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IFAC PapersOnLine 52-16 (2019) 292–297

Internal and External Linearization of Semi-Explicit Differential Algebraic Equations

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Abstract: In this paper, we study two kinds of linearization (internal and external) of nonlinear differential-algebraic equations DAEs of semi-explicit SE form. The difference of external and internal linearization is illustrated by an example of a mechanical system. Moreover, we define different levels of external equivalence for two SE DAEs. The proposed explicitation procedure allows us to treat a given SE DAE as a control system defined up to feedback transformation (a class of control systems). Then sufficient and necessary conditions, expressed via explicitation procedure, are given to describe when a given SE DAE is level-3 externally equivalent to a linear SE DAE of some specific forms. At last, we show by an example that level-2 external linearization of a DAE can be achieved if its explicitation is level-2 input-output linearizable.

Keywords: differential-algebraic equations, implicit systems, linearization, internal and external equivalence, geometric methods

1. INTRODUCTION

We study differential-algebraic equations DAEs of semi-explicit SE form

$$\Xi^{se}: \begin{cases} \mathcal{R}(x)\dot{x} = a(x) \\ 0 \quad\;\; = c(x), \end{cases} \tag{1}$$

where $\mathcal{R}(x)$, $a(x)$, and $c(x)$ are smooth maps with values in $\mathbb{R}^{r\times n}$, $\mathbb{R}^r$, and $\mathbb{R}^p$, respectively, and the word smooth will mean throughout $\mathscr{C}^\infty$-smooth, and where $x \in X$ is called the generalized state and X is an open subset of $\mathbb{R}^n$. A SE DAE of form (1) will be denoted by $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ or, simply, Ξ^{se}. A solution of Ξ^{se} is a curve $x(t) \in \mathscr{C}^1(I;X)$ with an open interval I such that for all $t \in I$, $x(t)$ solves (1). An admissible point of (1) is a point $x^0 \in X$ such that through x^0, there passes at least one solution. The motivation of studying SE DAEs is their presence in modeling of electrical circuits Riaza (2008), chemical processes Kumar and Daoutidis (1998), and constrained mechanical systems Campbell (1995).

Definition 1. (External equivalence). Consider two SE DAEs $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ and $\tilde{\Xi}^{se}_{n,r,p} = (\tilde{\mathcal{R}}, \tilde{a}, \tilde{c})$. If there exists a diffeomorphism $\psi : X \to \tilde{X}$ and a smooth invertible $r \times r$-matrix $Q^a(x)$ such that

$$\tilde{\mathcal{R}}(\psi(x)) = Q^a(x)\mathcal{R}(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1},$$
$$\tilde{a}(\psi(x)) = Q^a(x)a(x),$$

and if, additionally,

(i) there exists a smooth invertible $p \times p$-matrix $Q^c(x)$ such that $\tilde{c}(\psi(x)) = Q^c(x)c(x)$, we call Ξ^{se} and $\tilde{\Xi}^{se}$ externally equivalent, or shortly ex-equivalent, of level-1;

(ii) there exists a smooth invertible $p \times p$-matrix $Q^c(x)$ such that $\tilde{c}(\psi(x)) = Q^c(x)c(x)$ and $Q^c(x) = Q(c(x))$ for some invertible $Q(x)$, we call Ξ^{se} and $\tilde{\Xi}^{se}$ ex-equivalent of level-2;

(iii) there exists a constant invertible $p \times p$-matrix T such that $\tilde{c}(\psi(x)) = Tc(x)$, we call Ξ^{se} and $\tilde{\Xi}^{se}$ ex-equivalent of level-3.

The level-i ($i = 1, 2, 3$) ex-equivalence of two SE DAEs will be denoted by $\Xi^{se} \overset{ex-i}{\sim} \tilde{\Xi}^{se}$. If $\psi : X_0 \to \tilde{X}_0$ is a local diffeomorphism between neighborhoods X_0 of x^0 and $\tilde{X}_0$ of $\tilde{x}^0$, and $Q^a(x)$, $Q^c(x)$ are defined locally on X_0, we will speak about local ex-equivalence.

Remark 2. For SE DAEs, we propose three levels of external equivalence that correspond to three kinds of transformations of the constraint $c(x) = 0$. The interpretation of the three levels of ex-equivalence is as follows.

(i) Two constraints $0 = c(x)$ and $0 = \tilde{c}(x)$ are level-1 ex-equivalent if and only if $M_0 = \tilde{M}_0$, where $M_0 = \{x \,|\, c(x) = 0\}$ and $\tilde{M}_0 = \{x \,|\, \tilde{c}(x) = 0\}$;

(ii) Two constraints are level-2 ex-equivalent means that the foliations $M_d = \{x \,|\, c(x) = d\}$ and $\tilde{M}_{\tilde{d}} = \{x \,|\, \tilde{c}(x) = \tilde{d}\}$ coincide, where $d, \tilde{d} \in \mathbb{R}^p$, i.e., there exists a diffeomorphism ϕ such that $\tilde{M}_{\tilde{d}} = M_{\phi(d)}$. It also implies that the set of motions $x(t)$ respecting the constraint $c(x) = d$ (equivalently, $\mathrm{D}c(x(t)) \cdot \dot{x}(t) = 0$) coincides with that respecting $\tilde{c}(x) = \tilde{d}$;

(iii) Two constraints are level-3 ex-equivalent means the foliations M_d and $\tilde{M}_{\tilde{d}}$ coincide via a linear parameter transformation, i.e., $\tilde{M}_{\tilde{d}} = M_{Td}$.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2019.11.794

There are two kinds of equivalence relations for DAEs, namely, external and internal equivalence (for details of internal equivalence, we refer to Chen and Respondek (2018a) (linear DAEs) and Chen and Respondek (2018b) (nonlinear DAEs)). We will show the differences of these two equivalence relations in Section 3 by examples. Roughly speaking, the word "internal" means that we consider the DAE on its constrained submanifold only, Reich (1991) (also called invariant submanifold in Chen and Respondek (2018b) or configuration subspace in Steinbrecher (2006)), i.e., where the solutions of the DAE exist. Correspondingly, the word "external" means that we consider the DAE in a whole neighborhood and for some points in that neighborhood there may not exist solutions. More precisely, solutions of $\mathcal{R}\dot{x} = a(x)$ pass through each point of the neighborhood but some may not respect the algebraic constraint $c(x) = 0$. Therefore, external equivalence is interesting in all problems, where the nominal point does not respect the constraints but we want to steer the solution towards the constraint (in finite time or asymptotically). So the form of the DAE matters not only on the constraint set but in a neighborhood as well.

The purpose of this paper is to discuss when a SE DAE, given by (1), is locally equivalent to a linear SE DAE. Some results for linearization of DAEs can be found in Kawaji and Taha (1994), Jiandong and Zhaolin (2002), however, the concepts of external and internal equivalence are not distinguished in those papers. In the present paper, we will use a new tool named *explicitation* (see Definition 6) to represent DAEs as explicit control systems. As shown in the examples of Section 3, the internal linearizability has direct relations with the feedback linearizability of the explicit control system on its maximal output zeroing submanifold. For the external linearizability, we only consider level-3 and level-2 external equivalence, level-1 will be discussed in future. The level-3 external linearizability of SE DAEs is closely related to the involutivity of some distributions of an explicit control system (obtained via *explicitation*), as is shown in Section 4. Moreover, in Section 5 we provide an example of a system that is level-2 externally linearizable but not level-3 externally linearizable.

2. SOME RESULTS FOR THE LINEAR CASE

In this section, we introduce some concepts of linear semi-explicit DAEs of form

$$\Delta^{se}: \begin{cases} R\dot{x} = Ax \\ 0 = Cx, \end{cases} \tag{2}$$

where $R \in \mathbb{R}^{r\times n}$, $A \in \mathbb{R}^{r\times n}$, $C \in \mathbb{R}^{p\times n}$. We assume R to be of full row rank. A DAE of form (2) will be denoted by $\Delta^{se}_{r,n,p} = (R, A, C)$ or, simply, Δ^{se}. From the Kronecker canonical form KCF, see e.g. Kronecker (1890) or Berger and Trenn (2012), for matrix pencils $sE - H$ (or equivalently, for linear DAEs $E\dot{x} = Hx$), the canonical form SCF (see Proposition 1 below) can be deduced for linear SE DAEs. Definition 1 applied to linear systems says that two linear SE DAEs $\Delta^{se} = (R, A, C)$ and $\tilde{\Delta}^{se} = (\tilde{R}, \tilde{A}, \tilde{C})$ are ex-equivalent if there exists constant invertible matrices P, Q^a, Q^c such that $\tilde{R} = Q^a RP^{-1}$, $\tilde{A} = Q^a AP^{-1}$, $\tilde{C} = Q^c CP^{-1}$.

Proposition 1. Any linear SE DAE $\Delta^{se}_{r,n,p} = (R, A, C)$ is ex-equivalent to the following semi-explicit canonical form:

$$SCF: \begin{cases} \dot{z}^1 = A^1z^1 + B^1w^1 + K^1y \\ \dot{z}^2 = A^2z^2 \qquad\qquad\; + K^2y \\ \dot{z}^3 = A^3z^3 + B^3w^3 + K^3y \\ \dot{z}^4 = A^4z^4 \qquad\qquad\; + K^4y \\ 0 = w^0 \\ 0 = C^3z^3 \\ 0 = C^4z^4, \end{cases}$$

where $y = (y^0, y^3, y^4)$, $y^0 = w^0$, $y^3 = C^3z^3$ and $y^4 = C^4z^4$, and the system matrices satisfy $A^k = \mathrm{diag}\,[A^k_1, \dots, A^k_e]$ for $k = 1, 3, 4$, $B^k = \mathrm{diag}\,[B^k_1, \dots, B^k_e]$ for $k = 1, 3$ and B^k is empty for $k = 2, 4$, $C^k = \mathrm{diag}\,[C^k_1, \dots, C^k_e]$ for $k = 3, 4$ and C^k is empty for $k = 1, 2$, with

$$A^k_i = \begin{bmatrix} 0 & I_{\mu_i-1} \\ 0 & 0 \end{bmatrix},\ B^k_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\mu_i\times 1},\ C^k_i = [1\ 0] \in \mathbb{R}^{1\times\mu_i},$$

for $i = 1, \dots, e$, where e depends on k and is equal to a, b, c, d for $k = 1, 2, 3, 4$, respectively; A^2 is in the Jordan canonical form for real matrices.

Remark 3. If we regard the algebraic constraint as the zero output of the control system, the above SCF coincides with the Morse canonical form MCF for linear control systems (see Morse (1973)), modulo output injection terms K^iy.

Now let $\mathscr{M}^*$ be the largest subspace $\mathscr{M}$ such that

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathscr{M} \subseteq \begin{bmatrix} R \\ 0 \end{bmatrix} \mathscr{M}.$$

The Wong sequences (see Wong (1974)) of Δ^{se} are:

$$\mathscr{V}_0 := \mathbb{R}^n, \quad \mathscr{V}_{i+1} := \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \begin{bmatrix} R \\ 0 \end{bmatrix} \mathscr{V}_i, \quad i \in \mathbb{N},$$

$$\mathscr{W}_0 := \{0\}, \quad \mathscr{W}_{i+1} := \begin{bmatrix} R \\ 0 \end{bmatrix}^{-1} \begin{bmatrix} A \\ C \end{bmatrix} \mathscr{W}_i, \quad i \in \mathbb{N}.$$

The limits of $\mathscr{V}_i$ and $\mathscr{W}_i$ are denoted by $\mathscr{V}^*$ and $\mathscr{W}^*$. Notice that the solutions of Δ^{se} exist on $\mathscr{M}^*$ only and, moreover, $\mathscr{M}^* = \mathscr{V}^*$ (see, e.g. Chen and Respondek (2018a)). Now we introduce the following regularity and reachability concepts (compare Berger and Reis (2013)).

Definition 4. $\Delta^{se}_{r,n,p} = (R, A, C)$ is called

- ***internally regular***, if $\forall\ x^0 \in \mathscr{M}^*$, $\exists$ only one solution $x(t)$ for $t \geq 0$ such that $x(0) = x^0$,
- ***regular***, if it is *internally regular* and $r + p = n$,
- ***internally reachable***, if $\forall\ x^0, x^e \in \mathscr{M}^*$, $\exists\ t_e > 0$ and a solution $x(t)$ of Δ^{se} such that $x(0) = x^0$ and $x(t_e) = x^e$,
- ***constraint-free reachable***, if $\forall x^0, x^e \in \mathbb{R}^n$, $\exists\, t_e > 0$ and a solution $x(t)$ of $R\dot{x} = Ax$ such that $x(0) = x^0$ and $x(t_e) = x^e$.

Lemma 5. $\Delta^{se}_{r,n,p} = (R, A, C)$ is

(i) *internally regular* $\Leftrightarrow \dim \mathscr{V}^* = \dim(R\mathscr{V}^*) \Leftrightarrow \mathscr{V}^* \cap \mathscr{W}^* = 0 \Leftrightarrow z^1$-subsystem in the SCF is absent,
(ii) *regular* $\Leftrightarrow \mathscr{V}^* \cap \mathscr{W}^* = 0$ and $\mathscr{V}^* \oplus \mathscr{W}^* = \mathbb{R}^n \Leftrightarrow z^1$- and z^4-subsystems in the SCF are absent,
(iii) *internally reachable* $\Leftrightarrow \mathscr{V}^* \subseteq \mathscr{W}^* \Leftrightarrow z^2$-subsystem in the SCF is absent,
(iv) *constraint-free reachable* $\Leftrightarrow R\dot{x} = Ax$ is *internally reachable.*

The above lemma, proved in Berger and Reis (2013), can also be shown using the SCF described in Proposition 1. The purpose of this lemma is to show how the concepts of Definition 4 correspond to certain forms of linear SE DAEs and that they are closely related to the Wong sequences.

3. EXPLICITATION AND INTERNAL LINEARIZATION

We start this section by the definition of *explicitation* for SE DAEs. Throughout the paper, we will assume that $\mathcal{R}(x)$ is of full row rank equal to r in a neighborhood X_0 of a nominal point x^0.

Definition 6. (Explicitation) For $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$, set $m = n - r$. Then the *explicitation* of Ξ^{se}, denoted by $\mathbf{Expl}(\Xi^{se})$, is a class of control systems of the following form:

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)v \\ y = h(x), \end{cases} \quad (3)$$

where $v \in \mathbb{R}^m$ is called the driving variable, $h(x)$ is a smooth $\mathbb{R}^p$-valued function on X_0, and where $f, g_1, \ldots, g_m$ are smooth vector fields on X_0 satisfying

$$f(x) = \mathcal{R}^{\dagger}(x)a(x), \quad \mathrm{Im} g(x) = \ker \mathcal{R}(x), \quad h(x) = c(x).$$

Above $\mathcal{R}^{\dagger}(x)$ is a right inverse of $\mathcal{R}(x)$, i.e., $\mathcal{R}(x)\mathcal{R}^{\dagger}(x) = I_r$ and $g = (g_1, \ldots, g_m)$. We will denote control system (3) by $\Sigma_{n,m,p} = (f, g, h)$ or, simply, Σ.

Notice that $\mathbf{Expl}(\Xi^{se})$ is a class of control systems. Indeed, first, the distribution $\ker \mathcal{R}(x)$ spanned by $g_1, \ldots, g_m$ is given uniquely but not the vector fields $g_1, \ldots, g_m$ themselves and, secondly, f is given up to $\ker \mathcal{R}(x)$. We will use the notation $\Sigma \in \mathbf{Expl}(\Xi^{se})$ to indicate that control system (3) belongs to the *explicitation* class of Ξ^{se}. By setting $y = 0$ for system (3), we get a SE DAE parametrized by the driving variable v. The definition of f and g implies that $\dot{x} = f(x) + g(x)v$ and $\mathcal{R}(x)\dot{x} = a(x)$ have the same solutions. More precisely, if $(x(t), v(t))$, with $v \in \mathscr{C}^0(I)$, is a solution of $\dot{x} = f(x) + g(x)v$, then $x(t)$ is a $\mathscr{C}^1$-solution of $\mathcal{R}(x)\dot{x} = a(x)$ and, conversely, for any $\mathscr{C}^1$-solution $x(t)$ of $\mathcal{R}(x)\dot{x} = a(x)$, there exists a $\mathscr{C}^0$-function $v(t)$ such that $(x(t), v(t))$ satisfies $\dot{x} = f(x) + g(x)v$. Thus, via *explicitation*, we can study the solutions of Σ yielding a zero output instead of studying the solutions of Ξ^{se} directly. Since the *explicitation* allows to treat a SE DAE as a class of control systems, we give the definition of equivalence for control systems.

Definition 7. (System equivalence) Consider two control systems $\Sigma_{n,m,p} = (f, g, h)$ and $\tilde{\Sigma}_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$ defined on X and $\tilde{X}$, respectively. If there exists a diffeomorphism $\psi : X \to \tilde{X}$, an $\mathbb{R}^m$-valued function $\alpha(x)$, and an invertible $m \times m$-matrix-valued function $\beta(x)$ satisfying

$$\begin{aligned} \tilde{f}(\psi(x)) &= \tfrac{\partial \psi(x)}{\partial x} (f + g\alpha)(x), \\ \tilde{g}(\psi(x)) &= \tfrac{\partial \psi(x)}{\partial x} (g\beta)(x), \end{aligned}$$

and if, additionally,

(i) either there exists a constant invertible matrix T such that $\tilde{h}(\psi(x)) = Th(x)$, then we call Σ and $\tilde{\Sigma}$ system equivalent, shortly sys-equivalent, of level-3,
(ii) or there exists a diffeomorphism $\varphi : \mathbb{R}^p \to \mathbb{R}^p$ such that $\tilde{h}(\psi(x)) = \varphi(h(x))$, then we call the two control systems sys-equivalent of level-2.

The sys-equivalence of level-i ($i = 2, 3$) of two control systems will be denoted by $\Sigma \overset{sys-i}{\sim} \tilde{\Sigma}$. If $\psi : X_0 \to \tilde{X}_0$ is a local diffeomorphism between neighborhoods X_0 of x^0 and $\tilde{X}_0$ of x^0, φ is a local diffeomorphism around $h(x^0)$, and $\alpha(x)$, $\beta(x)$ are defined locally on X_0, we will speak about local sys-equivalence.

Actually the above defined system equivalence for two nonlinear control systems of form (3) is widely considered in nonlinear control theory, e.g., Isidori and Ruberti (1984), Marino et al. (1994), Isidori (1995), Nijmeijer and van der Schaft (1990). The following result is essential since it connects control systems with SE DAEs.

Proposition 2. (i) Consider two control systems $\Sigma_{n,m,p} = (f, g, h)$ and $\Sigma_{n,m,p} = (\tilde{f}, \tilde{g}, \tilde{h})$, that belong to the *explicitation* class of $\Xi^{se}_{n,r,p}$, i.e. $\Sigma, \tilde{\Sigma} \in \mathbf{Expl}(\Xi^{se})$. Then there exist an $\mathbb{R}^m$-valued function $\alpha(x)$ and an $m \times m$ invertible matrix $\beta(x)$ such that

$$\tilde{f}(x) = f(x) + g(x)\alpha(x), \;\; \tilde{g}(x) = g(x)\beta(x).$$

(ii) Two SE DAEs $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ and $\tilde{\Xi}^{se}_{n,r,p} = (\tilde{\mathcal{R}}, \tilde{a}, \tilde{c})$ are ex-equivalent of level-2 (respectively, level-3) if and only if two control systems $(f, g, h) = \Sigma \in \mathbf{Expl}(\Xi^{se})$ and $(\tilde{f}, \tilde{g}, \tilde{h}) = \tilde{\Sigma} \in \mathbf{Expl}(\tilde{\Xi}^{se})$ are sys-equivalent of level-2 (respectively, level-3).

Now we apply the above defined *explicitation* to the internal analysis of SE DAEs. For a SE DAE Ξ^{se}, a submanifold M^* is called a *maximal invariant submanifold* (for details, see Chen and Respondek (2018b)) if M^* is the largest submanifold of X such that $\forall\, x^0 \in M^*$, $\exists\, x(t)$ such that $x(0) = x^0$ and $x(t) \in M^*$, $t \in I$. M^* can be seen as a nonlinear generalization of the invariant space $\mathscr{M}^*$ for linear DAEs. But note that $\mathscr{M}^*$ always exists while M^* may not exist. Denote by $\Xi^{se}|_{M^*}$ a semi-explicit DAE Ξ^{se} restricted to its maximal invariant submanifold M^*.

Definition 8. (Internal equival.) Consider two SE DAEs $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ and $\tilde{\Xi}^{se}_{n,r,p} = (\tilde{\mathcal{R}}, \tilde{a}, \tilde{c})$. Let M^* and $\tilde{M}^*$ be their maximal invariant submanifolds. We call Ξ^{se} and $\tilde{\Xi}^{se}$ internally equivalent, shortly in-equivalent, if $\Xi^{se}|_{M^*}$ and $\tilde{\Xi}^{se}|_{\tilde{M}^*}$ are ex-equivalent.

Theorem 9. For $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$, the following are locally equivalent:

(i) Ξ^{se} is in-equivalent to a linear DAE Δ^{se} with internal reachability;
(ii) A (and then any) control system $(f^*, g^*) = \Sigma^* \in \mathbf{Expl}(\Xi^{se}|_{M^*})$ is feedback linearizable;
(iii) The linearizability distributions G_i, given by (14) below, of $\Sigma^* = (f^*, g^*)$, are involutive and of constant rank and $G^* = TM^*$.

The following example illustrates the above theorem. Note that in Chen and Respondek (2018b), it is proved that the maximal invariant submanifold M^* of DAEs coincides with the output zeroing submanifold of any control system in its *explicitation* class.

Example 10. (The Kapitsa pendulum with auxiliary controls). Consider the following equation of the Kapitsa pendulum taken from Fliess et al. (1995)

$$\begin{cases} \dot{\alpha} = p + \frac{u_1}{l}\sin\alpha \\ \dot{p} = (\frac{g}{l} - \frac{(u_1)^2}{l^2}\cos\alpha - \frac{(u_2)^2}{2l^2}\cos\alpha)\sin\alpha - \frac{u_1}{l}p\cos\alpha \\ \dot{z} = u_1. \end{cases} \quad (4)$$

We subject the system to two different holonomic constraints and analyze the constrained system from the DAE point of view.

Case 1: Consider the following holonomic constraint:

$$z + l\cos\alpha = c_{10}, \quad (5)$$

where c_{10} denotes a fixed constant. This holonomic constraint assures that the end joint of the pendulum keeps the same vertical position as its initial point. Now combine equations (4) and (5), and denote $x = (x_1, \ldots, x_5)$, where

$$x_1 = \alpha, \quad x_2 = p, \quad x_3 = z, \quad x_4 = u_1, \quad x_5 = u_2. \quad (6)$$

We get the following SE DAE:

$$\Xi_1^{se}:\begin{cases}\mathcal{R}_1(x)\dot{x} &= a_1(x)\\ 0 &= c_1(x),\end{cases} \quad (7)$$

where

$$\mathcal{R}_1(x)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\end{bmatrix},\quad c_1(x)=x_3+l\cos x_1-c_{10},$$

$$a_1(x)=\begin{bmatrix}x_2+\frac{x_4}{l}\sin x_1\\ (\frac{g}{l}-\frac{(x_4)^2}{l^2}\cos x_1-\frac{(x_5)^2}{2l^2}\cos x_1)\sin x_1-\frac{x_4}{l}x_2\cos x_1\\ x_4\end{bmatrix}.$$

Consider the above DAE around an admissible point $x^0=(x_{10},\dots,x_{50})$ such that $x_{50}\cos x_{10}\sin x_{10}\neq 0$. The *explicitation* of DAE (7) contains the following control system, see Definition 6, denoted by $\Sigma_1=(f_1,g_1,h_1)$, with driving variables $v_1=\dot{x}_4$, $v_2=\dot{x}_5$:

$$\Sigma_1:\begin{cases}\dot{x}=\begin{bmatrix}a_1(x)\\0\\0\end{bmatrix}+\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}\\ y=x_3+l\cos x_1-c_{10}.\end{cases} \quad (8)$$

Recall that the *explicitation* of our DAE is the above control system defined up to feedback transformation. By the zero dynamics algorithm (see Isidori (1995)), the maximal output zeroing submanifold of Σ_1, denoted by M_1^*, can be expressed as:

$$M_1^*=\left\{x\,|\,x_3+l\cos x_1-c_{10}=x_4\cos^2 x_1-lx_2\sin x_1=0\right\}.$$

Then system (8) restricted to M_1^* is

$$\begin{cases}\dot{x}_1=\frac{x_2}{\cos^2 x_1}\\ \dot{x}_2=(\frac{g}{l}-\frac{(x_2)^2}{\cos^3 x_1}-\frac{(x_5)^2}{2l^2}\cos x_1)\sin x_1\\ \dot{x}_5=v_2.\end{cases} \quad (9)$$

System (9) is locally static feedback equivalent to the following chained form around x^0:

$$\dot{\tilde{x}}_1=\tilde{x}_2,\quad \dot{\tilde{x}}_2=\tilde{x}_5,\quad \dot{\tilde{x}}_5=\tilde{v}_2,$$

where $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_5)$ are new coordinates and $\tilde{v}_2$ is a new control. It follows by Theorem 9 that Ξ_1^{se} is internally equivalent to the following linear DAE:

$$\begin{cases}\dot{\tilde{x}}_1=\tilde{x}_2\\ \dot{\tilde{x}}_2=\tilde{x}_5.\end{cases}$$

Case 2: Consider again system (4) but now under the following dummy holonomic constraints

$$\begin{cases}0=u_1\\ 0=\ln|\tan\frac{\alpha}{2}|+(k-1)z,\end{cases}$$

where $k\in\mathbb{R}$. Following the notations of Case 1, we write

$$\Xi_2^{se}:\begin{cases}\mathcal{R}_2(x)\dot{x} &= a_2(x)\\ 0 &= c_2(x),\end{cases} \quad (10)$$

where $\mathcal{R}_2(x)=\mathcal{R}_1(x)$, $a_2(x)=a_1(x)$ and

$$c_2(x)=\begin{bmatrix}x_4\\ \ln|\tan\frac{x_1}{2}|+(k-1)x_3\end{bmatrix}.$$

Consider Ξ_2^{se} around an admissible point x^0. Then the explicitation of Ξ_2^{se} gives a control system $\Sigma_2\in\mathbf{Expl}(\Xi_2^{se})$, where $\Sigma_2=(f_2,g_2,h_2)$ is given by

$$\Sigma_2:\begin{cases}\dot{x}=\begin{bmatrix}a_2(x)\\0\\0\end{bmatrix}+\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}\\ \begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}x_4\\ \ln|\tan\frac{x_1}{2}|+(k-1)x_3\end{bmatrix}.\end{cases} \quad (11)$$

The maximal output zeroing submanifold M_2^*, given by the zero dynamics algorithm applied to Σ_2, is:

$$M_2^*=\left\{x\,\middle|\,\begin{matrix}\ln|\tan\frac{x_1}{2}|+(k-1)x_3=x_2=x_4=0\\ 2lg-(x_5)^2\cos x_1=0\end{matrix}\right\},$$

which is the curve $x_3=\frac{\ln|\tan\frac{x_1}{2}|}{1-k}$, equipped with the coordinate x_1, on the plane $\{x_2=x_4=x_5=0\}$. The zero dynamics of Σ_2 is

$$\dot{x}_1=0, \quad (12)$$

and its solutions consist of fixed admissible points $x^0=(x_{10},0,x_{30},0,x_{50})$. Thus Ξ_2^{se} is in-equivalent to ODE (12).

4. LEVEL-3 EXTERNAL LINEARIZATION

We start by reviewing the results of linearization of input-output map for control systems, given in Isidori and Ruberti (1984). Denote by $r(A(x))$ the rank of the matrix $A(x)$ and denote by $r_{\mathbb{R}}(A(x))$ the dimension of the vector space spanned over $\mathbb{R}$ by the rows of $A(x)$ around x^0.

Theorem 11. (Isidori and Ruberti (1984), Cheng et al. (1988)) For a control system $\Sigma_{n,m,p}=(f,g,h)$, the following conditions are equivalent locally around x^0.

(i) System Σ is level-3 input-output linearizable;

(ii) The Toeplitz matrices

$$M_k=\begin{bmatrix}T_0(x)&T_1(x)&\cdots&T_k(x)\\0&T_0(x)&\cdots&T_{k-1}(x)\\ \cdots&\cdots&\cdots&\cdots\\0&0&\cdots&T_0(x)\end{bmatrix}$$

satisfy $r(M_k(x))=r_{\mathbb{R}}(M_k(x))$ for all $k\le 2n-1$, where $T_k(x)=L_gL_f^kh(x)$;

(iii) System Σ is level-3 sys-equivalent to

$$\begin{cases}\dot{\xi}^1 &= f^1(\xi)+g^1(\xi)v^1+g^3(\xi)v^3\\ \dot{\xi}^3 &= A^3\xi^3+B^3v^3 \qquad +K^3y\\ \dot{\xi}^4 &= f^4(\xi^4) \qquad +K^4y\\ y^3 &= C^3\xi^3\\ y^4 &= C^4\xi^4,\end{cases} \quad (13)$$

where $y=(y^3,y^4)$ and (A^3,B^3,C^3) is prime (see Morse (1973) for the definition of prime form).

Note that in Isidori and Ruberti (1984) and Cheng et al. (1988), the implication $(i)\Rightarrow(ii)$ is proved by *the structure algorithm*, from which a linearizing feedback can be constructed via a $r_{2n-1}\times m$ full row rank decoupling matrix $L_g\Gamma(x)$. Due to the reason of saving space, here we will not re-implement *the structure algorithm* but emphasize that this rank r_{2n-1} will be used for the external linearization problem below.

For a nonlinear control system $\Sigma_{n,m,p}=(f,g,h)$, define sequences of distributions G_i, S_i and codistributions P_i by (see Isidori (1995) and Nijmeijer and van der Schaft (1990) for those concepts, as well as for the definitions of Lie bracket, Lie derivative, and the notations $[f,G]$ and L_fG)

$$\begin{aligned}G_1 &:= G:=\mathrm{span}\{g_1,\dots,g_m\}\\ G_{i+1} &:= G_i+[f,G_i]\\ G^* &:= \sum_{i\ge1}G_i.\end{aligned} \quad (14)$$

$$\begin{aligned}S_1 &:= G,\\ S_{i+1} &:= S_i+[f,S_i\cap\ker dh]+\textstyle\sum_{j=1}^m[g_j,S_i\cap\ker dh]\\ S^* &:= \sum_{i\ge1}S_i.\end{aligned}$$

$$\begin{aligned}P_1 &:= \mathrm{span}\{dh_1,\dots,dh_p\},\\ P_{i+1} &:= P_i+L_f(P_i\cap G^\perp)+\textstyle\sum_{j=1}^m L_{g_j}(P_i\cap G^\perp)\\ P^* &:= \sum_{i\ge1}P_i.\end{aligned}$$

The above distributions and codistributions, together with $V_i:=P_i^\perp$, $V^*:=(P^*)^\perp$, play an important role in the problems of linearization and decoupling of nonlinear control systems.

Theorem 12. Consider $\Xi_{n,r,p}^{se}=(\mathcal{R},a,c)$ around a point x^0. Then in a neighborhood X_0 of x^0, Ξ^{se} is level-3 ex-equivalent to a linear SE DAE Δ^{se} with *internal regularity* and *constraint-free reachability* if and only if a (and then any) control system $\Sigma=(f,g,h)\in\mathbf{Expl}(\Xi^{se})$ satisfies the following conditions in X_0:

(i) Σ is level-3 input-output linearizable;

(ii) $G^*=TX_0$;

(iii) $[ad^k_{\tilde f}\tilde g_i, ad^l_{\tilde f}\tilde g_j] = 0$ for $1 \le i,j \le m,\ 0 \le l,k \le n$, where $\tilde f$ and $\tilde g_i$ are vector fields modified by a feedback transformation resulting from *the structure algorithm*;
(iv) $V^* \cap S^* = 0$.

Moreover, Δ^{se} is *regular* if and only if Ξ^{se} satisfies (i)-(iv) and, additionally, condition

(v) $V^* \oplus S^* = TX_0$.

Remark 13. (i) The distributions V^* and S^* are, obviously, the nonlinear generalizations of the limits of Wong sequences $\mathscr{V}^*$ and $\mathscr{W}^*$, respectively.

(ii) Condition (iv) above can be replaced by (iv)': the rank r_{2n-1} of the decoupling matrix $L_g\Gamma(x)$ in *the structure algorithm* equals m. Condition (v) can be replaced by (v)': $r + p = n$.

Observe that if the rank $r_{2n-1} = m$, which implies that the feedback transformation of *the structure algorithm* is unique, then condition (iii) of Theorem 12 is verifiable. However, if $r_{2n-1} < m$, which implies some inputs are not used for the purpose of input-output linearization, then condition (iii) may be difficult to check. We give the following theorem, in which the "unused" inputs serve to linearize the remaining part (contained in V^*) of the system and all conditions become checkable.

Theorem 14. Consider $\Xi^{se}_{n,r,p} = (\mathcal{R}, a, c)$ around a point x^0. Then in a neighborhood X_0 of x^0, Ξ^{se} is level-3 ex-equivalent to a linear SE DAE Δ^{se} of the form

$$\begin{cases} \dot z^1 = A^1 z^1 + B^1 w^1, & 0 = w^0 \\ \dot z^3 = A^3 z^3 + B^3 w^3 \ + K^3 y, & 0 = C^3 z^3, \end{cases} \tag{15}$$

where all matrices are in the SCF, if and only if a (and then any) control system $\Sigma \in \mathbf{Expl}(\Xi^{se})$ satisfies the following conditions in X_0:

(i) Σ is level-3 input-output linearizable;
(ii) S_i and G_i are involutive and of constant rank;
(iii) $S^* = TX_0$;
(iv) $S_i \cap V^* = G_i \cap V^*$.

Sketch of Proof: *Only if*. If Ξ^{se} is level-3 ex-equivalent to Δ^{se} given by (15), then any control system $\Sigma \in \mathbf{Expl}(\Xi^{se})$ is level-3 ex-equivalent to

$$\begin{cases} \dot z^1 = A^1 z^1 + B^1 w^1, & \dot w^1 = v^1, \\ \dot z^3 = A^3 z^3 + B^3 w^3 + K^3 y, & \dot w^3 = v^3, \\ y^0 = w^0, & \dot w^0 = v^0, \\ y^3 = C^3 z^3. \end{cases}$$

The above linear control system satisfies (i)-(iv) in an obvious way. Moreover, the invariance of S_i, G_i (clearly, G_i is involutive for the linear system), and V^* under level - 3 sys-equivalence, completes the proof of necessity.

If. Suppose $\Sigma \in \mathbf{Expl}(\Xi^{se})$ satisfies conditions (i)-(iv), then by condition (i) and Theorem 11, Σ is level-3 sys-equivalence to a control system of the form (13) via the structural algorithm. Subsequently, condition (iii) implies that there is no ξ_4 in system (13), i.e., after input-output linearization, Σ becomes

$$\begin{cases} \dot\xi^1 = f^1(\xi^1,\xi^3) + g^1(\xi^1,\xi^3)v^1 + g^3(\xi^1,\xi^3)v^3 \\ \dot\xi^3 = A^3\xi^3 + B^3 v^3 + K^3 y^3 \\ y^3 = C^3\xi^3. \end{cases} \tag{16}$$

For ease of proof, we assume that v^1 is of dimension 1. Denote

$$f = \begin{pmatrix} f^1(\xi) \\ A^3\xi^3 + K^3 y^3 \end{pmatrix}, \quad g_1 = \begin{pmatrix} g^1(\xi) \\ 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} g^3(\xi) \\ B^3 \end{pmatrix}.$$

In view of condition (iii), the key of the following proof is to find new coordinates $\tilde\xi^1$ and new control $\tilde v^1$ (we do not change ξ^3 and v^3) such that in $(\tilde\xi^1,\xi^3)$-coordinates and with the control $(\tilde v^1, v^3)$, the distributions G_i are rectified. Notice that from the involutivity of S_i in (ii), we have $S_{i+1} = S_i + [f, S_i \cap \ker dh]$. Now from $V^* = \operatorname{span}\left\{\frac{\partial}{\partial \xi^1}\right\}$, via condition (iv) and a direct calculation of S_i, we get for (16), $G_i \cap V^* = S_i \cap V^* = \operatorname{span}\left\{g_1, ad_f g_1, \ldots, ad_f^{i-1} g_1\right\}$. Then there exists a smallest number, denoted by ρ, such that $G_\rho \cap V^* = G^* \cap V^*$ (note that $\dim(G_\rho \cap V^*) - \dim(G_{\rho-1} \cap V^*) = 1$). Thus, from the involutivity of G_i, we can choose a scalar function $\psi(\xi^1,\xi^3)$ such that $d\psi \in (G_{\rho-1})^\perp$ and $d\psi \notin (G_\rho \cap V^*)^\perp = (V^*)^\perp$. The above construction implies that the dummy output $y^1 = \psi(\xi^1,\xi^3)$ has relative degree ρ and $L_{g_1}L_f^{\rho-1}\psi \neq 0$. Observe that $G_\rho \cap V^* = V^*$ and that $\operatorname{span}\left\{d\psi, \ldots, dL_f^{\rho-1}\psi\right\} \cap (V^*)^\perp = 0$. Thus $(\psi, \ldots, L_f^{\rho-1}\psi, \xi^3)$ form a local diffeomorphism (since $\operatorname{span}\left\{d\xi^3\right\} = (V^*)^\perp$ and $d\psi, \ldots, dL_f^{\rho-1}\psi$ are independent). Finally, via the change of coordinates $\tilde\xi^1_1 = \psi, \ldots, \tilde\xi^1_\rho = L_f^{\rho-1}\psi$ and the feedback transformation $\tilde v_1 = L_f^{\rho-1}\psi + v^1 L_{g_1}L_f^{\rho-1}\psi + v^3 L_{g_3}L_f^{\rho-1}\psi$, we get

$$\begin{cases} \dot{\tilde\xi}^1_1 = \tilde\xi^1_2, \quad \dot{\tilde\xi}^1_2 = \tilde\xi^1_3, \quad \ldots, \quad \dot{\tilde\xi}^1_\rho = \tilde v^1, \\ \dot\xi^3 = A^3\xi^3 + B^3 v^3 + K^3 y^3, \\ y^3 = C^3\xi^3. \end{cases}$$

□

Example 15. (Continuation of Example 10) Case 1: Consider Ξ^{se} around a point x^0 (not necessarily admissible). Assume $x_{50}\cos x_{10}\sin x_{10} \neq 0$. Then the control system Σ_1 satisfies conditions (i)-(iv) of Theorem 14 around x^0. In particular, via the change of coordinates

$$\begin{cases} \tilde x_3 = x_3 + l\cos x_1 - c_0, \quad \tilde x_4 = x_4\cos^2 x_1 - l x_2 \sin x_1, \\ \tilde x_1 = l\ln|\tan\frac{x_1}{2}| - x_3, \quad \tilde x_2 = \frac{l x_2}{\sin x_1}, \\ \tilde x_5 = g - \frac{\cos x_1 (l x_2 + x_4\sin x_1)^2}{l\sin^2 x_1} - \frac{(x_5)^2\cos x_1}{2l} \end{cases}$$

and the static feedback transformation

$$\begin{cases} \tilde v_1 = \tilde\alpha_1(x) + \cos^2 x_1 v_1, \\ \tilde v_2 = \tilde\alpha_2(x) - \frac{2(x_4\sin x_1 + l x_2)\cos x_1}{l\sin x_1}v_1 - \frac{x_5\cos x_1}{l}v_2, \end{cases}$$

where $\tilde\alpha_1(x) = L_f\tilde x_4(x)$ and $\tilde\alpha_2(x) = L_f\tilde x_5(x)$, Σ_1 is level-3 sys-equivalent to $\tilde\Sigma_1$ below. It follows from Proposition 2 that Ξ^{se}_1 is level-3 ex-equivalent to the following Δ^{se}_1 (since $\Sigma_1 \in \mathbf{Expl}(\Xi^{se}_1)$ and $\tilde\Sigma_1 \in \mathbf{Expl}(\Delta^{se}_1)$):

$$\tilde\Sigma_1 : \begin{cases} \dot{\tilde x}_3 = \tilde x_4, \ y = \tilde x_3 \\ \dot{\tilde x}_4 = \tilde v_1 \\ \dot{\tilde x}_1 = \tilde x_2 \\ \dot{\tilde x}_2 = \tilde x_5 \\ \dot{\tilde x}_5 = \tilde v_2 \end{cases} \Rightarrow \Delta^{se}_1 : \begin{cases} \dot{\tilde x}_1 = \tilde x_2 \\ \dot{\tilde x}_2 = \tilde x_5 \\ \dot{\tilde x}_3 = \tilde x_4 \\ 0 = \tilde x_3. \end{cases}$$

Note that the above transformation bringing $\tilde\Sigma_1$ into the linear DAE, given by Δ^{se}_1, is a dual procedure to that of *explicitation* and it is called implicitation of a control system (for details, see Chen and Respondek (2018a), Chen and Respondek (2018b)).

Case 2: We show that although Ξ^{se}_2 is internally equivalent to the ODE $\dot x_1 = 0$, it is ex-equivalent to a linear SE DAE. Consider Ξ^{se}_2 around x^0, which is not necessarily admissible. Assume $x_{50}\cos x_{10}\sin x_{10} \neq 0$. Since Σ_2 satisfies conditions (i)-(v) of Theorem 12 around x^0, it can be seen that Σ_2 is level-3 sys-equivalent to the following $\tilde\Sigma_2$ via the coordinates change

$$\begin{cases} \tilde{x}_3 = x_3, \quad \tilde{x}_4 = x_4, \quad \tilde{x}_1 = l\ln|\tan\frac{x_1}{2}| - x_3, \\ \tilde{x}_2 = \frac{lx_2}{\sin x_1}, \quad \tilde{x}_5 = g - \frac{\cos x_1(lx_2+x_4\sin x_1)^2}{l\sin^2 x_1} - \frac{(x_5)^2\cos x_1}{2l} \end{cases}$$

and the static feedback transformation

$$\begin{cases} \tilde{v}_1 = v_1, \\ \tilde{v}_2 = \tilde{\alpha}_2(x) - \frac{2(x_4\sin x_1 + lx_2)\cos x_1}{l\sin x_1}v_1 - \frac{x_5\cos x_1}{l}v_2, \end{cases}$$

where $\tilde{\alpha}_2(x) = L_f\tilde{x}_5(x)$. Moreover, since $\Sigma_2 \in \mathbf{Expl}(\Xi_2^{se})$ and obviously $\tilde{\Sigma}_2 \in \mathbf{Expl}(\Delta_2^{se})$, by Proposition 2, Ξ_2^{se} is level-3 ex-equivalent to the following Δ_2^{se}, which is *regular* and *constraint-free reachable*:

$$\tilde{\Sigma}_2 : \begin{cases} \dot{\tilde{x}}_3 = \tilde{x}_4 \\ \dot{\tilde{x}}_4 = \tilde{v}_1, \qquad y_1 = \tilde{x}_4 \\ \dot{\tilde{x}}_1 = \tilde{x}_2 + ky_1, \; y_2 = \tilde{x}_1 \\ \dot{\tilde{x}}_2 = \tilde{x}_5 \\ \dot{\tilde{x}}_5 = \tilde{v}_2 \end{cases} \Rightarrow \Delta_2^{se} : \begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 + k\tilde{x}_4 \\ \dot{\tilde{x}}_2 = \tilde{x}_5 \\ \dot{\tilde{x}}_3 = \tilde{x}_4 \\ 0 = \tilde{x}_4 \\ 0 = \tilde{x}_1. \end{cases}$$

5. AN EXAMPLE WHICH IS NOT LEVEL-3 EXTERNALLY LINEARIZABLE BUT SO IS LEVEL-2

Example 16. Consider a SE DAE $\Xi_3^{se} = (\mathcal{R}_3, a_3, c_3)$, described by

$$\mathcal{R}_3(x) = \begin{bmatrix} 1 & 0 & -x_1 & 0 & 0 & 0 \\ 0 & 0 & e^{3x_3} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; a_3(x) = \begin{bmatrix} 2(x_1e^{x_3})^{\frac{1}{2}}x_2 \\ -(x_5+ke^{x_3}) \\ x_6 \end{bmatrix},$$
$$c_3(x) = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix},$$

where $k \in \mathbb{R}$. We can choose a control system $(f_3, g_3, h_3) = \Sigma_3 \in \mathbf{Expl}(\Xi_3^{se})$, given by

$$\Sigma_3 : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 2(x_1e^{x_3})^{\frac{1}{2}}x_2 \\ 0 \\ 0 \\ x_5+ke^{x_3} \\ x_6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & x_1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{3x_3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ y_1 = x_3 \\ y_2 = x_4. \end{cases}$$

It is easy to verify that Σ_3 is not level-3 input-output linearizable (since the Toeplitz matrices $M_k(\Sigma_3)$ do not satisfy rank condition (ii) of Theorem 11). However, via a nonlinear coordinates change in the output space

$$\tilde{y}_1 = e^{y_1}, \qquad \tilde{y}_2 = y_2 - \frac{1}{3}e^{3y_1},$$

the system with the new outputs $\tilde{y}_1$, $\tilde{y}_2$ is level-3 input-output linearizable. Additionally, the transformed system satisfies conditions (i)-(iv) of Theorem 14. In fact, by choosing new coordinates

$$\begin{cases} \tilde{x}_1 = (x_1e^{-x_3})^{\frac{1}{2}}, \quad \tilde{x}_2 = x_2, \quad \tilde{x}_3 = e^{x_3}, \\ \tilde{x}_4 = x_4 - \frac{1}{3}e^{3x_3}, \quad \tilde{x}_5 = x_5, \quad \tilde{x}_6 = x_6, \end{cases}$$

and the feedback transformation $v_1 = \tilde{v}_1$, $v_2 = e^{-x_3}\tilde{v}_2$, $v_3 = \tilde{v}_3$, the system Σ_3 is level-2 sys-equivalent to the linear control system $\tilde{\Sigma}_3$ below. Moreover, since $\Sigma_3 \in \mathbf{Expl}(\Xi_3^{se})$, by Proposition 2, Ξ_3^{se} is level-2 ex-equivalent to the linear DAE Δ_3^{se} below

$$\tilde{\Sigma}_3 : \begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 \\ \dot{\tilde{x}}_2 = \tilde{v}_1 \\ \dot{\tilde{x}}_3 = \tilde{v}_2, \qquad \tilde{y}_1 = \tilde{x}_3 \\ \dot{\tilde{x}}_4 = \tilde{x}_5 + k\tilde{y}_1, \; \tilde{y}_2 = \tilde{x}_4 \\ \dot{\tilde{x}}_5 = \tilde{x}_6 \\ \dot{\tilde{x}}_6 = \tilde{v}_3 \end{cases} \Rightarrow \Delta_3^{se} : \begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 \\ \dot{\tilde{x}}_4 = \tilde{x}_5 + k\tilde{y}_1 \\ \dot{\tilde{x}}_5 = \tilde{x}_6 \\ 0 = \tilde{x}_3 \\ 0 = \tilde{x}_4. \end{cases}$$

In view of the example above, even if an explicit control system is not level-3 input-output linearizable, it may be so under level-2 sys-equivalence. Thus via further transformations, the original SE DAE is possibly level-2 externally linearizable. It suggests that the future work should be focused on level-2 input-output linearizability of control systems and corresponding SE DAEs.

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