Implicit function theorem for nonlinear time-delay systems with algebraic constraints

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***Abstract*— In this note, we discuss a generalization of the well-known implicit function theorem to the time-delay case. We show that the latter problem is closely related to the bicausal changes of coordinates of time-delay systems [4], [5]. An iterative algorithm is proposed to check the conditions and to construct the desired bicausal change of coordinates for the proposed implicit function theorem. Moreover, we show that our results can be applied to delayed differential-algebraic equations (DDAEs) to reduce their indices and to get their solutions. Some numerical examples are given to illustrate our results.**

***Index Terms*— nonlinear systems, time-delay, bicausal changes of coordinates, implicit function theorem, causality, differential-algebraic equations**

I. Introduction

We start from three different equations with time-delay variables:

$$a(\mathbf{x}_1,\mathbf{x}_2)=x_1(t)x_2(t-1)+x_2(t)x_2(t-1)+e_1=0,$$
$$b(\mathbf{x}_1,\mathbf{x}_2)=x_1(t)x_2(t-1)+x_1(t-1)x_2(t)x_2(t-2)+e_2=0,$$
$$c(\mathbf{x}_1,\mathbf{x}_2)=x_1(t)x_1(t-1)+x_2(t)x_2(t-1)+e_3=0,$$

where $(\mathbf{x}_1,\mathbf{x}_2)=(x_1(t),x_2(t),x_1(t-1),x_2(t-1),x_2(t-2))$ and e_1,e_2,e_3 are nonzero constants. The purpose is to express $x_1(t)$ as a function of $x_2(t)$ and its time-delays from each algebraic equation. For instance, it is clear to get $x_1(t)=\frac{-e_1-x_2(t)x_2(t-1)}{x_2(t-1)}$ for $x_2(t-1)\neq 0$ from the first equation, while it is not obvious if we can have similar conclusions for the other two equations. In the delay-free case, given some algebraic equations $\lambda(x_1,x_2)=0$, where $\lambda\in\mathcal{K}^p$, $x_1\in\mathbb{R}^p$, $x_2\in\mathbb{R}^{n-p}$ and $\mathcal{K}$ denotes the field of meromorphic functions, if the matrix $\frac{\partial\lambda}{\partial x_1}(x_1,x_2)\in\mathcal{K}^{p\times p}$ is invertible for all $(x_1,x_2)\in\mathbb{R}^n$ such that $\lambda(x_1,x_2)=0$ (or, a simpler but stronger condition, for all $(x_1,x_2)\in\mathbb{R}^n$), then by the classical implicit function theorem (see e.g. [19]), there exist functions $\gamma:\mathbb{R}^{n-p}\to\mathcal{K}^p$ such that $\lambda(x_1,x_2)=0$ implies $x_1=\gamma(x_2)$. We will study in this note a generalization of the implicit function theorem to algebraic equations with time-delays.

To deal with functions with time-delay variables, the algebraic framework proposed in [28] is a very useful tool. There are many applications of this framework see e.g., [2], [17], [22], [23], [29] for the problems like observations and structure analysis for time-delay systems, and more recently, the papers [3]–[6], [18] and the book [7] for extensions of the classical geometric control methods to nonlinear time-delay systems.

With the help of the algebraic framework, we will show in section II that although it is not possible to express $x_1(t)$ as a function of $x_2(t),x_2(t-1),x_2(t-2)$ from the last two equations, by a bicausal change of coordinates (see Definition 3 below) $(\tilde{x}_1,\tilde{x}_2)=\varphi(\mathbf{x}_1,\mathbf{x}_2)$, the equation $b(\varphi^{-1}(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2))=0$, i.e., $b(x_1,x_2)=0$ in $(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2)$-coordinates, implies $\tilde{x}_1=\gamma(\tilde{\mathbf{x}}_2)$ for some function γ (but we can not find such a bicausal change of coordinates for $c(\mathbf{x})=0$). The first problem is that when is it possible and how do we find such a bicausal coordinates transformation for a given time-delay algebraic equation? It turns out that such a problem is closely related to when functions with time-delay variables can be regarded as new bicausal coordinates and how to construct their complementary bicausal coordinates, the latter problems are discussed in [4], [5], [22]. We will recall some results from [5] and add two extra equivalent conditions as Theorem 1 in order to explain the relations. Then a generalization of implicit function theorem to time-delay equations is given as a corollary of Theorem 1.

To check the equivalent conditions of Theorem 1, we need to either construct the right-annihilator/kernel or the right-inverse of some polynomial matrix-valued functions, which can be done with the help of Smith canonical form for polynomial matrix-valued functions (see [22], also [15] for polynomial matrices with entries in $\mathbb{R}[\delta]$). We will discuss in section IV that the latter method has troubles when checking the necessity of those conditions. To deal with the latter problem, we propose an iterative algorithm by reducing the polynomial degree of the polynomial matrix-valued functions via bicausal changes of coordinates, which eventually allows to check the conditions of Theorem 1 and to construct the desired complementary bicausal coordinates.

Another contribution of this note is to apply the proposed implicit function theorem to delayed differential-algebraic equations (DDAEs), i.e., implicit time-delay equations (see e.g. [8], [13], [16], [26] for linear DDAEs and [1], [27], [30] for nonlinear DDAEs). It is well-known that for delay-free differential-algebraic equations (DAEs), the classical implicit function theorem is an essential tool for its index-reduction problem, e.g., given a semi-explicit DAE $\dot{x}_1=f(x_1,x_2), 0=g(x_1,x_2)$, if $\frac{\partial g(x_1,x_2)}{\partial x_2}\neq 0$ for all $(x_1,x_2)\in\mathbb{R}^2$ (i.e., the DAE is index-1), then to reformulate the DAE as an ordinary differential equation (ODE), we use the implicit function theorem to get $x_2=\eta(x_1)$ from the algebraic constraint for some functions η, then it results in an ODE $\dot{x}_1=f(x_1,\eta(x_1))$. For a high-index DAE, the geometric reduction method can be used to reduce the index, see e.g. [10], [11], [24], [25]. We will show below that by assuming that the algebraic constraints of DDAEs satisfy the conditions of the proposed implicit function theorem, a time-delay version of the geometric reduction method can be realized.

This note is organised as follows. Notations and the definitions of some notions in the algebraic framework are given in section II. The time-delay implicit function theorem is discussed in section III. The algorithm to check the conditions of the time-delay implicit function theorem is given in section IV. In section V, we discuss the index reduction algorithm and the solutions of nonlinear DDAEs by applying the results of sections III and IV. The conclusions and perspectives are put into section VI.

II. Notations and Preliminaries

We follow the algebraic framework of time-delay systems proposed in [28], the notations below are taken from those in e.g., [5], [7], [28]. In this note, we do not deal with singularities and assume throughout that $f(\mathbf{x})\equiv 0$ holds for no non-trivial meromorphic function f.

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I_r	identity matrix of $\mathbb{R}^{r\times r}$.
$x(\pm j)$	$x(t\pm j)$, $j\geq 0$.
$\mathbf{x}_{[\underline{j},\bar{j}]}$	$[x^T(-\underline{j}),\ldots,x^T(-\bar{j})]^T\in\mathbb{R}^{(\bar{j}-\underline{j}+1)n}$, $\underline{j}\leq\bar{j}$. Notice that both $\underline{j}$ and $\bar{j}$ can be negative indicating the forward time-shifts of $x(t)$.
$\mathbf{x}_{[\bar{j}]}$	$\mathbf{x}=\mathbf{x}_{[0,\bar{j}]}=[x^T,x^T(-1),\ldots,x^T(-\bar{j})]^T\in\mathbb{R}^{(\bar{j}+1)n}$, where $x=x(0)=x(t)\in\mathbb{R}^n$ and $\bar{j}\geq 0$.
$\mathcal{K}^*$	the field of causal and non-causal meromorphic functions $f(\mathbf{x}_{[\underline{j},\bar{j}]})$ with $\underline{j},\bar{j}\in\mathbb{N}$.
$\mathcal{K}$	the field of causal meromorphic functions $f(\mathbf{x}_{[\underline{j},\bar{j}]})$ with $\underline{j}=0$ and $\bar{j}\in\mathbb{N}$.
d	the differential operator: for $\xi(\mathbf{x}_{[\bar{j}]})\in\mathcal{K}$ and $\lambda(\mathbf{x}_{[\bar{j}]})\in\mathcal{K}^p$, $\mathrm{d}\xi(\mathbf{x}_{[\bar{j}]})=\sum_{j=0}^{\bar{j}}\frac{\partial\xi(\mathbf{x}_{[\bar{j}]})}{\partial x(-j)}\mathrm{d}x(-j)$ and $\mathrm{d}\lambda=\begin{bmatrix}\mathrm{d}\lambda_1\\ \cdots\\ \mathrm{d}\lambda_p\end{bmatrix}$.
δ	the backward time-shift operator: for $a(t),\xi(t)\in\mathcal{K}$, $\delta^j\xi(t)=\xi(-j)$ and $\delta^j(a(t)\mathrm{d}\xi(t))=a(-j)\mathrm{d}\xi(-j)$.
Δ	the forward time-shift operator: for $a(t),\xi(t)\in\mathcal{K}$, $\Delta^j\xi(t)=\xi(+j)$ and $\Delta^j(a(t)\mathrm{d}\xi(t))=a(+j)\mathrm{d}\xi(+j)$.
$\mathcal{K}^*(\delta]$	the left (Ore-)ring of polynomials in δ with entries in $\mathcal{K}^*$.
$\mathcal{K}(\delta]$	the left ring of polynomials in δ with entries in $\mathcal{K}$, any $\alpha(\mathbf{x},\delta)\in\mathcal{K}(\delta]$ has the form $\alpha(\mathbf{x},\delta)=\sum_{j=0}^{\bar{j}}\alpha^j(\mathbf{x})\delta^j$, where $\alpha^j(\mathbf{x})\in\mathcal{K}$.
$\deg(\cdot)$	the polynomial degree. For $\alpha(\mathbf{x},\delta)\in\mathcal{K}(\delta]$ above, $\deg(\alpha)=\bar{j}$. For $\beta(\mathbf{x},\delta)=[\beta_1(\mathbf{x},\delta),\ldots,\beta_n(\mathbf{x},\delta)]\in\mathcal{K}^n(\delta]$, $\deg(\beta)=\max\{\deg(\beta_i),1\leq i\leq n\}$.
$\wedge$	exterior product

The sums and multiplications for any two elements of $\mathcal{K}(\delta]$ are well-defined [28], so are the row rank and the column rank of a matrix $A(\cdot,\delta)\in\mathcal{K}^{r\times m}(\delta]$ over $\mathcal{K}(\delta]$ (note that unlike non-polynomial matrices, the two ranks could be different for the same matrix $A(\cdot,\delta)$). Remark that a polynomial matrix-valued function $A(\cdot,\delta)\in\mathcal{K}^{r\times r}(\delta]$ is of full (row or column) rank over $\mathcal{K}(\delta]$ does not necessarily mean that $A(\cdot,\delta)$ has a polynomial inverse over $\mathcal{K}(\delta]$, the following notion of unimodularity generalizes that of invertibility of non-polynomial matrices.

Definition 1 ([22], [28]). A matrix $A(\cdot,\delta)\in\mathcal{K}^{r\times r}(\delta]$ is called *unimodular* if there exists a matrix $B(\cdot,\delta)\in\mathcal{K}^{r\times r}(\delta]$ such that $A(\cdot,\delta)B(\cdot,\delta)=B(\cdot,\delta)A(\cdot,\delta)=\mathrm{I}_r$.

Denote the vector space generated by the differentials $\mathrm{d}x(-j)$, $j\geq 0$ over $\mathcal{K}$ by $\mathcal{E}$. An element $\omega\in\mathcal{E}$ is called one-form. The one-form ω is *exact*, i.e., there exists $\lambda\in\mathcal{K}$ such that $\omega=\mathrm{d}\lambda$, if and only if $\mathrm{d}\omega=0$ (Poincaré lemma [21]). A p-dimensional codistribution $\mathrm{span}_{\mathcal{K}}\{\omega_1,\ldots,\omega_p\}$ is *integrable*, i.e., there exist $\lambda_1,\ldots,\lambda_p\in\mathcal{K}$ such that $\mathrm{span}_{\mathcal{K}}\{\omega_1,\ldots,\omega_p\}=\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_p\}$, if and only if $\mathrm{d}\omega_i\wedge\omega_1\wedge\cdots\wedge\omega_p=0$, for $1\leq i\leq p$ (Frobenius theorem [21]). The sets of one-forms defined over the ring $\mathcal{K}(\delta]$ have both the structure of a vector space $\mathcal{E}$ over $\mathcal{K}$ and the structure of a (left)-module, $\mathcal{M}=\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}x\}$. A (left)-submodule of $\mathcal{M}$ consists of all possible linear combinations of given one forms over the ring $\mathcal{K}(\delta]$. Denote $\mathcal{O}:=\mathrm{span}_{\mathcal{K}(\delta]}\{\omega_1,\ldots,\omega_p\}\subseteq\mathcal{M}$ the submodule generated by one forms $\omega_1,\ldots,\omega_p$ over $\mathcal{K}(\delta]$. The *right-annihilator* (or the kernel) of the submodule $\mathcal{O}$ is spanned by all vectors $\tau(\cdot,\delta)\in\mathcal{K}^{*n}(\delta]$ such that $\omega_i(\cdot,\delta)\tau(\cdot,\delta)=0$ for $1\leq i\leq p$. The right-annihilator is called *causal* if there are no forward time-shifts on the variables of the coefficients of $\tau(\cdot,\delta)$, i.e., $\tau(\cdot,\delta)\in\mathcal{K}^n(\delta]$. The closure of the submodules of $\mathcal{M}$ recalled below will play an important role.

Definition 2 ([28]). Given a finite generated module $\mathcal{M}$, let $\mathcal{N}$ be a submodule of $\mathcal{M}$ of dimension r over $\mathcal{K}(\delta]$, the closure of $\mathcal{N}$ is the submodule

$$\overline{\mathcal{N}}:=\{\omega\in\mathcal{M}\mid\exists\,0\neq\alpha(\cdot,\delta)\in\mathcal{K}(\delta],\ \alpha(\cdot,\delta)\omega\in\mathcal{N}\},$$

or equivalently, $\overline{\mathcal{N}}$ is the largest submodule of $\mathcal{M}$ which contains $\mathcal{N}$ and is of rank r over $\mathcal{K}(\delta]$. The submodule $\mathcal{N}$ is called *closed* if $\mathcal{N}=\overline{\mathcal{N}}$.

Definition 3 (bicausal coordinates changes [5], [23]). Consider a system (differential or not) with state coordinates $x\in\mathbb{R}^n$. A mapping $z=\varphi(\mathbf{x}_{[\bar{j}]})\in\mathcal{K}^n$, is called a bicausal change of coordinates if there exist an integer $\bar{j}_z\geq 0$ and a mapping $\varphi^{-1}\in\mathcal{K}^n$ such that $x=\varphi^{-1}(\mathbf{z}_{[\bar{j}_z]})$.

Remark that a mapping $z=\varphi(\mathbf{x})$ is a bicausal change of coordinates if and only if $T(\mathbf{x},\delta)\in\mathcal{K}^{n\times n}(\delta]$ is a unimodular matrix [7], where $\mathrm{d}z=\mathrm{d}\varphi(\mathbf{x})=T(\mathbf{x},\delta)\mathrm{d}x$. For a function $\lambda(\mathbf{x})\in\mathcal{K}$, we will simply write λ in z-coordinates as

$$\lambda(\mathbf{z}):=\lambda(\varphi^{-1}(\mathbf{z}),\ldots,\varphi^{-1}(\mathbf{z}(-\bar{j}))).$$

III. Implicit Function Theorem for Time-Delay Algebraic Equations

Now consider the time-delay algebraic equations $\lambda(\mathbf{x})=\lambda(\mathbf{x}_1,\mathbf{x}_2)=0$ with $\lambda\in\mathcal{K}^p$. The differentials of λ satisfy $\mathrm{d}\lambda(\mathbf{x})=T_1(\mathbf{x},\delta)\mathrm{d}x_1+T_2(\mathbf{x},\delta)\mathrm{d}x_2=0$. If $T_1(\mathbf{x},\delta)\in\mathcal{K}^{p\times p}(\delta]$ is unimodular, then $\mathrm{d}x_1=-T_1^{-1}T_2(\mathbf{x},\delta)\mathrm{d}x_2$. Thus by Poincaré lemma, there exist functions $\gamma\in\mathcal{K}^p$ such that $x_1=\gamma(\mathbf{x}_2)$. The last analysis explains why we can get x_1 as a function of x_2 and $x_2(-1)$ from $a(\mathbf{x})=0$ in section I, clearly, $\mathrm{d}a=x_2(-1)\mathrm{d}x_1+(x_1\delta+x_2\delta+x_2(-1))\mathrm{d}x_2$ and $x_2(-1)$ is unimodular. To have a similar result for $b(\mathbf{x})=0$, we have to use bicausal changes of coordinates as shown in Example 1 below. In general, we have the following theorem, in which items (i) and (ii) are taken from Theorem 2 of [5], items (iii) and (iv) are new and serve to our problem.

We use the following condition **(C)** for any submodule $\mathcal{N}\subseteq\mathcal{M}$:
(C): $\mathcal{N}$ is closed and its right-annihilator is causal.

Theorem 1. *Consider p-functions $\lambda_k(\mathbf{x})\in\mathcal{K}$, $1\leq k\leq p$, of the variables $x\in\mathbb{R}^n$ and its time-delays. Define the submodule $\mathcal{L}:=\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_k(\mathbf{x}),1\leq k\leq p\}$ and assume that $\mathrm{rank}_{\mathcal{K}(\delta]}\mathcal{L}=p$. Then the following statements are equivalent:*

(i) *$\mathcal{L}$ satisfies **(C)**.*
(ii) *There exist $n-p$ functions $\theta_1(\mathbf{x}),\ldots,\theta_{n-p}(\mathbf{x})$ such that $\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_p,\mathrm{d}\theta_1,\ldots,\mathrm{d}\theta_{n-p}\}=\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}x\}$, i.e., $\tilde{x}=[\lambda_1(\mathbf{x}),\ldots,\lambda_p(\mathbf{x}),\theta_1(\mathbf{x}),\ldots,\theta_{n-p}(\mathbf{x})]^T$ is a bicausal change of coordinates.*
(iii) *$L(\mathbf{x},\delta)\in\mathcal{K}^{p\times n}(\delta]$, where $\mathrm{d}\lambda(\mathbf{x})=L(\mathbf{x},\delta)\mathrm{d}x$ and $\lambda=[\lambda_1,\ldots,\lambda_p]^T$, has a polynomial right-inverse, i.e., $\exists L^\dagger(\mathbf{x},\delta)\in\mathcal{K}^{n\times p}(\delta]$ such that $LL^\dagger=\mathrm{I}_p$.*
(iv) *There exists a bicausal change of coordinates $\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix}=\varphi(\mathbf{x})$ with $\tilde{x}_1\in\mathbb{R}^p$ and $\tilde{x}_2\in\mathbb{R}^{n-p}$ such that $L_1(\cdot,\delta)$ is unimodular and $L_2(\cdot,\delta)\not\equiv 0$, where $L_1(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2,\delta)\mathrm{d}\tilde{x}_1+L_2(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2,\delta)\mathrm{d}\tilde{x}_2=\mathrm{d}\lambda(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2)$.*

Remark 1. The added condition (iii) is in some cases easier to be checked than condition (i) because the right-annihilator could be rendered causal even some non-causal terms shows up in the initial calculation of the kernel. Take $\lambda=x_1(-1)x_2+x_2^2$ from Example 3.6 of [5], in which it is claimed that the right-annihilator of $\mathcal{L}=\mathrm{span}_{\mathcal{K}(\delta]}\{x_2\delta\mathrm{d}x_1+(x_1(-1)+2x_2)\mathrm{d}x_2\}$ is *not* causal because the non-causal functions $r(\mathbf{x}_{[-1,0]},\delta)=\begin{bmatrix}-2x_2(+1)-x_1\\ x_2\delta\end{bmatrix}$ belongs to the right-annihilator. However, $L(\mathbf{x},\delta)=[\,x_1\delta,\ x_1(-1)+2x_2\,]$ has a polynomial right-inverse

$$L^{\dagger}(\mathbf{x},\delta)=\left[\begin{smallmatrix}0\\ \frac{1}{x_1(-1)+2x_2}\end{smallmatrix}\right]$$

(for $x_1(-1)+2x_2\neq 0$). In fact, $r(\mathbf{x},\delta)$ can be rendered as $\left[\begin{smallmatrix}-1\\ \frac{x_2}{x_1(-1)+2x_2}\delta\end{smallmatrix}\right]$ proving that the right-annihilator is actually causal. Indeed, choose $\theta=x_1$, we have $\left[\begin{smallmatrix}\lambda(\mathbf{x})\\ \theta(\mathbf{x})\end{smallmatrix}\right]$ is a bicausal change of coordinates since $\left[\begin{smallmatrix}\mathrm{d}\lambda(\mathbf{x})\\ \mathrm{d}\theta(\mathbf{x})\end{smallmatrix}\right]=\Theta(\mathbf{x},\delta)\left[\begin{smallmatrix}\mathrm{d}x_1\\ \mathrm{d}x_2\end{smallmatrix}\right]$ and $\Theta(\mathbf{x},\delta)=\left[\begin{smallmatrix}x_2\delta & x_1(-1)+2x_2\\ 1 & 0\end{smallmatrix}\right]$ is unimodular, hence by the equivalence of items (i) and (ii), the right-annihilator of $\mathcal{L}$ is actually causal.

Proof. The proof of $(i)\Leftrightarrow(ii)$ can be found in [5] and [22].

(i)⇒(iii): Assume that item (i) holds, then by Lemma 12 of [22], there exist two (causal) unimodular matrices $P(\mathbf{x},\delta)\in\mathcal{K}^{p\times p}(\delta]$ and $Q(\mathbf{x},\delta)\in\mathcal{K}^{n\times n}(\delta]$ such that $P(\mathbf{x},\delta)L(\mathbf{x},\delta)Q(\mathbf{x},\delta)=[\mathrm{I}_p\ 0]$. It follows that $L(\mathbf{x},\delta)Q(\mathbf{x},\delta)=[P^{-1}(\mathbf{x},\delta)\ 0]$ and thus $L(\mathbf{x},\delta)Q_1(\mathbf{x},\delta)=P^{-1}(\mathbf{x},\delta)$, where $Q=\left[\,Q_1\ Q_2\,\right]$ and $Q_1(\mathbf{x},\delta)\in\mathcal{K}^{n\times p}(\delta]$. Hence $L(\mathbf{x},\delta)Q_1(\mathbf{x},\delta)P(\mathbf{x},\delta)=\mathrm{I}_p$ and $L^{\dagger}(\mathbf{x},\delta)=Q_1(\mathbf{x},\delta)P(\mathbf{x},\delta)$ is a polynomial right-inverse of $L(\mathbf{x},\delta)$.

(iii)⇒ (i): Assume that there exists $L^{\dagger}(\mathbf{x},\delta)\in\mathcal{K}^{n\times p}(\delta]$ such that $L(\mathbf{x},\delta)L^{\dagger}(\mathbf{x},\delta)=\mathrm{I}_p$. Then by Lemma 4 of [22], there always exists a unimodular matrix $U(\mathbf{x},\delta)=\left[\begin{smallmatrix}U_1(\mathbf{x},\delta)\\ U_2(\mathbf{x},\delta)\end{smallmatrix}\right]\in\mathcal{K}^{n\times n}(\delta]$ such that $\left[\begin{smallmatrix}U_1(\mathbf{x},\delta)\\ U_2(\mathbf{x},\delta)\end{smallmatrix}\right]L^{\dagger}(\mathbf{x},\delta)=\left[\begin{smallmatrix}R(\mathbf{x},\delta)\\ 0\end{smallmatrix}\right]$ with $R(\mathbf{x},\delta)\in\mathcal{K}^{p\times p}(\delta]$ being of full row rank over $\mathcal{K}(\delta]$. Then by $L(\mathbf{x},\delta)L^{\dagger}(\mathbf{x},\delta)=\mathrm{I}_p$ and $U_1(\mathbf{x},\delta)L^{\dagger}(\mathbf{x},\delta)=R(\mathbf{x},\delta)$, we get that $U_1(\mathbf{x},\delta)=R(\mathbf{x},\delta)L(\mathbf{x},\delta)+T(\mathbf{x},\delta)U_2(\mathbf{x},\delta)$ for some matrix $T(x,\delta)\in\mathcal{K}^{p\times(n-p)}(\delta]$. It follows that $\left[\begin{smallmatrix}U_1(\mathbf{x},\delta)\\ U_2(\mathbf{x},\delta)\end{smallmatrix}\right]=\left[\begin{smallmatrix}R(\mathbf{x},\delta)L(\mathbf{x},\delta)+T(\mathbf{x},\delta)U_2(\mathbf{x},\delta)\\ U_2(\mathbf{x},\delta)\end{smallmatrix}\right]=\left[\begin{smallmatrix}\mathrm{I} & T(\mathbf{x},\delta)\\ 0 & \mathrm{I}\end{smallmatrix}\right]\left[\begin{smallmatrix}R(\mathbf{x},\delta)L(\mathbf{x},\delta)\\ U_2(\mathbf{x},\delta)\end{smallmatrix}\right]$ is unimodular and thus $\left[\begin{smallmatrix}R(\mathbf{x},\delta)L(\mathbf{x},\delta)\\ U_2(\mathbf{x},\delta)\end{smallmatrix}\right]$ is unimodular as well. So $\mathrm{span}_{\mathcal{K}(\delta]}\{R(\mathbf{x},\delta)L(\mathbf{x},\delta)\mathrm{d}x\}$ satisfies **(C)** by Theorem 13 of [22]. Notice that $\mathrm{span}_{\mathcal{K}(\delta]}\{R(\mathbf{x},\delta)L(\mathbf{x},\delta)\mathrm{d}x\}$ and $\mathcal{L}$ have the same right-annihilator and $\mathrm{span}_{\mathcal{K}(\delta]}\{R(\mathbf{x},\delta)L(\mathbf{x},\delta)\mathrm{d}x\}\subseteq\mathcal{L}$. Hence $\mathcal{L}$ is closed and its right-annihilator is causal.

(ii)⇒(iv): Assume that item (ii) holds. Define $\tilde{x}_1:=\lambda(\mathbf{x})+\eta(\boldsymbol{\theta}(\mathbf{x}))$, where η is any function in $\mathcal{K}^p$ of $\theta=[\theta_1,\dots,\theta_{n-p}]^T$ and their delays, and $\tilde{x}_2:=\theta(\mathbf{x})$. Then $\left[\begin{smallmatrix}\mathrm{d}x_1\\ \mathrm{d}x_2\end{smallmatrix}\right]=\left[\begin{smallmatrix}\mathrm{I}_p & E(\mathbf{x},\delta)\\ 0 & \mathrm{I}_{n-p}\end{smallmatrix}\right]\Theta(\mathbf{x},\delta)\mathrm{d}x$, where $E(\mathbf{x},\delta)\mathrm{d}\theta=E(\boldsymbol{\theta},\delta)\mathrm{d}\theta=\mathrm{d}\eta(\theta)$ and $\Theta(\mathbf{x},\delta)\mathrm{d}x=\left[\begin{smallmatrix}\mathrm{d}\lambda(\mathbf{x})\\ \mathrm{d}\theta(\mathbf{x})\end{smallmatrix}\right]$. Since $\Theta(\mathbf{x},\delta)$ is unimodular as $\left[\begin{smallmatrix}\lambda(\mathbf{x})\\ \theta(\mathbf{x})\end{smallmatrix}\right]$ defines a bicausal change of coordinates, we have that $\left[\begin{smallmatrix}\mathrm{I}_p & E(\mathbf{x},\delta)\\ 0 & \mathrm{I}_{n-p}\end{smallmatrix}\right]\Theta(\mathbf{x},\delta)$ is unimodular and $\left[\begin{smallmatrix}\tilde{x}_1\\ \tilde{x}_2\end{smallmatrix}\right]$ defines a bicausal change of coordinates as well. Hence by $\lambda(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2)=\tilde{x}_1-\eta(\tilde{\mathbf{x}}_2)$, the matrix $L_1(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2,\delta)=\mathrm{I}_p$ is unimodular.

(iv)⇒(iii): Suppose that item (iv) holds, then $\mathrm{d}\lambda=L(\mathbf{x},\delta)\mathrm{d}x=L(\mathbf{x},\delta)\Psi^{-1}(\mathbf{x},\delta)\Psi(\mathbf{x},\delta)\mathrm{d}x=L(\mathbf{x},\delta)\Psi^{-1}(\mathbf{x},\delta)\left[\begin{smallmatrix}\mathrm{d}\tilde{x}_1\\ \mathrm{d}\tilde{x}_2\end{smallmatrix}\right]=\left[\,L_1(\mathbf{x},\delta)\ L_2(\mathbf{x},\delta)\,\right]\left[\begin{smallmatrix}\mathrm{d}\tilde{x}_1\\ \mathrm{d}\tilde{x}_2\end{smallmatrix}\right]$, where $\Psi(\mathbf{x},\delta)\mathrm{d}x=\mathrm{d}\varphi(\mathbf{x})$ and $\Psi(\mathbf{x},\delta)\in\mathcal{K}^{n\times n}(\delta]$ is unimodular. Because $L_1(\mathbf{x},\delta)$ is unimodular, we have $\left[\,L_1(\mathbf{x},\delta)\ L_2(\mathbf{x},\delta)\,\right]\left[\begin{smallmatrix}L_1^{-1}(\mathbf{x},\delta)\\ 0\end{smallmatrix}\right]=L(\mathbf{x},\delta)\Psi^{-1}(\mathbf{x},\delta)\left[\begin{smallmatrix}L_1^{-1}(\mathbf{x},\delta)\\ 0\end{smallmatrix}\right]=\mathrm{I}_p$. It follows that $L^{\dagger}(\mathbf{x},\delta)=\Psi^{-1}(\mathbf{x},\delta)\left[\begin{smallmatrix}L_1^{-1}(\mathbf{x},\delta)\\ 0\end{smallmatrix}\right]$ is a polynomial right-inverse of $L(\mathbf{x},\delta)$.

□

The results of Theorem 1 can be extended to functions with dependent differentials via the results of (strong) integrability of left-submodules in [18]. In the delay-free case [12], for s-functions $\lambda_k(x)\in\mathcal{K}$, $1\leq k\leq s$, if the rank of $\mathrm{d}\lambda$ over $\mathcal{K}$ is $p\leq s$, then we can choose p-functions (whose differentials are independent over $\mathcal{K}$) from $\lambda_k(x)$ as new coordinates. While in the time-delay case, for functions with dependent differentials over $\mathcal{K}(\delta]$, even the conditions of Theorem 1 are satisfied, we can *not* always choose p functions from $\lambda_k(x,\delta)$ as new bicausal coordinates. For example, take $\lambda_1(\mathbf{x}_{1,[1]},\mathbf{x}_{2,[2]})=x_1(-1)+x_2(-2)$ and $\lambda_2(\mathbf{x}_{1,[1]},\mathbf{x}_{2,[2]})=(x_1+x_2(-1))(x_1(-1)+x_2(-2))$, we have $\mathrm{d}\lambda_1=\delta\mathrm{d}x_1+\delta^2\mathrm{d}x_2$ and $\mathrm{d}\lambda_2=(x_1(-1)+x_2(-2)+(x_1+x_2(-1))\delta)\mathrm{d}x_1+((x_1(-1)+x_2(-2))\delta+(x_1+x_2(-1))\delta^2)\mathrm{d}x_2$, it can be seen that $\mathrm{d}\lambda_1$ and $\mathrm{d}\lambda_2$ are dependent over $\mathcal{K}(\delta]$, and the submodule $\mathcal{L}=\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_1,\mathrm{d}\lambda_2\}$ is closed and its right-annihilator $\mathrm{span}_{\mathcal{K}(\delta]}\left\{\left[\begin{smallmatrix}\delta\\ -1\end{smallmatrix}\right]\right\}$ is causal, but we can *not* choose either λ_1 or λ_2 as a new bicausal coordinate since neither $\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_1\}$ nor $\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_2\}$ is closed. Observe that we may still construct $\tilde{\lambda}=x_1+x_2(-1)$ as a new bicausal coordinate and $\tilde{\mathcal{L}}=\mathrm{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}\tilde{\lambda}\right\}=\mathcal{L}$. In general, the following results hold:

Proposition 1. *Consider s-functions $\lambda_i(\mathbf{x})\in\mathcal{K}$, $1\leq i\leq s$, with $\mathrm{rank}_{\mathcal{K}(\delta]}\mathcal{L}=p\leq s$, where $\mathcal{L}:=\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_k,1\leq k\leq s\}$. If $\mathcal{L}$ satisfies* **(C)***, then we can find p-functions $\tilde{\lambda}_1,\dots,\tilde{\lambda}_p\in\mathcal{K}$, which do not necessarily belong to $\{\lambda_1,\dots,\lambda_s\}$, such that $\tilde{\mathcal{L}}=\mathrm{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}\tilde{\lambda}_k(\mathbf{x}),1\leq k\leq p\right\}=\mathcal{L}$ and $\lambda(\mathbf{x})=[\lambda_1(\mathbf{x}),\dots,\lambda_s(\mathbf{x})]^T=0$ is equivalent to $\tilde{\lambda}(\mathbf{x})=[\tilde{\lambda}_1(\mathbf{x}),\dots,\tilde{\lambda}_p(\mathbf{x})]^T=0$, i.e., $x(t)$ satisfies $\lambda(\mathbf{x})=0$ if and only if it satisfies $\tilde{\lambda}(\mathbf{x})=0$.*

Proof. Choose any p-functions $\lambda_1(\mathbf{x}),\dots,\lambda_p(\mathbf{x})$ from $\lambda(\mathbf{x})$ such that the differentials $\mathrm{d}\lambda_k$, $1\leq k\leq p$, are independent over $\mathcal{K}(\delta]$. Then the submodule $\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_1,\dots,\mathrm{d}\lambda_p\}$ is (strongly) integrable in the sense of [18]. Thus its closure $\overline{\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_1,\dots,\mathrm{d}\lambda_p\}}$, which coincides with $\mathcal{L}$ (because $\mathcal{L}$ is closed), is (strongly) integrable as well by Lemma 2 of [18]. So there exist p-functions $\bar{\lambda}_1,\dots,\bar{\lambda}_p$ such that $\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\bar{\lambda}_1,\dots,\mathrm{d}\bar{\lambda}_p\}=\overline{\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_1,\dots,\mathrm{d}\lambda_p\}}=\mathcal{L}$. However, it is not necessarily true that $\lambda(\mathbf{x})=0$ if and only if $\bar{\lambda}(\mathbf{x})=0$. Since $\mathcal{L}$ satisfies **(C)**, we can choose $x_1=\bar{\lambda}_1$, …, $x_p=\bar{\lambda}_p$, $x_{p+1}=\theta_1$, …, $x_n=\theta_{n-p}$ as new bicausal coordinates by Theorem 1. It follows that λ_k, $1\leq k\leq s$ depends only on $(x_1,\dots,x_p)$ and their delays, i.e., $\lambda=\lambda(\mathbf{x}_1,\dots,\mathbf{x}_p)$. For $1\leq k\leq p$, fix $x_k=x_k(-1)=\cdots=x_k(-\bar{j})=c_k$, where c_k are constants, and solve the algebraic equations $\lambda(c_1,\dots,c_p)=0$. Then by setting $\tilde{\lambda}_k=x_k-c_k=\bar{\lambda}_k(\mathbf{x})-c_k$, $1\leq k\leq p$, we have $\lambda(\mathbf{x}_1,\dots,\mathbf{x}_p)=0$ if and only if $\tilde{\lambda}(\mathbf{x}_1,\dots,\mathbf{x}_p)=0$. □

We are now ready to present a generalization of the implicit function theorem for time-delay algebraic equations.

Corollary 1 (implicit function theorem)**.** *Consider s-algebraic equations $\lambda(\mathbf{x})=0$ and $\mathcal{L}:=\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_k,1\leq k\leq s\}$. Let $\mathrm{rank}_{\mathcal{K}(\delta]}\mathcal{L}=p\leq s$, if $\mathcal{L}$ satisfies* **(C)***, then there exists a bicausal change of coordinates $\left[\begin{smallmatrix}\tilde{x}_1\\ \tilde{x}_2\end{smallmatrix}\right]=\varphi(\mathbf{x})$ with $\tilde{x}_1\in\mathbb{R}^p$ and $\tilde{x}_2\in\mathbb{R}^{n-p}$ such that $\lambda(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2)=0$ implies $\tilde{x}_1=\eta(\tilde{\mathbf{x}}_2)$.*

Proof. If $p<s$, then we use the results of Proposition 1 to replace $\lambda(\mathbf{x})=0$ by $\tilde{\lambda}(\mathbf{x})=0$. Because $\tilde{\mathcal{L}}=\mathcal{L}$ satisfies item (i) of Theorem 1, we have $\mathrm{d}\tilde{x}_1=L_1^{-1}L_2(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2,\delta)\mathrm{d}\tilde{x}_2$ by item (iv). Hence by Poincaré lemma, there always exist functions $\eta\in\mathcal{K}^p$ such that $\tilde{x}_1=\eta(\tilde{\mathbf{x}}_2)$. □

Remark 2. The result of Corollary 1 is sufficient but not necessary, take the following example, $\lambda(\mathbf{x}_{1,[1]},\mathbf{x}_{2,[1]})=(x_1+x_1(-1))/x_2(-1)+e=0$ with a constant $e\neq 0$, we have

$$\mathrm{d}\lambda=(\frac{1}{x_2(-1)}+\frac{1}{x_2(-1)}\delta)\mathrm{d}x_1-(\frac{x_1+x_1(-1)}{x_2^2(-1)}\delta)\mathrm{d}x_2.$$

It can be seen by using Lemma 1 below that the right-annihilator of $\mathcal{L} = \text{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda\}$ is *not* causal (as $\Delta x_1 = x_1(+1)$ is not causal). However, $\lambda(\mathbf{x}) = (x_1 + x_1(-1))/x_2(-1) + e = 0$ is equivalent to $\hat{\lambda}(\mathbf{x}) = (x_1 + x_1(-1)) + ex_2(-1) = 0$ (for $x_2(-1) \neq 0$), and $\text{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}\hat{\lambda}\right\}$ satisfies **(C)**. In fact, by the bicausal change of coordinates $\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1+ex_2 \end{bmatrix}$, we have $\hat{\lambda}(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) = \tilde{x}_1 + \tilde{x}_2(-1) = 0$ implying that $\tilde{x}_1 = -\tilde{x}_2(-1)$. Observe that $\mathcal{L}$ does not satisfy **(C)** for all $x(t)$ but it satisfies **(C)** for all $x(t)$ such that $\lambda(\mathbf{x}) = 0$ because $\mathcal{L}$ restricted to $\{\mathbf{x} \mid \lambda(\mathbf{x}) = 0\}$ is $\text{span}_{\mathcal{K}(\delta]}\left\{\frac{1+\delta}{x_2(-1)}\mathrm{d}x_1 + \frac{e}{x_2(-1)}\delta \mathrm{d}x_2\right\}$, which coincides with $\text{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}\hat{\lambda}\right\}$ and satisfies **(C)**. Remark that when and how we can find $\hat{\lambda}$ in the general case is an interesting problem, but we will not discuss that in details as the purpose of the remaining note is to show how to check the condition of Corollary 1 (section IV) and to use it to solve DDAEs (section V)

IV. An Algorithm for Checking the Condition of the Implicit Function Theorem

To construct the right-annihilator of a left-submodule is, in general, not an easy task (see e.g., Remark 1), which makes the conditions of Theorem 1 and Corollary 1 difficult to be checked. A conventional way to find the kernel of a polynomial matrix-valued function $L(\mathbf{x}, \delta) \in \mathcal{K}^{p\times n}(\delta]$ is to transform $L(\mathbf{x}, \delta)$ into its Smith canonical form $Q(\mathbf{x}, \delta)L(\mathbf{x}, \delta)P(\mathbf{x}, \delta) = [L_1(\mathbf{x}, \delta), 0]$ by two (causal) unimodular matrices Q and P (see e.g., [4], [5], [22]). However, the existence of (causal) unimodular matrices to transform $L(\mathbf{x}, \delta)$ into its Smith canonical form requires already its kernel to be causal [22]. Therefore, the necessity of item (i) of Theorem 1 is uncheckable by the last method, i.e., if the kernel of $L(\mathbf{x}.\delta)$ is not causal, we can not transform $L(\mathbf{x}.\delta)$ into its Smith form via (causal) unimodular matrices in order to verify if the kernel is indeed not causal.

The following lemma provides some easily checkable necessary conditions for the causality of the right-annihilator of a submodule generated by the differential of a function. Consider a function $\lambda(\mathbf{z}_{[0,\bar{j}]}) \in \mathcal{K}$ of the variables $z = [z_1, \ldots, z_q]^T \in \mathbb{R}^q$ and its time-delays. Let $\alpha \mathrm{d}z = [\alpha_1, \ldots, \alpha_q]\mathrm{d}z = \mathrm{d}\lambda$, where $\alpha_i(\mathbf{z}, \delta) = \sum_{j=0}^{\bar{j}_i} \alpha_i^j(\mathbf{z})\delta^j \in \mathcal{K}(\delta]$, $1 \leq i \leq q$, and denote $\bar{j} = \deg(\alpha) = \max\{\bar{j}_i, 1 \leq i \leq q\}$.

Lemma 1. *If the right-annihilator of $\alpha(\mathbf{z}, \delta)$ is causal, then there exists a permutation of α_i (by that of z_i) such that 1). $\alpha_1 \not\equiv 0$; 2). $0 \leq \bar{j}_1 \leq \bar{j}_2$; 3). the right-annihilator of $[\alpha_1(\mathbf{z}, \delta), \alpha_2(\mathbf{z}, \delta)]$ is causal as well.*

Moreover, if that is causal, then rewrite (separate the highest-order terms $\alpha_1^{\bar{j}_1}\delta^{\bar{j}_1}$ and $\alpha_2^{\bar{j}_2}\delta^{\bar{j}_2}$ from the remaining terms $\bar{\alpha}_1$ and $\bar{\alpha}_2$)

$$[\alpha_1, \alpha_2] = [\bar{\alpha}_1, \bar{\alpha}_2] + \alpha_1^{\bar{j}_1}[\hat{\alpha}_1, \hat{\alpha}_2],$$

where $\hat{\alpha}_1(\delta) = \delta^{\bar{j}_1}$, $\hat{\alpha}_2(\mathbf{z}, \delta) = \frac{\alpha_2^{\bar{j}_2}(\mathbf{z})}{\alpha_1^{\bar{j}_1}(\mathbf{z})}\delta^{\bar{j}_2}$, we have that

(i) *the delays of the variables $\mathbf{z}$ of $\hat{\alpha}_2(\mathbf{z}, \delta) = \hat{\alpha}_2(\mathbf{z}_{[\bar{j}_1,\bar{j}]}, \delta)$ are at least $\bar{j}_1$, i.e., $[\Delta^{\bar{j}_1}\hat{\alpha}_1, \Delta^{\bar{j}_1}\hat{\alpha}_2] = [1, \Delta^{\bar{j}_1}\hat{\alpha}_2]$ is causal.*

(ii) *Let $(\xi_1, \xi_2) = \mathbf{z}_{[\bar{j}_1,\bar{j}]}$ with $\xi_2 = (z_1(-\bar{j}_1), z_2(-\bar{j}_2))$. Then by fixing ξ_1 as constants, the codistribution*

$$\mathcal{D} := \text{span}_{\mathcal{K}}\left\{\mathrm{d}z_1(-\bar{j}_1) + \frac{\alpha_2^{\bar{j}_2}(\mathbf{z}_{[\bar{j}_1,\bar{j}]})}{\alpha_1^{\bar{j}_1}(\mathbf{z}_{[\bar{j}_1,\bar{j}]})}\mathrm{d}z_2(-\bar{j}_2)\right\}$$

is integrable. That is, there exists a function $\hat{\lambda}(\mathbf{z}_{[\bar{j}_1,\bar{j}]}) \in \mathcal{K}$ such that

$$\mathcal{D} = \text{span}_{\mathcal{K}}\left\{\frac{\partial\hat{\lambda}(\mathbf{z}_{[\bar{j}_1,\bar{j}]})}{\partial\xi_2}\mathrm{d}\xi_2\right\}. \tag{1}$$

(iii) *$\tilde{z} = \varphi(\mathbf{z}) = [\tilde{z}_1, \ldots, \tilde{z}_q]^T$, where*

$$\tilde{z}_1 = \Delta^{\bar{j}_1}\hat{\lambda}(\mathbf{z}_{[\bar{j}_1,\bar{j}]}), \quad \tilde{z}_2 = z_2, \quad \ldots, \quad \tilde{z}_q = z_q,$$

defines a bicausal change of coordinates and $\alpha \mathrm{d}z$ under $\tilde{z}$-coordinates, i.e., $\tilde{\alpha}\mathrm{d}\tilde{z} = [\tilde{\alpha}_1, \ldots, \tilde{\alpha}_q]\mathrm{d}\tilde{z}$ with $\tilde{\alpha}(\tilde{\mathbf{z}}, \delta) = \alpha(\mathbf{z}, \delta)\Psi^{-1}(\mathbf{z}, \delta)$, where $\Psi(\mathbf{z}, \delta)\mathrm{d}z = \mathrm{d}\varphi(\mathbf{z})$, satisfies $\deg(\tilde{\alpha}_1) = \bar{j}_1$, $\deg(\tilde{\alpha}_2) < \bar{j}_2$ and $\deg(\tilde{\alpha}_i) = \bar{j}_i$ for $3 \leq i \leq q$, that is, the polynomial degree of α_2 is reduced by the bicausal coordinates change.

Proof. The proof of Lemma 1 is given after that of Theorem 2. □

The above lemma shows a way to reduce the polynomial degree of the differential of a delayed function via bicausal changes of coordinates. With the help of Lemma 1 and inspired by the classical method to transform a polynomial matrix into its triangular normal form (or Hermite form, see e.g., [15]), we propose Algorithm 1 below, which can be used to check the equivalent conditions of Theorem 1 and to construct the desired complementary bicausal coordinates $(\theta_1, \ldots, \theta_{n-p})$.

Theorem 2. *The functions $\lambda_k(\mathbf{x})$,$1 \leq k \leq p$, satisfy the equivalent conditions in Theorem 1 if and only if Algorithm 1 returns to YES. Moreover, if Algorithm 1 returns to YES, then let $\tilde{z}_2, \ldots, \tilde{z}_q$ with $q = n-p+1$, be the functions from the last iteration, i.e., $[\tilde{z}_2, \ldots, \tilde{z}_q]^T = Q_p\varphi_p \circ \cdots \circ Q_1\varphi_1$, where, for each $1 \leq k \leq p$,*

$$\varphi_k = \varphi_k^{l_k} \circ P_k^{l_k} \cdots \varphi_k^2 \circ P_k^2\varphi_k^1 \circ P_k^1 \in \mathcal{K}^{n-k+1} \tag{2}$$

and $Q_k = [0, \mathrm{I}_{n-k}] \in \mathbb{R}^{(n-k)\times(n-k+1)}$ selects the last $n-k$ rows of φ_k, and l_k denotes the number of iterations for λ_k, we have that $[\lambda_1, \ldots, \lambda_p, \theta_1, \ldots, \theta_{n-p}]^T$ is a bicausal change of x-coordinates, where $\theta_1 = \tilde{z}_2$, …, $\theta_{n-p} = \tilde{z}_q$.

Remark 3. Algorithm 1 and Theorem 2 provide another way to prove $(i) \Rightarrow (ii)$ of Theorem 1, the original proof in [5] uses a contradiction with the help of the extended Lie brackets. Algorithm 1 proves $(i) \Rightarrow (ii)$ by directly constructing the complementary bicausal coordinates $(\theta_1, \ldots, \theta_{n-p})$ in Theorem 1 (ii) using condition **(C)** and Lemma 1.

Proof of Theorem 2. "*Only if*:" Assume that $\mathcal{L}$ satisfies **(C)**. Then by Theorem 13 of [22], the latter assumption is equivalent to that there exists a matrix $\Theta(\mathbf{x}, \delta) \in \mathcal{K}^{(n-p)\times n}(\delta]$ such that $\begin{bmatrix} L(\mathbf{x},\delta) \\ \Theta(\mathbf{x},\delta) \end{bmatrix}$ is unimodular, where $L(\mathbf{x}, \delta)\mathrm{d}x = \mathrm{d}\lambda(\mathbf{x})$. It follows that $\mathcal{L}_k := \text{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_k\}$ for all $1 \leq k \leq p$ satisfy **(C)** because we can always find Θ_k such that $\begin{bmatrix} L_k(\mathbf{x},\delta) \\ \Theta_k(\mathbf{x},\delta) \end{bmatrix}$ is unimodular, where $L_k(\mathbf{x}, \delta)\mathrm{d}x = \begin{bmatrix} \mathrm{d}\lambda_1(\mathbf{x}) \\ \cdots \\ \mathrm{d}\lambda_k(\mathbf{x}) \end{bmatrix}$. Remark that the property that $\mathcal{L}_k$ satisfies **(C)** is invariant under bicausal changes of coordinates. Now consider $k = 1$, i.e., in each $1 \leq l \leq l_1$-iteration of Algorithm 1, the right-annihilator of $\mathcal{L}_1$ is causal and thus by Lemma 1, we can always find P_1^l such that $\Delta^{\bar{j}_1}[\hat{\alpha}_1, \hat{\alpha}_2]$ is causal. By reducing the polynomial degree of α_2 and permutations, $\tilde{\alpha}_i$ for all $i \geq 2$ eventually become 0 at $l = l_1$. Moreover, $\deg(\tilde{\alpha}_1) = 0$ for $l = l_1$ as $\mathcal{L}_1$ is closed. So the algorithm does not returns to *NO* in the the first l_1-iterations. Suppose that the algorithm does not return to *NO* for $k = 1, \ldots, k^* - 1$, i.e., after $(l_1 + \cdots + l_{k^*-1})$-iterations, then $\begin{bmatrix} \lambda_1 \\ \cdots \\ \lambda_{k^*} \end{bmatrix}$ becomes $\begin{bmatrix} \lambda_1(\mathbf{x}_1) \\ \cdots \\ \lambda_{k^*-1}(\mathbf{x}_1, \ldots, \mathbf{x}_{k^*-1}) \\ \lambda_{k^*}(\mathbf{x}_1, \ldots, \mathbf{x}_{k^*-1}, \mathbf{z}_1, \ldots, \mathbf{z}_q) \end{bmatrix}$ with $q = n -$

Algorithm 1

Input: $\lambda_1(\mathbf{x}),\ldots,\lambda_p(\mathbf{x})$
Output: *YES/NO*

1: Set $k \leftarrow 1$, $l \leftarrow 1$, $q \leftarrow n$, $z=[z_1,\ldots,z_q]^T \leftarrow [x_1,\ldots,x_n]^T$.
2: **if** $k>1$ **then**
3: Fix $x_1,\ldots,x_{k-1}$ as constants, set $q \leftarrow n-k+1$ and set $z=[z_1,\ldots,z_q]^T \leftarrow [x_k,\ldots,x_n]^T$ to regard $\lambda_k(\mathbf{x}) = \lambda_k(\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{z}) = \lambda_k(\mathbf{z})$ as a function of z-variables and its time-delays.
4: **end if**
5: Set $\alpha(\mathbf{z},\delta)\mathrm{d}z = \mathrm{d}\lambda_k(\mathbf{z},\delta)$ to get $\alpha=[\alpha_1,\ldots,\alpha_q]\in\mathcal{K}^q(\delta]$.
6: Find a permutation matrix $P_k^l \in \mathbb{R}^{q\times q}$ such that $\alpha_1 \not\equiv 0$, $\bar{j}_1 \le \bar{j}_2$ and $[\Delta^{\bar{j}_1}\hat{\alpha}_1, \Delta^{\bar{j}_1}\hat{\alpha}_2]$ is causal after $z \leftarrow P_k^l z$ and $\alpha \leftarrow \alpha(P_k^l)^{-1}$.
7: **if** $\nexists P_k^l$ **then**
8: **return** *NO*
9: **else**
10: Find $\hat{\lambda}(\mathbf{z}_{[\bar{j}_1,\bar{j}]}) \in \mathcal{K}$ such that (1) holds.
11: Set $\tilde{z}_1 \leftarrow \Delta^{\bar{j}_1}\hat{\lambda}$, $\tilde{z}_2 \leftarrow z_2$, …, $\tilde{z}_q \leftarrow z_q$.
12: Define a bicausal change of z-coordinates $\tilde{z} = \varphi_k^l(\mathbf{z}) = [\tilde{z}_1,\tilde{z}_2,\ldots,\tilde{z}_q]^T \in \mathcal{K}^q$.
13: Set $\Psi_k(\mathbf{z},\delta)\mathrm{d}z \leftarrow \mathrm{d}\varphi_k^l(\mathbf{z})$, $\tilde{\alpha}(\mathbf{z},\delta) \leftarrow \alpha(\mathbf{z},\delta)\Psi_k^{-1}(\mathbf{z},\delta)$ and $z \leftarrow (\varphi_k^l)^{-1}(\tilde{\mathbf{z}})$ to have $\tilde{\alpha}(\tilde{\mathbf{z}},\delta) = [\tilde{\alpha}_1(\tilde{\mathbf{z}},\delta),\ldots,\tilde{\alpha}_q(\tilde{\mathbf{z}},\delta)]$ and $\lambda_k(\tilde{\mathbf{z}})$.
14: **if** $\exists 2 \le i \le q : \tilde{\alpha}_i \not\equiv 0$ **then**
15: Set $\alpha \leftarrow \tilde{\alpha}$ and $z \leftarrow \tilde{z}$, $l \leftarrow l+1$ and go to line 5.
16: **else**
17: **if** $\deg(\tilde{\alpha}_1(\tilde{\mathbf{z}},\delta)) \ne 0$ **then**
18: **return** *NO*
19: **else**
20: **if** $k=p$ **then**
21: **return** *YES*
22: **else**
23: Set $[x_k,\ldots,x_n]^T \leftarrow [\tilde{z}_1,\ldots,\tilde{z}_q]^T$.
24: $k \leftarrow k+1$, $l \leftarrow 1$.
25: Go to line 2
26: **end if**
27: **end if**
28: **end if**
29: **end if**

k^*+1 and
$$\begin{bmatrix}\mathrm{d}\lambda_1\\ \cdots \\ \mathrm{d}\lambda_{k^*-1}\\ \mathrm{d}\lambda_{k^*}\end{bmatrix} = \begin{bmatrix} c_1 & 0 & \cdots & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots\\ \star & \cdots & c_{k^*-1} & 0 & \cdots & 0\\ \star & \cdots & \star & \alpha_1 & \cdots & \alpha_q\end{bmatrix}\begin{bmatrix}\mathrm{d}x_1\\ \vdots\\ \mathrm{d}x_{k^*-1}\\ \mathrm{d}z_1\\ \vdots\\ \mathrm{d}z_q\end{bmatrix},$$
where $c_k \not\equiv 0$ and $\deg(c_k)=0$ for all $1\le k\le k^*-1$, "$\star$" denotes some irrelevant terms. Thus by $\mathcal{L}_{k^*}$ satisfies **(C)**, we get that $\mathrm{span}_{\mathcal{K}(\delta]}\{\alpha(\mathbf{z})\mathrm{d}z\}$ satisfies **(C)** (when fixing $(x_1,\ldots,x_{k^*-1})$ as constants) as well, which indicates that Algorithm 1 does not return to *NO* for $k=k^*$. Hence the algorithm returns to *YES* once $k=p$.

"*If*:" Suppose that the algorithm returns to *YES*. Then, we can construct the following bicausal changes of x-coordinates:
$$\psi_1=\varphi_1,\ \psi_2=\begin{bmatrix}M_1\psi_1\\ \varphi_2(N_1\psi_1)\end{bmatrix},\ \ldots,\ \psi_p=\begin{bmatrix}M_{p-1}\psi_{p-1}\\ \varphi_p(N_{p-1}\psi_{p-1})\end{bmatrix},$$
where φ_k, $1\le k\le p$, are defined by (2), $M_k=[\,\mathrm{I}_k\ 0\,]\in\mathbb{R}^{k\times n}$ and $N_k=[\,0\ \mathrm{I}_{n-k}\,]\in\mathbb{R}^{(n-k)\times n}$. Indeed, each ψ_k defines a bicausal change of coordinates on $\mathcal{K}^n$ because $\mathrm{d}\psi_k = \begin{bmatrix}\mathrm{I}_k & 0\\ \star & \Psi_k\end{bmatrix}\mathrm{d}\psi_{k-1}$, where $\Psi_k(\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{z},\delta)\mathrm{d}z = \mathrm{d}\varphi_k(\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{z})$ and Ψ_k, by the constructions in Algorithm 1, is unimodular. Then define the following bicausal change of coordinates $\tilde{x}=[\tilde{x}_1,\ldots,\tilde{x}_n]^T = \psi(\mathbf{x}) = \psi_p\circ\cdots\circ\psi_1(\mathbf{x})$, thus in $\tilde{x}$-coordinates we have that
$$\begin{bmatrix}\mathrm{d}\lambda_1\\ \cdots\\ \mathrm{d}\lambda_p\end{bmatrix} = \begin{bmatrix} c_1 & 0 & \cdots & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots\\ * & \cdots & c_p & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \vdots\\ \tilde{x}_p\\ \tilde{x}_{p+1}\\ \vdots\\ \tilde{x}_n\end{bmatrix},$$
where $c_i=c_i(\tilde{\mathbf{x}})\not\equiv 0$ and $\deg(c_i)=0$ for all $1\le i\le p$. It follows that $[\lambda_1,\ldots,\lambda_p,\tilde{x}_{p+1},\ldots,\tilde{x}_n]^T$ is a bicausal change of coordinates because $T(x,\delta)$, where $T\mathrm{d}\tilde{x} = \begin{bmatrix}\mathrm{d}\lambda\\ \mathrm{d}\tilde{x}_{p+1}\\ \vdots\\ \mathrm{d}\tilde{x}_n\end{bmatrix}$, is a unimodular matrix. Thus item (ii) of Theorem 1 holds with $\theta_1=\tilde{x}_{p+1}$, …, $\theta_{n-p}=\tilde{x}_n$. Moreover, by using φ_k and Q_k, we can express $[\tilde{x}_{p+1},\ldots,\tilde{x}_n]^T = Q_p\varphi_p\circ\cdots\circ Q_1\varphi_1$. □

Proof of Lemma 1. We need to prove that there exist two integers $1\le r\le q-1$ and $r+1\le s\le q$ such that the right-annihilator of $[\alpha_r,\alpha_s]$ is causal. Suppose that the right-annihilators of $[\alpha_r,\alpha_s]$ are not causal for all $1\le r\le q-1$, $r+1\le s\le q$. Let $\begin{bmatrix}\beta_{l(r,s)}\\ \gamma_{l(r,s)}\end{bmatrix}\in\mathcal{K}^2(\delta]$, where $l(r,s)=(r-1)(q-\frac{r}{2})+s-r$ and $1\le l\le l^*=\frac{q(q-1)}{2}$, be a basis for the right-annihilator of $[\alpha_r,\alpha_s]$, then define $\tau_l := [0,\ldots,0,\beta_l,0,\ldots,0,\gamma_l,0,\ldots,0]^T$, where β_l and γ_l are in the r-th and s-th rows of τ_l, respectively. It follows that $\alpha\tau_l=0$ for all $1\le l\le l^*$. Thus the right-submodule $\mathcal{T}=\mathrm{span}_{\mathcal{K}(\delta]}\{\tau_1,\ldots,\tau_{l^*}\}$ is in the right-annihilator of $\mathrm{span}_{\mathcal{K}(\delta]}\{\alpha\mathrm{d}z\}$, so $\dim\mathcal{T}\le q-1$. Recall that the right-annihilator of a left-submodule is always closed (see [7]). By the construction, $\mathcal{T}$ is closed and $\dim\mathcal{T}\ge q-1$, which implies $\mathcal{T}$ coincides with the right-annihilator of $\mathrm{span}_{\mathcal{K}(\delta]}\{\alpha\mathrm{d}z\}$ because they have the same dimension $q-1$ and are both closed. If τ_l, for all $1\le l\le\frac{q(q-1)}{2}$, are not causal, we have that the right-annihilator of α is not causal. Hence if the right-annihilator of α is causal, then there must exist r,s such that τ_l is causal.

(i) If the right-annihilator of $[\alpha_1(\mathbf{z},\delta),\alpha_2(\mathbf{z},\delta)]$, generated by $\begin{bmatrix}\beta(\mathbf{z},\delta)\\ \gamma(\mathbf{z},\delta)\end{bmatrix}$, is causal, then the right-annihilator of $[\alpha_1^{\bar{j}_1}(\mathbf{z})\delta^{\bar{j}_1},\alpha_2^{\bar{j}_2}(\mathbf{z})\delta^{\bar{j}_2}]$ is also causal. Indeed, write $\beta(\mathbf{z},\delta)=\sum_{j=1}^{\bar{j}_\beta}\beta^j(\mathbf{z})\delta^j$ and $\gamma(\mathbf{z},\delta)=\sum_{j=1}^{\bar{j}_\gamma}\gamma^j(\mathbf{z})\delta^j$, we can deduce both $\bar{j}_1+\bar{j}_\beta=\bar{j}_2+\bar{j}_\gamma$ and $[\alpha_1^{\bar{j}_1}(\mathbf{z})\delta^{\bar{j}_1},\alpha_2^{\bar{j}_2}(\mathbf{z})\delta^{\bar{j}_2}]\begin{bmatrix}\beta^{\bar{j}_\beta}(\mathbf{z})\delta^{\bar{j}_\beta}\\ \gamma^{\bar{j}_\gamma}(\mathbf{z})\delta^{\bar{j}_\gamma}\end{bmatrix}=0$ from $\alpha_1\beta+\alpha_2\gamma=0$. It follows that the right-annihilator of $[\hat{\alpha}_1,\hat{\alpha}_2]$ is causal. Then because $\hat{\alpha}_1=\delta^{\bar{j}_1}$, by a direct calculation, the right-annihilator of $[\hat{\alpha}_1,\hat{\alpha}_2]$ is generated by $[\Delta^{\bar{j}_1}\hat{\alpha}_2,-1]^T$. Hence $[\Delta^{\bar{j}_1}\hat{\alpha}_1,\Delta^{\bar{j}_1}\hat{\alpha}_2]=[1,\Delta^{\bar{j}_1}\hat{\alpha}_2]$ is causal.

(ii) Let $(\tilde{\xi}_1,\xi_2)=\mathbf{z}_{[0,\bar{j}]}$ and $\xi_2=(z_1(-\bar{j}_1),z_2(-\bar{j}_2))$. If we fix $\tilde{\xi}_1$ as constants, then $\lambda(\mathbf{z}_{[0,\bar{j}]})=\lambda(\tilde{\xi}_1,\xi_2)=\lambda(\xi_2)$ can be seen as a function of ξ_2. It follows that the one form $\hat{\omega}=\alpha_1^{\bar{j}_1}(\tilde{\xi}_1,\xi_2)\mathrm{d}z_1(-\bar{j}_1)+\alpha_2^{\bar{j}_2}(\tilde{\xi}_1,\xi_2)\mathrm{d}z_2(-\bar{j}_1)$ is exact (by fixing $\tilde{\xi}_1$). Then by Frobenius theorem, the codistribution $\mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}z(-\bar{j}_1)+\hat{\alpha}_2^{\bar{j}_2}(\mathbf{z}_{[\bar{j}_1,\bar{j}]})\mathrm{d}z(-\bar{j}_2)\right\}=\mathrm{span}_{\mathcal{K}}\{\hat{\omega}\}$, where $\hat{\alpha}_2^{\bar{j}_2}=\frac{\alpha_2^{\bar{j}_2}}{\alpha_1^{\bar{j}_1}}$ depends only on $\mathbf{z}_{[\bar{j}_1,\bar{j}]}$ by item (i), is integrable when fixing ξ_1, where $(\xi_1,\xi_2)=\mathbf{z}_{[\bar{j}_1,\bar{j}]}$. Hence there exists a function $\hat{\lambda}=\hat{\lambda}(\bar{z}_{[\bar{j}_1,\bar{j}]})\in\mathcal{K}$ such that (1) holds

(iii) By construction, we have $\mathrm{d}\hat{\lambda}(\mathbf{z}_{[\bar{j}_1,\bar{j}]})=\hat{\beta}(\mathbf{z}_{[\bar{j}_1,\bar{j}]},\delta)\mathrm{d}z+c\hat{\alpha}_1(\mathbf{z}_{[\bar{j}_1,\bar{j}]},\delta)\mathrm{d}z_1+c\hat{\alpha}_2(\mathbf{z}_{[\bar{j}_1,\bar{j}]},\delta)\mathrm{d}z_2$ for some function $c=c(\mathbf{z}_{[\bar{j}_1,\bar{j}]})\in\mathcal{K}$, where $\hat{\beta}=[\hat{\beta}_1,\ldots,\hat{\beta}_q]$, $\hat{\beta}_1\equiv 0$, $\deg(\hat{\beta}_2)\le\bar{j}_2-1$, and for $3 \le i \le q$, $\deg(\hat{\beta}_i) = \bar{j}_i$ if $\bar{j}_i \ge \bar{j}_1$ and $\hat{\beta}_i \equiv 0$ if $\bar{j}_i < \bar{j}_1$. Now let $\varphi(\mathbf{z}) = [\Delta^{\bar{j}_1}\hat{\lambda}(\mathbf{z}), z_2, \dots, z_q]^T$ and $\Psi \mathrm{d}z = \mathrm{d}\varphi(\mathbf{z})$, we get

$$\Psi = \begin{bmatrix} \Delta^{\bar{j}_1}(c\hat{\alpha}_1) & \Delta^{\bar{j}_1}(\hat{\beta}_2 + c\hat{\alpha}_2) & \Delta^{\bar{j}_1}\hat{\beta}_3 & \cdots & \Delta^{\bar{j}_1}\hat{\beta}_q \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

which is upper triangular and $\Delta^{\bar{j}_1}(c\hat{\alpha}_1) = \Delta^{\bar{j}_1}(c\delta^{\bar{j}_1}) = c(+\bar{j}_1)$ is of polynomial degree zero, and thus $\Psi(\mathbf{z}, \delta)$ is unimodular and $\varphi(\mathbf{z}, \delta)$ is a bicausal change of coordinates. Then we have $[\tilde{\alpha}_1, \dots, \tilde{\alpha}_q] = \alpha\Psi^{-1} =$

$$\begin{bmatrix} \bar{\alpha}_1 + \alpha_1^{\bar{j}_1}\delta^{\bar{j}_1} \\ \bar{\alpha}_2 + \alpha_1^{\bar{j}_1}\hat{\alpha}_2 \\ \alpha_3 \\ \cdots \\ \alpha_q \end{bmatrix}^T \begin{bmatrix} \frac{1}{c(+\bar{j}_1)} & -(\frac{\Delta^{\bar{j}_1}\hat{\beta}_2}{c(+\bar{j}_1)} + \Delta^{\bar{j}_1}\hat{\alpha}_2) & -\frac{\Delta^{\bar{j}_1}\hat{\beta}_3}{c(+\bar{j}_1)} & \cdots & -\frac{\Delta^{\bar{j}_1}\hat{\beta}_q}{c(+\bar{j}_1)} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

By a direct calculation, we have $\tilde{\alpha}_1 = \frac{\bar{\alpha}_1}{c(+\bar{j}_1)} + \frac{\alpha_1^{\bar{j}_1}}{c}\delta^{\bar{j}_1}$, $\tilde{\alpha}_2 = -(\bar{\alpha}_1 + \alpha_1^{\bar{j}_1}\delta^{\bar{j}_1})\frac{\Delta^{\bar{j}_1}\hat{\beta}_2}{c(+\bar{j}_1)} - \bar{\alpha}_1\Delta^{\bar{j}_1}\hat{\alpha}_2 + \bar{\alpha}_2$ and $\tilde{\alpha}_i = \alpha_i - (\bar{\alpha}_1 + \alpha_1^{\bar{j}_1}\delta^{\bar{j}_1})(\frac{\Delta^{\bar{j}_1}\hat{\beta}_i}{c(+\bar{j}_1)})$ for $3 \le i \le q$. Notice that $\deg(\bar{\alpha}_1 + \alpha_1^{\bar{j}_1}\delta^{\bar{j}_1}) = \bar{j}_1$, $\deg(\frac{\Delta^{\bar{j}_1}\hat{\beta}_2}{c(+\bar{j}_1)}) \le \bar{j}_2 - 1 - \bar{j}_1$ and $\deg(-\bar{\alpha}_1\Delta^{\bar{j}_1}\hat{\alpha}_2 + \bar{\alpha}_2) \le \bar{j}_2 - 1$. Hence $\deg(\tilde{\alpha}_1) = \bar{j}_1$, $\deg(\tilde{\alpha}_2) < \bar{j}_2$ and $\deg(\tilde{\alpha}_i) = \deg(\alpha_i) = \bar{j}_i$, $\forall\, i \ge 3$.

□

Example 1. 1). Consider $b(\mathbf{x}) = 0$ in section I, we apply Algorithm 1 to $b(\mathbf{x})$. For $k = 1$, $l = 1$,

$$\alpha = \left[\, x_2(-1) + x_2x_2(-2)\delta,\ x_1(-1)x_2(-2) + x_1\delta + x_1(-1)x_2\delta^2 \,\right].$$

It is seen that $P_1^1 = \mathrm{I}_2$ and $[\hat{\alpha}_1, \hat{\alpha}_2] = [\delta, \frac{x_1(-1)}{x_2(-2)}\delta^2]$. Thus $\Delta^{\bar{j}_1}[\hat{\alpha}_1, \hat{\alpha}_2] = [1, \frac{x_1}{x_2(-1)}\delta]$ with $\bar{j}_1 = 1$ is causal. Then $\mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}x_1(-1) + \frac{x_1(-1)}{x_2(-2)}\mathrm{d}x_2(-2)\right\}$ is integrable and we find the function $\hat{\lambda} = x_1(-1)x_2(-2)$ satisfying (1) (ξ_1 is absent and $\xi_2 = (x_1(-1), x_2(-2))$). The bicausal coordinate transformation is $\varphi_1^1 = \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} = \begin{bmatrix} x_1x_2(-1) \\ x_2 \end{bmatrix}$ as $\Delta^1\hat{\lambda} = x_1x_2(-1)$. Thus in $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$-coordinates, $b = b(\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2) = \tilde{z}_1 + \tilde{z}_1(-1)\tilde{z}_2$ and $\tilde{\alpha} = [1 + \tilde{z}_2\delta, \tilde{z}_1(-1)]$. So $\tilde{\alpha}_2 \not\equiv 0$ and we go to the second iteration (i.e., line 15→line 5). For $k = 1$, $l = 2$, we use the permutation matrix $P_1^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to have $\tilde{\alpha}\begin{bmatrix} \mathrm{d}\tilde{z}_1 \\ \mathrm{d}\tilde{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{z}_1(-1) & 1 + \tilde{z}_2\delta \end{bmatrix}\begin{bmatrix} \mathrm{d}\tilde{z}_2 \\ \mathrm{d}\tilde{z}_1 \end{bmatrix}$. Define new coordinates $\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = P_1^2\begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}$ to have $b(\tilde{x}_1, \tilde{x}_2) = \tilde{x}_2 + \tilde{x}_2(-1)\tilde{x}_1 + e_2$ and $\mathrm{d}b = \begin{bmatrix} \tilde{x}_2(-1) & 1 + \tilde{x}_1\delta \end{bmatrix}\begin{bmatrix} \mathrm{d}\tilde{x}_1 \\ \mathrm{d}\tilde{x}_2 \end{bmatrix}$. Now although $1 + \tilde{x}_1\delta \not\equiv 0$, we can already conclude that $b(\tilde{x}_1, \tilde{x}_2)$ satisfies item (iv) of Theorem 1 without continuing the algorithm because $\tilde{x}_2(-1)$ is unimodular. Moreover, we get $\varphi_1 = P_1^2\varphi_1^1$ by (2) and the complementary coordinate $\theta = Q_1\varphi_1 = x_1x_2(-1)$. It can be checked that $\begin{bmatrix} b(\mathbf{x}) \\ \theta(\mathbf{x}) \end{bmatrix}$ is indeed a bicausal change of coordinates. Moreover, by Corollary 1, $b(\mathbf{x}) = 0$ implies $\tilde{x}_1 = \frac{-e_2 - \tilde{x}_2}{\tilde{x}_2(-1)}$.

2). Consider $c(\mathbf{x}) = 0$ in section I and apply Algorithm 1 to $c(\mathbf{x})$. For $l = 1$, $\alpha = [x_1(-1) + x_1\delta, x_2(-1) + x_2\delta]$ and $\hat{\alpha} = [\delta, \frac{x_2}{x_1}\delta]$, it is seen that $\bar{j}_1 = 1$ and $\Delta^1\hat{\alpha} = [1, \frac{x_1(+1)}{x_2(+1)}]$ is not causal. Thus Algorithm 1 returns to *NO*, meaning that $c(\mathbf{x})$ can not be regarded as a bicausal coordinate and there does not exist a bicausal coordinates transformation such that Theorem 1 (iv) holds.

3). As the third example, we consider two functions together:

$$\begin{cases} \lambda_1 = x_2x_1(-2) + x_3(-1)x_2(-1) \\ \lambda_2 = x_3(-1)x_2(-1)x_1(-1) + x_2x_1(-2)x_1 + x_3(-1)x_2(-1)x_1 \end{cases}$$

and apply Algorithm 1. For $k = 1$, $l = 1$, $\alpha = [x_2\delta^2, x_1(-2) + x_3(-1)\delta, x_2(-1)\delta]$, we find $P_1^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $\alpha(P_1^1)^{-1}P_1^1\mathrm{d}x = \begin{bmatrix} x_2(-1)\delta & x_1(-2) + x_3(-1)\delta & x_2\delta^2 \end{bmatrix}\begin{bmatrix} \mathrm{d}x_3 \\ \mathrm{d}x_2 \\ \mathrm{d}x_1 \end{bmatrix}$. Thus we have $\alpha_1 = x_2(-1)\delta$, $\alpha_2 = x_1(-2) + x_3(-1)\delta$ and $\bar{j}_1 = \bar{j}_2 = 1$. So $[\Delta^1\hat{\alpha}_1, \Delta^1\hat{\alpha}_2] = [1, \frac{x_2}{x_3}]$ is causal. Then we find $\hat{\lambda} = x_2(-1)x_3(-1)$ and $\varphi_1^1 = [\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]^T = [x_2x_3, x_2, x_1]^T$ to get $\tilde{\alpha} = [\delta, \tilde{x}_3(-2), \tilde{x}_2\delta^2]$ and $\lambda_1 = \tilde{x}_1(-1) + \tilde{x}_2\tilde{x}_3(-2)$. Since both $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ are not zero, we drop all the tildes and go to next iteration (i.e., line 15→line 5). For $k = 1$, $l = 2$, $\lambda_1 = x_1(-1) + x_2x_3(-2)$, we find $P_1^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\varphi_1^2 = [\tilde{x}_1, \tilde{x}_2, \tilde{x}_3] = [x_2x_3(-2) + x_1(-1), x_1, x_3]^T$. Then $\tilde{\alpha} = [1, 0, 0]$, we have $\deg(\tilde{\alpha}_1) = 0$ and go to $k = 2$ (i.e., line 25→line 2). Notice that $\varphi_1 = \varphi_1^2 \circ P_1^2\varphi_1^1 \circ P_1^1 = [\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]^T = [x_2x_1(-2) + x_2(-1)x_3(-1), x_2x_3, x_1]^T$ and in φ_1-coordinates, we have $\lambda_2 = \tilde{x}_1\tilde{x}_3 + \tilde{x}_2(-1)\tilde{x}_3(-1)$.

Now we are at line 2 and we restart the procedure. For $k = 2$, $l = 1$, set $z_1 = \tilde{x}_2$ and $z_2 = \tilde{x}_3$ to have $\lambda_2(z_1, z_2) = \tilde{x}_1z_2 + z_1(-1)z_2(-1)$ and $\alpha = [z_2(-1)\delta, \tilde{x}_1 + z_1(-1)\delta]$. We find $P_2^1 = \mathrm{I}_2$ and $\varphi_2^1 = \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} = \begin{bmatrix} z_1z_2 \\ z_2 \end{bmatrix}$. It follows that $\tilde{\alpha} = [\delta, \tilde{x}_1]$ and $\lambda_2 = \tilde{z}_1(-1) + \tilde{x}_1\tilde{z}_2$. Drop the tildes of $\tilde{z}_1(-1)$ and $\tilde{z}_2(-1)$. For $k = 2$, $l = 2$, $\lambda_2 = z_1(-1) + \tilde{x}_1z_2$, we find $P_2^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\varphi_2^2 = \begin{bmatrix} z_1(-1) + \tilde{x}_1z_2 \\ z_1 \end{bmatrix}$. Thus $\tilde{\alpha} = [1, 0]$ and the algorithm returns to *YES*. Moreover, we have $\varphi_2 = \varphi_2^2 \circ P_2^2\varphi_2^1 \circ P_2^1 = \begin{bmatrix} \tilde{x}_1z_2 + z_1(-1)z_2(-1) \\ z_1 \end{bmatrix}$. Thus the complementary coordinate $\theta = z_1 = \tilde{x}_2 = x_2x_3$, we can check that $\begin{bmatrix} \mathrm{d}\lambda_1 \\ \mathrm{d}\lambda_2 \\ \mathrm{d}\theta \end{bmatrix}$ is indeed a unimodular matrix.

V. Applications to Nonlinear Differential-Algebraic Equations with Time-Delays

Consider a delayed differential-algebraic equation (DDAE) of the following form:

$$\Xi: \quad \sum_{j=0}^{\bar{j}} E^j(\mathbf{x}_{[\bar{i}]})\delta^j\dot{x} = F(\mathbf{x}_{[\bar{i}]}) \qquad (3)$$

with an initial-value function $x(s) = \xi_x(s)$, $s \in [-\bar{i}, 0]$, where $E^j : \mathbb{R}^{(\bar{i}+1)n} \to \mathcal{K}^{p\times n}$ and $F : \mathbb{R}^{(\bar{i}+1)n} \to \mathcal{K}^p$, where $\bar{i}$ and $\bar{j}$ denote the maximal delay of x and $\dot{x}$, respectively. We can shortly rewrite (3) as $E(\mathbf{x}, \delta)\dot{x} = F(\mathbf{x})$, where $E(\mathbf{x}, \delta) = \sum_{j=0}^{\bar{j}} E^j(\mathbf{x}_{[\bar{i}]})\delta^j \in \mathcal{K}^{p\times n}(\delta]$. Remark that the form (3) is general enough to describe many constrained physical models under delay effects as in e.g. [1], [16], [27], the DDAE Ξ reduces to a delay-free DAE of the form $E(x)\dot{x} = F(x)$ [10], [20], [24] when $\bar{j} = \bar{i} = 0$.

Definition 4. A function $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of Ξ with the initial-value function ξ_x if there exists $T > 0$ such that $x(t)$ is continuously differentiable on $[-\bar{i}, T)$ and satisfies (3) for all $t \in [0, T)$.

We will call Ξ an index-0 DDAE if $E(\mathbf{x}, \delta)$ is of full row rank over $\mathcal{K}(\delta]$. An index-0 DDAE is very close to a delayed ODE, the latter has the classifications of the retarded, the neutral and the advanced types, and can be solved via the step method (see e.g., [14]). For an index-0 DDAE Ξ, if $p = n$ and $\mathrm{rank}_{\mathcal{K}}E^0(\mathbf{x}) = p$, we can always rewrite Ξ as a delayed ODE of the neutral type:

$$\dot{x} = (E^0)^{-1}F(\mathbf{x}) - \sum_{j=1}^{\bar{j}}(E^0)^{-1}E^j(\mathbf{x})\delta^j\dot{x} = f(\mathbf{x}_{[0,\bar{i}]}, \dot{\mathbf{x}}_{[1,\bar{j}]}).$$

Remark that if $\mathrm{rank}_{\mathcal{K}} E^0(\mathbf{x}) \neq p$, then an index-0 DDAE results in a mixed type, or in particular, an advanced type delayed ODE, for which, in general, is hard to define a smooth solution unless the initial-value functions satisfy some restrictive conditions. In the present note, we are only interested in delayed ODEs of neutral or retarded types, so below we make the assumption that $\mathrm{rank}_{\mathcal{K}} E^0(\mathbf{x}) = p$ for index-0 DDAEs.

Now given a DDAE $\Xi : E(\mathbf{x}, \delta)\dot{x} = F(\mathbf{x})$, which may not be index-0, we propose the following algorithm to reduce its index with the help of the results in sections III and IV. Algorithm 2 generalizes the geometric reduction algorithm for delay-free DAEs in [10].

Algorithm 2 DDAE reduction algorithm

Input: $E(\mathbf{x}, \delta)$ and $F(\mathbf{x})$
Output: $E_{k^*}(\mathbf{z}_{k^*}, \delta)$ and $F_{k^*}(\mathbf{z}_{k^*})$

1: Set $k \leftarrow 0$, $z_k \leftarrow x$, $E_k \leftarrow E$, $F_k \leftarrow F$, $r_{k-1} = p$, $n_{k-1} = n$
2: **if** the row rank of $E_k(\mathbf{z}_k, \delta)$ over $\mathcal{K}(\delta]$ is r_{k-1} **then**
3: **return** $k^* \leftarrow k$, $z_{k^*} \leftarrow z_k$, $E_{k^*} \leftarrow E_k$, $F_{k^*} \leftarrow F_k$
4: **else**
5: Denote the row rank of $E_k(\mathbf{z}_k, \delta)$ over $\mathcal{K}(\delta]$ by $r_k < r_{k-1}$.
6: Find a unimodular matrix $Q_k(\mathbf{z}_k, \delta) \in \mathcal{K}^{r_{k-1} \times r_{k-1}}(\delta]$ such that $Q_k(\mathbf{z}_k, \delta)E_k(\mathbf{z}_k, \delta) = \begin{bmatrix} E_{k1}(\mathbf{z}_k,\delta) \\ 0 \end{bmatrix}$, where $E_{k1}(\cdot, \delta) \in \mathcal{K}^{r_k \times r_{k-1}}(\delta]$ and its row rank over $\mathcal{K}(\delta]$ is r_k.
7: Denote $Q_k(\mathbf{z}_k, \delta)F_k(\mathbf{z}_k) = \begin{bmatrix} F_{k1}(\mathbf{z}_k) \\ F_{k2}(\mathbf{z}_k) \end{bmatrix}$, where $F_{k2}(\mathbf{z}_k) \in \mathcal{K}^{r_{k-1} - r_k}$.
8: Define the submodule $\mathcal{F}_k := \mathrm{span}_{\mathcal{K}(\delta]} \{\mathrm{d}F_{k2}(\mathbf{z}_k)\}$. Denote $\mathrm{rank}_{\mathcal{K}(\delta]} \mathcal{F}_k = n_{k-1} - n_k \leq r_{k-1} - r_k$.
9: Assume that $\mathcal{F}_k$ satisfies **(C)**.
10: **if** $n_{k-1} - n_k < r_{k-1} - r_k$ **then**
11: Find functions $\tilde{F}_{k2}(\mathbf{z}_k) \in \mathcal{K}^{n_{k-1} - n_k}$ such that $\mathrm{span}_{\mathcal{K}(\delta]} \left\{ \mathrm{d}\tilde{F}_{k2} \right\} = \mathcal{F}_k$ and $\tilde{F}_{k2}(\mathbf{z}_k) = 0$ is equivalent to $F_{k2}(\mathbf{z}_k) = 0$.
12: $F_{k2} \leftarrow \tilde{F}_{k2}$
13: **end if**
14: Find functions $\theta(\mathbf{z}_k) \in \mathcal{K}^{n_k}$ such that $\begin{bmatrix} z_{k+1} \\ \bar{z}_{k+1} \end{bmatrix} = \varphi_k(\mathbf{z}_k) = \begin{bmatrix} \theta(\mathbf{z}_k) \\ F_{k2}(\mathbf{z}_k) \end{bmatrix}$ is a bicausal change of z_k-coordinates.
15: Set $[\tilde{E}_{k1}, \tilde{E}_{k2}] \leftarrow E_{k1}\Psi_k^{-1}$ and $z_k \leftarrow \varphi_k^{-1}(\mathbf{z}_{k+1}, \bar{\mathbf{z}}_{k+1})$, where $\Psi_k \mathrm{d}z_k = \mathrm{d}\varphi_k$ and $\tilde{E}_{k1}(\mathbf{z}_{k+1}, \bar{\mathbf{z}}_{k+1}, \delta) \in \mathcal{K}^{r_k \times n_k}(\delta]$.
16: Set $E_{k+1}(\mathbf{z}_{k+1}, \delta) \leftarrow \tilde{E}_{k1}(\mathbf{z}_{k+1}, 0, \delta)$ and $F_{k+1}(\mathbf{z}_{k+1}) \leftarrow F_{k1}(\mathbf{z}_{k+1}, 0)$.
17: Set $k \leftarrow k + 1$ and go to line 2.
18: **end if**

Theorem 3. *Consider a DDAE Ξ, given by (3). Assume that the submodule $\mathcal{F}_k$ of Algorithm 2 satisfies* **(C)** *for $k \geq 0$. We have that*

(i) *there exists an integer $0 \leq k^* \leq p$ such that Algorithm 2 returns to $E_{k^*}(\mathbf{z}_{k^*}, \delta)$ and $F_{k^*}(\mathbf{z}_{k^*})$.*

(ii) *The DDAE*

$$\Xi^* : \ E_{k^*}(z_{k^*}, \delta)\dot{z}_{k^*} = F_{k^*}(\mathbf{z}_{k^*})$$

is index-0, and Ξ^* and Ξ have isomorphic solutions, i.e., there exists a bicausal change of coordinates $\Phi(\mathbf{x}) = [z_{k^*}, \bar{z}_{k^*}, \ldots, \bar{z}_1]^T$ such that $z_{k^*}(t)$ is a solution of Ξ^* with the initial-value function $\xi_{z_{k^*}}$ if and only if $x(t) = \Phi^{-1}(\mathbf{z}_{k^*}(t), 0, \ldots, 0)$ is a solution of Ξ with the initial-value function $\xi_x = \Phi^{-1}(\boldsymbol{\xi}_{z_{k^*}}, 0, \ldots, 0)$.

(iii) *For $E_{k^*}(\mathbf{z}_{k^*}, \delta) = \sum\limits_{j=0}^{\bar{j}_{z^*}} E^j_{k^*}(\mathbf{z}_{k^*})\delta^j$, suppose that $\mathrm{rank}_{\mathcal{K}} E^0_{k^*}(\mathbf{z}_{k^*}) = r_{k^*}$, then Ξ has a unique solution with the initial-value function ξ_x if and only if $r_{k^*} = n_{k^*}$.*

Proof. (i) By using the results of Lemma 4 of [22], Corollary 1 and Theorem 1 above, respectively, we can guarantee the existences of the unimodular matrix $Q_k(\mathbf{z}_k, \delta)$ of line 6, the functions $\tilde{F}_{k2}(\mathbf{z}_k)$ of line 11 and the functions $\theta(\mathbf{z}_k)$ of line 14. Thus the algorithm does not stop until $r_{k^*} = r_{k^*-1}$. Then by $p \geq r_0 > r_1 > \cdots > r_{k^*-1} = r_{k^*} \geq 0$, it can be deduced that $0 \leq k^* \leq p$.

(ii) Ξ^* is index-0 because $E_{k^*}(\mathbf{z}_{k^*}, \delta)$ is of full row rank over $\mathcal{K}(\delta]$. Now consider the $1, \ldots, k$ steps of Algorithm 2, the unimodular matrice $Q_k(\mathbf{z}_k, \delta)$ for each k does not change solutions, we have that $z_{k+1}(t)$ is a solution of $E_{k+1}(\mathbf{z}_{k+1}, \delta)\dot{z}_{k+1} = F_{k+1}(\mathbf{z}_{k+1})$ if and only if $x(t) = z_0(t)$ is a solution of Ξ, where

$$x(t) = z_0(t) = \varphi_0^{-1}(\mathbf{z}_1(t), 0), \cdots, z_k(t) = \varphi_k^{-1}(\mathbf{z}_{k+1}(t), 0). \quad (4)$$

Each bicausal change of coordinates $\varphi_k(z_k, \delta)$ is defined on $\mathcal{K}^{n_{k-1}}$, we extend it to $\mathcal{K}^n$ by setting $\Phi_k = [\varphi_k, \bar{z}_k, \bar{z}_{k-1}, \ldots, \bar{z}_1]^T$ (set $\Phi_0 = \varphi_0$). Notice that if $k^* = 0$, then Ξ^* coincides with Ξ and item (ii) holds immediately. If $k^* > 0$, assume without loss of generality that $k + 1 = k^*$, then $\Phi := \Phi_{k^*} = [z_{k+1}, \bar{z}_{k+1}, \ldots, \bar{z}_1]^T = [z_{k^*}, \bar{z}_{k^*}, \ldots, \bar{z}_1]^T$ maps any solution $x(t)$ (and its delays) of Ξ to $(z_{k+1}(t), 0, \ldots, 0)$ by (4), where $z_{k+1}(t) = z_{k^*}(t)$ is a solution of Ξ^*.

(iii) Rewite Ξ^* as $E^0_{k^*}(\mathbf{z}_{k^*})\dot{z}_{k^*} = F_{k^*}(\mathbf{z}_{k^*}) - \sum\limits_{j=1}^{\bar{j}_{z_{k^*}}} E^j_{k^*}(\mathbf{z}_{k^*})\delta^j \dot{z}_{k^*}$. If $\mathrm{rank}_{\mathcal{K}} E^0_{k^*}(\mathbf{z}_{k^*}) = r_{k^*}$, then we can always find the right-inverse $(E^0_{k^*})^\dagger(\mathbf{z}_{k^*})$ of $E^0_{k^*}(\mathbf{z}_{k^*})$ over $\mathcal{K}$. Thus all solutions of Ξ^* are solutions of the followings delayed ODE corresponding to all choices of free variables $v = v(t)$:

$$\dot{z}_{k^*} = (E^0_{k^*})^\dagger F_{k^*}(\mathbf{z}_{k^*}) - \sum_{j=1}^{\bar{j}} (E^0_{k^*})^\dagger E^j_{k^*}(\mathbf{z}_{k^*})\delta^j \dot{z}_{k^*} + g(\mathbf{z}_{k^*})v,$$

where $g(\mathbf{z}_{k^*}) \in \mathcal{K}^{n_{k^*} \times (n_{k^*} - r_{k^*})}$ is of full column rank over $\mathcal{K}$ and $E^0_{k^*}(\mathbf{z}_{k^*})g(\mathbf{z}_{k^*}) = 0$. So Ξ^* has a unique solution with an initial-value function $\xi_{z_{k^*}}$ if and only if the free variables v is absent, i.e., $n_{k^*} - r_{k^*} = 0$. Finally, since Ξ^* and Ξ have isomorphic solutions by item (ii), we have that item (iii) holds. $\square$

Example 2. Consider the following nonlinear DDAE Ξ, given by

$$\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2^3 x_1(-1)/(\ln c - x_1) \\ e^{x_1(-3) + x_3(-2)x_2(-3)} - c \\ x_1(-1) - x_1(-2) + x_3x_2(-1) - x_3(-1)x_2(-2) \end{bmatrix}.$$

where $c > 0$ is a constant. We apply Algorithm 2 to Ξ. For $k = 0$, $E_0 = E \in \mathbb{R}^{5\times 4}$ is constant and $\mathrm{rank}_{\mathcal{K}} E_0 = r_0 = 3$. Let $Q_0 = \mathrm{I}_5$, we get $F_{02} = \begin{bmatrix} e^{x_1(-3) + x_3(-2)x_2(-3)} + c \\ x_1(-1) - x_1(-2) + x_3x_2(-1) - x_3(-1)x_2(-2) \end{bmatrix}$ (i.e., line 7). By a direct calculation, it is found that $\mathcal{F}_0 = \mathrm{span}_{\mathcal{K}(\delta]} \{\mathrm{d}F_{02}\}$ is closed and $\mathrm{rank}_{\mathcal{K}(\delta]} \mathrm{span}_{\mathcal{K}(\delta]} \{\mathrm{d}F_{02}\} = 1 < p - r_0 = 2$. Thus we use the results of Proposition 1 to find $\bar{F}_{02} = x_1(-1) + x_3x_2(-1)$ such that $\mathrm{span}_{\mathcal{K}(\delta]} \{\mathrm{d}\bar{F}_{02}\} = \mathcal{F}_0$. It can be checked by applying Algorithm 1 to $\bar{F}_{02}$ that $\mathcal{F}_0$ satisfies **(C)**. We modify $\bar{F}_{02}$ to $\tilde{F}_{02} = x_1(-1) + x_3x_2(-1) - \ln c$ such that $\tilde{F}_{02} = 0$ is equivalent to $F_{02} = 0$ (i.e., line 11). Then $\bar{z}_1 = \tilde{F}_{02} = x_1(-1) + x_3x_2(-1) - \ln c$, $z_1 = [x_1, x_2, x_4]^T$ is a bicausal change of coordinates and in $(z_1, \bar{z}_1)$-coordinates, Ξ becomes (i.e., line 15) $\begin{bmatrix} \tilde{E}_{01}(\mathbf{z}_1, \bar{\mathbf{z}}_1, \delta) & \tilde{E}_{02}(\mathbf{z}_1, \bar{\mathbf{z}}_1, \delta) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{\bar{z}}_1 \end{bmatrix} = \begin{bmatrix} F_{01}(\mathbf{z}_1, \bar{\mathbf{z}}_1) \\ F_{02}(\mathbf{z}_1, \bar{\mathbf{z}}_1) \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{x_2(-1)}\delta & \frac{\bar{z}_1 - \ln c + x_1(-1)}{x_2^2(-1)}\delta & 0 & -\frac{1}{x_2(-1)}\delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{\bar{z}}_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{x_2^3 x_1(-1)}{\ln c - x_1} \\ -x_4x_1(-1) \\ ce^{\bar{z}_1(-2)} - c \\ \bar{z}_1 - \bar{z}_1(-1) \end{bmatrix}.$$

Thus by setting $\bar{\mathbf{z}}_1 = 0$, we get (i.e., line 16)

$$E_1 = \tilde{E}_{01} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{x_2(-1)}\delta & \frac{-\ln c + x_1(-1)}{x_2^2(-1)}\delta & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} x_2 \\ \frac{x_2^3 x_1(-1)}{\ln c - x_1} \\ -x_4 x_1(-1) \end{bmatrix}.$$

Now set $k = 1$ and go from line 17 to line 2. We have that the row rank of E_1 over $\mathcal{K}(\delta]$ is $r_1 = 2 < r_0$. Choose $Q_2(\mathbf{z}_1, \delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{x_2(-1)}\delta & \frac{\ln c - x_1(-1)}{x_2^2(-1)}\delta & 1 \end{bmatrix}$ to define $F_{12} = 1 + x_2(-1)x_1(-2) - x_4 x_1(-1)$. We can check via Algorithm 1 that $\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}F_{12}\}$ satisfies **(C)** and $\bar{z}_2 = F_{12}$, $z_2 = [\tilde{x}_1, \tilde{x}_2]^T = [x_1, x_2 x_1(-1)]^T$ define a bicausal change of z_1-coordinates. By similar calculations as for $k = 0$, we have that (line 16)

$$E_2 = \begin{bmatrix} 1 & 0 \\ \frac{\tilde{x}_2}{\tilde{x}_1(-1)}\delta & \tilde{x}_1(-1) \end{bmatrix}, \quad F_2 = \begin{bmatrix} \frac{\tilde{x}_2}{\tilde{x}_1(-1)} \\ \frac{\tilde{x}_2^3}{\tilde{x}_1^2(-1)(\ln c - \tilde{x}_1)} \end{bmatrix}.$$

Go from line 16 to line 2 and set $k = 2$, we have that the row rank of E_2 over $\mathcal{K}(\delta]$ is $r_2 = 2 = r_1$, thus Algorithm 2 returns to $k^* = 2$ and $z^* = z_2$. The DDAE Ξ^* : $E_2(\mathbf{z}^*, \delta)\dot{z}^* = F_2(\mathbf{z}^*)$ is clearly index-0 and we can rewrite it as a delayed ODE of the neutral-type:

$$\dot{\tilde{x}}_1 = \frac{\tilde{x}_2}{\tilde{x}_1(-1)}, \quad \dot{\tilde{x}}_2 = \frac{\tilde{x}_2^3}{\tilde{x}_1^3(-1)(\ln c - \tilde{x}_1)} - \frac{\tilde{x}_2}{\tilde{x}_1^2(-1)}\dot{\tilde{x}}_1(-1). \quad (5)$$

Given initial-value conditions $\tilde{x}_1(s) = \xi_{\tilde{x}_1}(s)$, $s \in [-1, 0]$ and $\tilde{x}_2(s) = \xi_{\tilde{x}_2}(s)$, $s \in [-1, 0]$, we can calculate the solution $(\tilde{x}_1(t), \tilde{x}_2(t))$ of (5) with respect to $(\xi_{\tilde{x}_1}, \xi_{\tilde{x}_2})$ by the step method. Hence by Theorem 3 (ii), $\Phi^{-1}(\tilde{x}_1(t), \tilde{x}_2(t), 0, 0)$ is the solution of Ξ with the initial-value conditions $\Phi^{-1}(\xi_{\tilde{x}_1}, \xi_{\tilde{x}_2}, 0, 0)$, where $\Phi = [x_1, x_2 x_1(-1), \bar{z}_2, \bar{z}_1]^T = [x_1, x_2 x_1(-1), 1 + x_2(-1)x_1(-2) - x_4 x_1(-1), x_1(-1) + x_3 x_2(-1) - \ln c]^T$ is a bicausal change of coordinates.

VI. Conclusions and Perspectives

In order to generalize the implicit function theorem to the time-delay case, we have proposed two extra equivalent conditions to the results of bicausal changes of coordinates in [5]. A technical lemma and an iterative algorithm are given to check those equivalent conditions. Moreover, we show that the generalized implicit function theorem can be used for reducing the index and for solving time-delay differential-algebraic equations.

There are some further problems can be investigated based on our results. The example in Remark 2 shows that it is possible to find a weaker condition for the time-delay implicit function theorem. Another problem is to extend Algorithm 2 to the general case when $\mathcal{F}_k$ does not satisfies **(C)**. Moreover, an interesting observation from Example 2, which has already been pointed out in [1], [8], is that even the original DDAE Ξ has a form of retarded type, the resulting delayed ODE can still be of neutral type (or even advanced type in general), the problem of finding when a given DDAE can be reformulated as a delayed ODE of retarded, neutral, or advanced type, is open and challenging. Another topic is to use our results to design reduce-order observers for both the states [6] and the inputs [9] of time-delay systems.

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