. Monatsb, (1968), p. 310–338.

[187] J. Willems, *Almost invariant subspaces: an approach to high gain feedback design–Part I: Almost controlled invariant subspaces*, IEEE Transactions on Automatic Control, 26 (1981), pp. 235–252.

[188] ——, *Almost invariant subspaces: an approach to high gain feedback design–Part II: Almost conditionally invariant subspaces*, IEEE Transactions on Automatic Control, 27 (1982), pp. 1071–1085.

[189] J. C. WILLEMS, *Input-output and state-space representations of finite-dimensional linear time-invariant systems*, Linear Algebra and its Applications, 50 (1983), pp. 581–608.

[190] J. C. WILLEMS AND C. COMMAULT, *Disturbance decoupling by measurement feedback with stability or pole placement*, SIAM Journal on Control and Optimization, 19 (1981), pp. 490–504.

[191] K.-T. WONG, *The eigenvalue problem* $\lambda Tx + Sx$, Journal of Differential Equations, 16 (1974), pp. 270–280.

[192] W. WONHAM AND A. MORSE, *Feedback invariants of linear multivariable systems*, Automatica, 8 (1972), pp. 93–100.

[193] W. M. WONHAM, *Linear multivariable control*, in Optimal Control Theory and its Applications, Springer, 1974, pp. 392–424.

[194] W. M. WONHAM AND A. S. MORSE, *Decoupling and pole assignment in linear multivariable systems: a geometric approach*, SIAM Journal on Control, 8 (1970), pp. 1–18.

[195] L. XIAOPING, *On linearization of nonlinear singular control systems*, in American Control Conference, June 1993, pp. 2284–2287.

[196] L. XIAOPING, *Asymptotic output tracking of nonlinear differential-algebraic control systems*, Automatica, 34 (1998), pp. 393–397.

[197] L. XIAOPING AND S. CELIKOVSKY, *Feedback control of affine nonlinear singular control systems*, International Journal of Control, 68 (1997), pp. 753–774.

[198] W. YIM AND S. N. SINGH, *Feedback linearization of differential-algebraic systems and force and position control of manipulators*, Dynamics and Control, 3 (1993), pp. 323–352.

[199] X. YUN AND N. SARKAR, *Unified formulation of robotic systems with holonomic and nonholonomic constraints*, IEEE Transactions on Robotics and Automation, 14 (1998), pp. 640–650.

[200] E. ZEEMAN, *Differential equations for the heartbeat and nerve impulse*, Towards a Theoretical Biology, 4 (1972), pp. 8–67.

Some Notations and Notions from Differential Geometry

dh	the differential of a smooth function $h : X \to \mathbb{R}$. In coordinates $x = [x_1, \dots, x_n]^T$, we have $dh = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dx_i = [\frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_n}]$
$< x, g >$	the inner product of a co-vector $x = [x_1, \dots, x_n]$ and vector $g = [g^1, \dots, g^n]^T$, i.e., $\sum_{i=1}^{n} x_i g^i$
$L_f h$	the lie derivative (direction derivative) of a smooth function $h : X \to \mathbb{R}$ with respect to a vector field f. In coordinates x, $L_f h(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(x) f(x) = \frac{\partial h}{\partial x}(x) f(x) =< dh(x), f(x) >$
$[f, g]$	the lie bracket of two vector fields f and g. In coordinates x, $[f, g](x) = \frac{\partial g}{\partial x}(x) f(x) - \frac{\partial f}{\partial x}(x) g(x) = \sum_j \left(\sum_i \frac{\partial g_j}{\partial x_i}(x) f_i(x) - \frac{\partial f_j}{\partial x_i}(x) g_i(x) \right) \frac{\partial}{\partial x_j}$, where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ denote the Jacobi matrices of g and f
$ad_f g$	$[f, g]$
$\wedge$	exterior product
$d\xi_1 \wedge d\xi_2$	$d\xi_1^1 \wedge \cdots \wedge \xi_1^{n_1} \wedge d\xi_2 \wedge \cdots \wedge \xi_2^{n_2}$, where $\xi_1 = (\xi_1^1, \dots, \xi_1^{n_1})$ and $\xi_2 = (\xi_2^2, \dots, \xi_2^{n_2})$
TX	the tangent bundle of a smooth manifold X
$T_x M$	the tangent space of a submanifold M of $\mathbb{R}^n$ at $x \in M$
distribution $\mathscr{D}$	a map attaching to any $x \in X$ a linear subspace $\mathscr{D}(x) \subseteq T_x X$
codistribution $\mathscr{D}^\perp$	consists of all linear forms (co-vectors) $\omega(x)$ such that $< \omega(x), g(x) >= 0$, for any $g(x) \in \mathscr{D}(x)$
involutive $\mathscr{D}$	a distribution $\mathscr{D}$ is involutive if for any $f, g \in \mathscr{D}$, we have $[f, g] \in \mathscr{D}$
Foliation M_α	a p-dimensional foliation of an n-dimensional manifold X is a decomposition of X into a union of disjoint connected submanifolds $\{M_\alpha\}_{\alpha \in A}$, called the leaves of the foliation, with the following property: Every point in X has a neighborhood U and a system of local coordinates $x = (x_1, \dots, x_n) : U \to \mathbb{R}^n$ such that for each leaf M_α, the components of $U \cap M_\alpha$ are described by the equations $x_{p+1} = const., \dots, x_n = const.$

Abbreviations

DAE	differential-algebraic equation
DAECS	differential-algebraic equation control system
EM-equivalent	extended Morse equivalent
EMCF	extended Morse canonical form
EMNF	extended Morse normal form
EMTF	extended Morse triangular form
ex-equivalent	externally equivalent
ex-fb-equivalent	externally feedback equivalent
FBCF	feedback canonical form
in-equivalent	internally equivalent
in-fb-equivalent	internally feedback equivalent
KCF	Kronecker canonical form
M-equivalent	Morse equivalent
MCF	Morse canonical form
MCISF	maximal controlled invariant submanifold form
MNF	Morse normal form
MTF	Morse triangular form
NWF	nonlinear generalization of the Weierstrass form
ODE	ordinary differential equation
ODECS	ordinary differential equation control system
QL	quasi-linear
SE	semi-explicit
SMCISF	special maximal controlled invariant submanifold form
sys-equivalent	system equivalent
sys-fb-equivalent	system feedback equivalent
WF	Weierstrass form