# Strong left-invertibility and strong input-observability of nonlinear time-delay systems

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***Abstract*— In this paper, we study the problem of unknown inputs reconstruction for nonlinear time-delay systems. First we define two notions called strong left-invertibility and strong input-observability and the word "strong" is to address the causality properties of those two notions. Then necessary and sufficient conditions for the strong left-invertibility and the strong input-observability are given under the algebraic framework proposed in [1]. We find that a sequence of inputs submodules plays an important role for the strong left-invertibility of time-delay systems. A structure algorithm is provided to construct that sequence and to formulate an input reconstructor. At last, several examples are given to illustrate how to check the strong left-invertibility and the strong input-observability by applying the proposed structure algorithm, and to show how to recover the inputs via causal outputs and the initial value functions of states (strong left-invertibility) or only via causal outputs (strong input-observability).**

***Index Terms*— nonlinear systems, time delays, left-invertibility, input-observability, structure algorithm**

## I. Introduction

We consider a nonlinear time-delay control system, denoted by $\Sigma$, with commensurable delays in states, inputs and outputs, which is given by

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t-i\tau), 0 \leq i \leq \bar{i}) \\ \qquad + \displaystyle\sum_{j=0}^{\bar{j}} g^j(x(t-i\tau), 0 \leq i \leq \bar{i})u(t-j\tau), \\ y(t) = h(x(t-i\tau), 0 \leq i \leq \bar{i}), \\ x(s) = \varphi_x(s), \ s \in [-\bar{i}\tau, 0], \\ u(s) = \varphi_u(s), \ s \in [-\bar{j}\tau, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^m$ are vectors of states, inputs and outputs, respectively; $\tau$ is a constant delay and we assume below that $\tau = 1$ for convenience; the integers $\bar{i}, \bar{j} \geq 0$ are the indices representing the maximal delay of the states and of the inputs, respectively, if $\bar{i} = \bar{j} = 0$, we will call $\Sigma$ a delay-free control system; the vector fields $f : \mathbb{R}^{(\bar{i}+1)n} \to \mathbb{R}^n$, $g_k^j : \mathbb{R}^{(\bar{i}+1)n} \to \mathbb{R}^n$, $1 \leq k \leq m$, and the function $h : \mathbb{R}^{(\bar{i}+1)n} \to \mathbb{R}^m$ are meromorphic in their arguments and we denote $\mathcal{K}$ the field of meromorphic functions; $\varphi_x$ and $\varphi_u$ are the initial value functions of the states and the inputs, respectively. A function $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of $\Sigma$ with the initial value function $\xi_x$ if there exists $T > 0$ such that $x(t)$ is continuously differentiable on $[-\bar{i}, T)$ and satisfies (1) for all $t \in [0, T)$. In this paper, we do not deal with singularities, e.g., when we say that a matrix-value function $g(x(t-i), 0 \leq i \leq \bar{i})$ is of constant rank, we mean that the rank is constant for all $t > 0$.

The invertibility analysis of control systems has drawn attentions of researchers for decades. For a linear delay-free system, its left- (or right-) invertibility [2], [3] is usually defined by that of its transfer function matrix, which is equivalent to the injectivity (or surjectivity) of the input-output map. In some cases, $u(t)$ depends only on the outputs $y(t)$ and their derivatives but does not depend on the initial condition $x_0$ [4] and we will use the terminology "input-observable" for a system being left-invertible without the knowledge of $x_0$. For a nonlinear delay-free system, different approaches were used to characterize its invertibilty, e.g., the nonlinear generalizations of the Silverman's structure algorithm [5], [6]; the geometric methods and the dynamic extension algorithm [7], [8], which are related to input-output decoupling and linearization problems; and also the differential-algebraic methods [9], [10] and differential flatness [11]. For the studies of invertibility of time-delay systems, much less results can be found. The authors of [12] extended the structure algorithm to construct a left-inversion for time-delay systems, normal forms and new outputs constructions were used in [13], [14] for the observations of both states and unknown inputs. Some other topics related to the invertibility of delayed systems are e.g., the input-output linearization [15], [16], zero dynamics [17], observability [18] and accessibility analysis [19], [20].

In the present paper, we study the strong left-invertibility and the strong input-observability of time-delay systems. The word "strong" is to emphasize the causality property which is desired for recovering the inputs $u(t)$. Note that in the state-observability analysis of delayed system [1], [18], the words "weak" and "strong" are also used for the properties of recovering the states by non-causal and causal outputs [13], respectively. Our practical motivation for studying left-invertibility is to recover the unknown inputs (e.g., perturbations, fault, unknown parameters, secret messages) in physical systems e.g. the disturbances or torques estimations of electrical motors

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[21] via the outputs and their time derivatives, the latter can be constructed via the differentiators by sliding modes technique [22], [23].

The paper is organised as follows: We recall the algebraic framework for studying time-delay systems in section II-A. The definitions of strong left-invertibility and strong input-observability are given in section II-B. The structural algorithm and the equivalent conditions for strong left-invertibility and strong input-observability are shown in section III. Some examples are given in section IV to illustrate the results of section III. The conclusions and perspectives end the paper.

## II. Preliminaries and definitions

### A. Algebraic framework

We will use the algebraic setting for nonlinear time-delay systems proposed in [12], [1]. The symbol "d" denotes the standard differential operator. For a function $y(t) = h(x(t), \ldots, x(t-\bar{i}))$, its differential is

$$\mathrm{d}y = h_{x(t)}\mathrm{d}x + \ldots h_{x(t-\bar{i})}\mathrm{d}x(t-\bar{i}) = \sum_{i=0}^{\bar{i}} h_{x(t-i)}\mathrm{d}x(t-i),$$

where $h_{x(t-i)} = \frac{\partial h}{\partial x(t-i)}$, $0 \leq i \leq \bar{i}$. Given two functions $\xi(t), a(t) \in \mathcal{K}$, the backward time-shift operator (or the delay operator, see e.g., [1], [24]) is defined as follows:

$$\delta^i\xi(t) = \xi(t-i) \quad \text{and} \quad \delta^i(a(t)\mathrm{d}\xi(t)) = a(t-i)\mathrm{d}\xi(t-i).$$

Due to the delay operator $\delta$, we can write the differential above as

$$\mathrm{d}y = \left(h_x + h_{\delta x}\delta + \ldots + h_{\delta^{\bar{i}}x}\delta^{\bar{i}}\right)\mathrm{d}x.$$

Let $\mathcal{K}(\delta]$ denotes the ring of non-commutative polynomials of $\delta$ with coefficients in $\mathcal{K}$, then $\mathcal{K}(\delta]$ is a left ore ring [1]. The system (1) can be reformulated into the following form with the help of the delay operator $\delta$:

$$\Sigma : \begin{cases} \dot{x} = f(x,\delta) + G(x,\delta)u, \\ y = h(x,\delta), \\ x(s) = \varphi_x(s), \;\; s \in [-\bar{i}, 0], \\ u(s) = \varphi_u(s), \;\; s \in [-\bar{j}, 0], \end{cases} \tag{2}$$

where $f(x,\delta) = f(\delta^i x, 0 \leq i \leq \bar{i}) \in \mathcal{K}^n$, $h(x,\delta) = h(\delta^i x, 0 \leq i \leq \bar{i}) \in \mathcal{K}^m$ and $g^j(x,\delta) = g^j(\delta^i x, 0 \leq i \leq \bar{i})$, $G(x,\delta) := \sum_{j=0}^{\bar{j}} g^j(x,\delta)\delta^j \in \mathcal{K}(\delta]^{n\times m}$. For $1 \leq l \leq m$, define $L_f h_l(x,\delta) := \sum_{r=0}^{n}\sum_{i=0}^{\bar{i}} \frac{\partial h_l(x,\delta)}{\partial(\delta^i x_r)}\delta^i f_r(x,\delta)$ and $L_G h_l := \sum_{s=0}^{m} L_{G_s}h_l$, where $G_s$ is the s$^{th}$ column of $G$. With the latter notations, we can calculate the derivative $\dot{y}_l$ of the output $y_l$ with respective to $t$ as

$$\dot{y}_l = L_f h_l(x,\delta) + L_G h_l(x,\delta)u.$$

Denote by $u^{(r)}$ the $r$-th order time-derivative of a function $u \in \mathcal{K}$. The sets of one forms $\mathcal{M} := \mathrm{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{(k)}, \forall 1 \leq k \leq \infty\right\}$ generated by the differentials of functions over $\mathcal{K}(\delta]$ are called modules. The rank of any module over $\mathcal{K}(\delta]$ is well defined, see [1] for the definitions of multiplications and additions over $K(\delta]$. The right-annihilator of a submodule $\mathcal{N} = \mathrm{span}_{\mathcal{K}(\delta]}\left\{\nu_1(\cdot,\delta), \ldots, \nu_m(\cdot,\delta)\right\} \subseteq M$ is spanned by all vectors $\beta(\cdot,\delta)$ with coefficients in $\mathcal{K}(\delta]$ such that $\nu_i(\cdot,\delta)\beta(\cdot,\delta) = 0$ for $1 \leq i \leq p$. The closure of the submodules of $\mathcal{M}$ and the unimodular matrices over the ring $\mathcal{K}(\delta]$ recalled below will play important roles for the strong left-invertibility analysis of time-delay systems.

**Definition 1** ([1]). Given a finite generated module $\mathcal{M}$, let $\mathcal{N}$ be a submodule of $\mathcal{M}$ of rank $r$ over $\mathcal{K}(\delta]$, the closure of $\mathcal{N}$ is the submodule

$$\overline{\mathcal{N}} := \{\omega \in \mathcal{M} \mid \exists\, 0 \neq a(\cdot,\delta) \in \mathcal{K}(\delta],\; a(\cdot,\delta)\omega \in \mathcal{N}\},$$

or equivalently, $\overline{\mathcal{N}}$ is the largest submodule of $\mathcal{M}$ which contains $\mathcal{N}$ and is of rank $r$. The submodule $\mathcal{N}$ is called *closed* if $\mathcal{N} = \overline{\mathcal{N}}$.

**Definition 2** ([1], [24]). A matrix $A(\cdot,\delta) \in \mathcal{K}^{r\times r}(\delta]$ is called *unimodular* if there exists a matrix $B(\cdot,\delta) \in \mathcal{K}^{r\times r}(\delta]$ such that $A(\cdot,\delta)B(\cdot,\delta) = B(\cdot,\delta)A(\cdot,\delta) = I_r$, where $I_r$ denotes the identity matrix of $\mathbb{R}^{r\times r}$.

The following lemma is crucial for our main results.

**Lemma 3** ([24]). *Consider a full row rank matrix $R(\cdot,\delta) \in \mathcal{K}^{r\times m}(\delta]$ over $\mathcal{K}(\delta]$ and the submodule $\mathcal{R} = \mathrm{span}_{\mathcal{K}(\delta]}\{R_i(\cdot,\delta), 1 \leq i \leq r\}$ generated by the rows $R_i$ of $R$. There exist two unimodular matrices $P(\cdot,\delta) \in \mathcal{K}^{r\times r}(\delta]$ and $Q(\cdot,\delta) \in \mathcal{K}^{m\times m}(\delta]$ such that $P(\cdot,\delta)R(\cdot,\delta)Q(\cdot,\delta) = [I_r\ 0]$ if and only if the submodule $\mathcal{R}$ is closed and its right-annihilator is causal.*

### B. Strong left-invertibility and strong input-observability of delayed systems

A delay-free system is called left-invertible (cf. [5]) if two outputs $y(t, x_0, u(t)) = \tilde{y}(t, x_0, \tilde{u}(t))$ for $t \in [0, T)$ and a small enough $T > 0$ implies that $u(t) = \tilde{u}(t)$ for all $t \in [0, T)$, providing initial conditions $x_0$; if a system is left-invertible without knowing the initial conditions, we call it input-observable. Now we generalize the notions of left-invertibility and input-observability for systems with delays.

**Definition 4** (strong left-invertibility and strong input-observability). The system $\Sigma$, given by (1), is called strongly left-invertible if there exist integers $\bar{i}_y, \bar{i}_x \geq 0$ such that $y(t, \varphi_x, \varphi_u, u) = \tilde{y}(t, \varphi_x, \varphi_{\tilde{u}}, \tilde{u})$ for all $t \in [-\bar{i}_y, T)$ for a small enough $T$, implies that $u(t) = \tilde{u}(t)$ for all $t \in [0, T)$ providing the initial value functions of the states $x$, i.e., $\varphi_x(t)$, $t \in [-\bar{i}_x, 0]$. The system $\Sigma$ is called strongly input-observable if $\Sigma$ is strongly left-invertible without the knowledge of $\varphi_x$.

**Remark 5.** (i) In the above definition, the indices $\bar{i}_y, \bar{i}_x$ indicate the maximal delays on $y$- and $x$- variables (and their derivatives), respectively, to obtain an injective input-output map or to recover the input $u(t)$, $t \in [0, T)$, from the outputs and their time derivatives and delays, i.e., $y^{(k)}(t-i)$, $k \geq 0$, $0 \leq i \leq \bar{i}_y$, and the states with their delays $x(t-i)$, $0 \leq i \leq \bar{i}_x$. If the indices $\bar{i}_y = \bar{i}_x = 0$, then the above definitions reduce to those in the delay-free cases. Note that $\bar{i}_x$ is not necessarily equal to $\bar{i}$ of (1) because by differentiating the outputs, larger delays of $x$ can be present.

(ii) The two notions in Definition 4 do not require the knowledge of the inputs initial value functions $\varphi_u$ to recover the inputs $u(t)$ for $t \in [0,T)$, the difference of strong left-invertibility and strong input-observability only comes from their requirements for the states initial value functions $\varphi_x$.

(iii) A linear time-delay system, $\dot{x} = A(\delta)x + B(\delta)u$, $y = C(\delta)x$ is strongly left-invertible if and only if there exists a matrix $T^{-1}(s,\delta) \in \mathbb{R}(s)[\delta]$ such that $T^{-1}(s,\delta)T(s,\delta) = I$, where $T(s,\delta) = C(\delta)(sI - A(\delta))^{-1}B(\delta)$ is the transfer functions matrix and $\mathbb{R}(s)[\delta]$ is the ring of rational fractions of $s$ and polynomials of $\delta$ with coefficients in $\mathbb{R}$.

We give some simple examples to illustrate the two notions in Definition 4.

$$\Sigma_1 : \begin{cases} \dot{x}_1(t) = x_2(t) + u(t-1), \\ \dot{x}_2(t) = u(t), \\ y(t) = x_1(t), \end{cases} \quad \Sigma_2 : \begin{cases} \dot{x}_1(t) = x_2(t-1)u(t), \\ \dot{x}_2(t) = u(t-1), \\ y(t) = x_1(t), \end{cases}$$
$$\Sigma_3 : \begin{cases} \dot{x}_1(t) = x_1(t-1) + x_2(t), \\ \dot{x}_2(t) = x_1(t-1)u(t), \\ y(t) = x_1(t), \end{cases} \tag{3}$$

The system $\Sigma_1$ above is not strongly left-invertible because we can not solve $u(t)$ for $t \in [0,T)$ based on the knowledge of $y(t)$ for $t \in [0,T)$ unless the inputs initial value functions $u(s) = \varphi_u(s)$ for $s \in [-1,0]$ is known. The system $\Sigma_2$ is strongly left-invertible because we can recover $u(t)$ by $u = \frac{\dot{y}}{\delta x_2}$ and the delayed ODE $\dot{x}_2 = \frac{\delta \dot{y}}{\delta^2 x_2}$ providing $y(t)$, $t \in [-1,T)$ and $x_2(s) = \varphi_{x_2}(s)$, $s \in [-2,0]$ (thus $\bar{i}_x = 2$, $\bar{i}_y = 1$ and $x_2(t-2) = 0$ is a singularity), but $\Sigma_2$ is not strongly input-observable because we can not express $\delta x_2$ (and thus $u$) by the outputs with their derivatives and delays. The system $\Sigma_3$ is strongly input-observable since $u = \frac{\ddot{y} - \delta \dot{y}}{\delta y}$ with $\bar{i}_y = 1$.

## III. Main results

For a vector $x \in \mathbb{R}^n$, we write $\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}x\} = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$. Denote $\mathcal{X} = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}x\}$, $\mathcal{U} = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}u\}$ and $\mathcal{Y} = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}y\}$. Let $\tilde{\mathcal{Y}}$ (resp., $\tilde{\mathcal{U}}$) denote the submodule consisting of $\mathcal{Y}$ (resp., of $\mathcal{U}$) and the finite order time derivatives of the components of $\mathcal{Y}$ (resp., of $\mathcal{U}$).

Now we define successively the following sequence of indices $\rho^k := (\rho_1^k, \ldots, \rho_m^k)$ and that of submodules $\mathcal{U}_k \subseteq \mathcal{U}$ for $k \geq 1$. Set $\rho^1 := (\rho_1^1, \ldots, \rho_m^1)$, where $\rho_l^1$, for each $1 \leq l \leq m$, is the integer such that $\mathrm{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}y_l, \ldots, \mathrm{d}y_l^{(\rho_l^1 - 1)}\right\} \subseteq \mathcal{X}$ and $\mathrm{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}y_l, \ldots, \mathrm{d}y_l^{(\rho_l^1)}\right\} \not\subseteq \mathcal{X}$, and define the submodule

$$\mathcal{U}_1 := \mathcal{U} \cap \left(\mathrm{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}y_1^{(\rho_1^1)}, \ldots, \mathrm{d}y_m^{(\rho_m^1)}\right\} + \mathcal{X}\right). \tag{4}$$

For $k > 1$, assume that $\rho^{k-1}$ and $\tilde{\mathcal{U}}_{k-1}$ have already been defined, let $\rho^k := (\rho_1^k, \ldots, \rho_m^k)$, where $\rho_l^k \geq \rho_l^{k-1}$ is the integer such that

$$\mathrm{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}y_l, \ldots, \mathrm{d}y_l^{(\rho_l^k - 1)}\right\} \subseteq \mathcal{X} + \tilde{\mathcal{U}}_{k-1}$$
$$\mathrm{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}y_l, \ldots, \mathrm{d}y_l^{(\rho_l^k)}\right\} \not\subseteq \mathcal{X} + \tilde{\mathcal{U}}_{k-1},$$

where $\tilde{\mathcal{U}}_{k-1}$ consists of $\mathcal{U}_{k-1}$ and the time derivatives of the components of $\mathcal{U}_{k-1}$. Define the submodule

$$\mathcal{U}_k := \mathcal{U} \cap \left(\mathrm{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}y_1^{(\rho_1^k)}, \ldots, \mathrm{d}y_m^{(\rho_m^k)}\right\} + \mathcal{X} + \tilde{\mathcal{U}}_{k-1}\right).$$

The following structure algorithm provides a practical construction of the sequence of submodules $\mathcal{U}_1, \ldots, \mathcal{U}_k$ under the assumption that each submodule $\mathcal{U}_k$ is closed and its right-annihilator is causal. Note that the latter assumption is also a necessary condition for the strong left-invertibility of $\Sigma$ as shown in Theorem 9(iii) below.

**Algorithm 6** (structure algorithm)**.** ***Step 1:*** *Set $\bar{u}_0 := u$ and $m_0 = m$. Define $\rho^1 = (\rho_1^1, \ldots, \rho_m^1)$, where $\rho_l^1$, $1 \leq l \leq m$, is the smallest integer such that $y_l^{(\rho_l^1)}$ depends explicitly on $u = \bar{u}_0$, i.e., $L_G L_f^k h_l = 0$, for $0 \leq k \leq \rho_l^1 - 2$ and $L_G L_f^{\rho_l^1 - 1} h_l \neq 0$. Then, we have*

$$y^{(\rho^1)} = \begin{bmatrix} y_1^{(\rho_1^1)} \\ \vdots \\ y_m^{(\rho_m^1)} \end{bmatrix} = a^1(x,\delta) + b^1(x,\delta)\bar{u}_0. \tag{5}$$

*Denote the rank of $b^1(x,\delta)$ over $\mathcal{K}(\delta]$ by a constant $r_1$. The submodule $\mathcal{U}_1$ defined in (4) is given by $\mathcal{U}_1 = \mathrm{span}_{\mathcal{K}(\delta]}\left\{b_1^1 \mathrm{d}u, \ldots, b_m^1 \mathrm{d}u\right\}$. Assume that $\mathcal{U}_1$ is closed and its right-annihilator is causal. Then by Lemma 3, there always exist two unimodular matrices $P^1(x,\delta) \in \mathcal{K}^{m\times m}(\delta]$ and $Q^1(x,\delta) \in \mathcal{K}^{m_0 \times m_0}(\delta]$ such that*

$$P^1(x,\delta)b^1(x,\delta)Q^1(x,\delta) = \begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix}.$$

*Define new inputs $\tilde{u}_1 \in \mathbb{R}^{r_1}$ and $\bar{u}_1 \in \mathbb{R}^{m_1}$, where $m_1 := m_0 - r_1$, by*

$$\begin{bmatrix} \tilde{u}_1 \\ \bar{u}_1 \end{bmatrix} = (Q^1(x,\delta))^{-1}\bar{u}_0. \tag{6}$$

*By (5) and (6), the inputs $\tilde{u}_1$ can be expressed as*

$$\tilde{u}_1 = P_1^1(x,\delta)(y^{(\rho^1)} - a^1(x,\delta)) = \psi_1(x, y^{(\rho^1)}, \delta),$$

*where $P_1^1$ consists of the first $r_1$ rows of $P^1$. The submodule $\mathcal{U}$ can be expressed as $\mathcal{U}_1 = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\tilde{u}_1\}$. Moreover, we define the following system*

$$\begin{cases} \dot{x} = f(x,\delta) + \hat{G}(x,\delta)\begin{bmatrix} \psi_1(y^{(\rho^1)}, x, \delta) \\ \bar{u}_1 \end{bmatrix} \\ \quad = f_1(x, y^{(\rho^1)}, \delta) + G_1(x,\delta)\bar{u}_1, \\ y^{(\rho^1)} = a^1(x,\delta) + \hat{b}^1(x,\delta)\psi_1(y^{(\rho^1)}, x, \delta) \\ \quad = \xi_1(x, y^{(\rho^1)}, \delta). \end{cases}$$

*where $\hat{G} = GQ^1$ and $[\hat{b}^1, 0] = b^1 Q^1$. Let $\tilde{Y}_1$ denotes $y$ and its time derivatives of order at most $\rho^1$, we can write $f_1 = f_1(x, \tilde{Y}_1, \delta)$, $G_1 = G_1(x, \tilde{Y}_1, \delta)$ and $\xi_1 = \xi_1(x, \tilde{Y}_1, \delta)$.*

***Step** $k$ ($k > 1$)**:** Suppose in Step $k-1$ that the indices $\rho^{k-1} = (\rho_1^{k-1}, \ldots, \rho_m^{k-1})$, the closed submodule $\mathcal{U}_{k-1} = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\tilde{u}_1, \ldots, \mathrm{d}\tilde{u}_{k-1}\}$ with causal right-annihilator and the following system have already been constructed:*

$$\begin{cases} \dot{x} = f_{k-1}(x, \tilde{Y}_{k-1}, \delta) + G_{k-1}(x, \tilde{Y}_{k-1}, \delta)\bar{u}_{k-1}, \\ y^{(\rho^{k-1})} = \xi_{k-1}(x, \tilde{Y}_{k-1}, \delta), \end{cases}$$

*where $\bar{u}_{k-1} \in \mathbb{R}^{m_{k-1}}$ and $\tilde{Y}_{k-1}$ consists of $y$ and its time derivatives of order at most $\rho^{k-1}$. Let $\rho_l^k$ ($\rho_l^k \geq \rho_l^{k-1}$) be the smallest integer such that $y_l^{(\rho_l^k)}$ explicitly depends on $\bar{u}_{k-1}$, denote $\rho^k = (\rho_1^k, \ldots, \rho_m^k)$. Then*

$$y^{(\rho^k)} = \begin{bmatrix} y_1^{(\rho_1^k)} \\ \vdots \\ y_m^{(\rho_m^k)} \end{bmatrix} = a^k(x, \tilde{Y}_k, \delta) + b^k(x, \tilde{Y}_k, \delta)\bar{u}_{k-1},$$

*where $\tilde{Y}_k$ consists of $y$ and its time derivatives with order at most $\rho^k$. Denote the rank of $b^k(x, \tilde{Y}_k, \delta)$ over $\mathcal{K}(\delta]$ by a constant $r_k$. The submodule $\mathcal{U}_k = \mathrm{span}_{\mathcal{K}(\delta]}\left\{b_1^k \mathrm{d}\bar{u}_{k-1}, \ldots, b_m^k \mathrm{d}\bar{u}_{k-1}\right\} + \mathcal{U}_{k-1}$. Assume that $\mathcal{U}_k$ is closed and its right-annihilator is causal. Then there always exist two unimodular matrices $P^k(\cdot, \delta) \in \mathcal{K}^{m\times m}(\delta]$ and $Q^k(\cdot, \delta) \in \mathcal{K}^{m_{k-1}\times m_{k-1}}(\delta]$ such that*

$$P^k(x, \tilde{Y}_k, \delta) b^k(x, \tilde{Y}_k, \delta) Q^k(x, \tilde{Y}_k, \delta) = \begin{bmatrix} I_{r_k} & 0 \\ 0 & 0 \end{bmatrix}.$$

*Define the new inputs $\tilde{u}_k \in \mathbb{R}^{r_k}$ and $\bar{u}_k \in \mathbb{R}^{m_k}$, where $m_k := m_{k-1} - r_k$, by*

$$\begin{bmatrix} \tilde{u}_k \\ \bar{u}_k \end{bmatrix} = (Q^k(x, \tilde{Y}_k, \delta))^{-1}\bar{u}_{k-1}. \tag{7}$$

*Thus $\tilde{u}_k$ can be expressed by*

$$\tilde{u}_k = P_1^k(x, \tilde{Y}_k, \delta)(y^{(\rho^k)} - a^k(x, \tilde{Y}_k, \delta)) = \psi_k(x, \tilde{Y}_k, \delta), \tag{8}$$

*where $P_1^k$ consists of the first $r_k$ rows of $P^k$. The submodule $\mathcal{U}_k$ can be rewrote as $\mathcal{U}_k = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\tilde{u}_k, \mathrm{d}\tilde{u}_{k-1}, \ldots, \mathrm{d}\tilde{u}_1\}$. Then we get the following system:*

$$\begin{cases} \dot{x} = f_{k-1}(\cdot, \delta) + \hat{G}_{k-1}(\cdot, \delta)\begin{bmatrix} \psi_k(x, \tilde{Y}_k, \delta) \\ \bar{u}_k \end{bmatrix} \\ \quad = f_k(x, \tilde{Y}_k, \delta) + G_k(x, \tilde{Y}_k, \delta)\bar{u}_k, \\ y^{(\rho^k)} = a^k(x, \tilde{Y}_k, \delta) + \hat{b}^k(x, \tilde{Y}_k, \delta)\psi_k(x, \tilde{Y}_k, \delta) \\ \quad = \xi_k(x, \tilde{Y}_k, \delta). \end{cases}$$

**Remark 7.** (i) The above defined sequence of submodules $\mathcal{U}_k$ is clearly non-decreasing, i.e., $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \cdots \subseteq \mathcal{U}_m \subseteq \mathcal{U}_{m+1} \subseteq \cdots$. It follows that

$$0 \leq \dim \mathcal{U}_1 \leq \ldots \leq \dim \mathcal{U}_m \leq \cdots \leq \dim \mathcal{U} = m.$$

By definition, the dimension of $\mathcal{U}_k$ stabilizes once stop increasing. It follows that there exists a smallest integer $1 \leq \tilde{k}^* \leq m$ such that $\dim \mathcal{U}_{\tilde{k}^*} = \dim \mathcal{U}_{\tilde{k}^*+j}$, $j \geq 1$. Recall that $\mathcal{U}_k$ are submodules of $\mathcal{M}$ over $\mathcal{K}(\delta]$, thus by Theorems 1 and A.2 of [1], the sequence $\mathcal{U}_k$ stabilizes in a finite step $k^* \geq 1$. It is worth to note that, in general, $k^*$ is *not* necessarily equal to $\tilde{k}^*$ and may be larger that $m$. Take the system $\Sigma_3$ above for example, we have $\mathcal{U}_1 = \mathrm{span}_{\mathcal{K}(\delta]}\{\delta \mathrm{d}u\}$ and $\mathcal{U}_3 = \mathcal{U}_2 = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}u\}$, so $\tilde{k}^* = 1 = m$ and $k^* = 2 > m$.

(ii) The multi-indices $\rho^1 \leq \rho^2 \leq \cdots \leq \rho^{k^*}$ are called the *structure at infinity* for delay-free systems [7], [10]. Note that the integer $\rho_l^k$ may become infinity at certain Step $k \geq 1$, which means that by differentiating $y_i^{(\rho_l^{k-1})}$ ($\rho_l^0 = 0$), it is not possible to reach any new input from $\bar{u}_{k-1}$. The integer $r_k$ is the number of new inputs reached at each step $k$, we have $\dim \mathcal{U}_k = r_1 + \cdots + r_k$.

(iii) Given $\mathcal{U}_1 = \mathrm{span}_{\mathcal{K}(\delta]}\left\{b^1(\cdot, \delta)\mathrm{d}\bar{u}\right\}$, we can always find a unimodular matrix $P^1(\cdot, \delta)$ such that $P^1(\cdot, \delta)b^1(\cdot, \delta) = \begin{bmatrix} \bar{b}^1(\cdot, \delta) \\ 0 \end{bmatrix}$ and $\bar{b}^1(\cdot, \delta)$ is of full row rank over $\mathcal{K}(\delta]$. However, if the right-annihilator of $\mathcal{U}_1$ is not causal, then there does not exist a (causal) unimodular matrix $Q^1(\cdot, \delta)$ such that $\bar{b}^1(\cdot, \delta)Q^1(\cdot, \delta) = [\hat{b}^1(\cdot, \delta), 0]$. Suppose that such $Q^1$ exists, then the matrix $\hat{b}^1(\cdot, \delta)$ is unimodular if and only if $\mathcal{U}_1$ is closed. We can do similar analysis for the other $\mathcal{U}_k$.

Algorithm 6 is inspired by the dynamic extension algorithm [7], [8] and the classical structure algorithm [10] of delay-free systems but the technical details are very different and are adjusted to deal with the causality problem. A major difference is that in the delay-free case, once an output $y_l$ reaches some new inputs, we stop differentiating $y_l$, then by a row permutation of $y$, we keep differentiating the rest outputs. While in each Step $k$ of Algorithm 6, even under the assumption that $\mathcal{U}_k$ is closed and its right-annihilator is causal, we can *not* always get a sub-matrix $b_1^k$ from $b^k$ such that $\mathrm{span}_{\mathcal{K}(\delta]}\left\{b^k \mathrm{d}\bar{u}_{k-1}\right\} = \mathrm{span}_{\mathcal{K}(\delta]}\left\{b_1^k \mathrm{d}\bar{u}_{k-1}\right\}$ by just row permutations of $y$, thus we have to keep differentiating all outputs $y$. Below we compare our structure algorithm with that in [12] and the new outputs construction procedure in [13] for time-delay systems.

**Remark 8.** Roughly speaking, both the method in [12] and that in [13] try to construct some dummy outputs to avoid differentiating directly the inputs $u$ once some of the inputs appear when differentiating the outputs $y$. The dummy outputs in [12] depend on the whole states $x$, which can change completely the structure at infinity $(\rho^1, \ldots, \rho^{k^*})$ of the system. The structure algorithm in [12] actually follows similar lines as the zero dynamics algorithm [6], [7] of the delayed-free case, which was designed to find the inputs $u$ which render the outputs to be zero, but not for recovering $u(t)$ from knowing $y(t)$. The new outputs in [13] depend only on the original outputs and their derivatives, which does not require extra information on the states but the conditions to find those new outputs can not be easily satisfied.

Now we are ready to give the main result of the paper.

**Theorem 9.** *For a time-delay system $\Sigma$, given by (1), the following statements are equivalent:*

(i) *$\Sigma$ is strongly left-invertible.*
(ii) *$\mathcal{U} \subseteq \mathcal{X} + \tilde{\mathcal{Y}}$.*
(iii) *$\mathcal{U}_{k^*} = \mathcal{U}$ and $\mathcal{U}_k$ are closed in $\mathcal{M}$ and its right-annihilators are causal for all $1 \leq k \leq k^*$, where $k^* \leq m$ is the smallest integer such that $\mathcal{U}_{k^*+j} = \mathcal{U}_{k^*}$ for all $j \geq 1$.*
(iv) *There exists an input reconstructor of the form (9) such that $\hat{u}(t) - u(t) = 0$ for all $t \in [0, T)$, where*

$$\begin{cases} \dot{z} = F(z, \tilde{Y}_{k^*}, \delta), \quad z(s) = \varphi_z(s), \quad s \in [-\bar{i}_z, 0], \\ \hat{u} = H(z, \tilde{Y}_{k^*}, \delta), \end{cases} \tag{9}$$

*where $\tilde{Y}_{k^*}$ consists of $y$ and its time derivatives of order at most $\rho^{k^*}$ and $z$ depends on $x$.*

*Proof.* $(i) \Leftrightarrow (ii)$: By Definition 4, the system $\Sigma$ is strongly left-invertible if and only if there exist integers $\bar{i}_y, \bar{i}_x > 0$

and an injective map $H:(y^{(k)}(t-i_y),x(t-i_x))\mapsto u(t), 0\le k\le\rho^{(k^*)}, 0\le i_y\le\bar{i}_y, 0\le i_x\le\bar{i}_x$, so $u=H(\dot{y},\ldots,y^{(\rho^{k^*})},x,\delta)$, which means that $\mathcal{U}\subseteq\tilde{\mathcal{Y}}+\mathcal{X}$.

$(ii)\Rightarrow(iii)$ Assume $\mathcal{U}\subseteq\tilde{\mathcal{Y}}+\mathcal{X}$, i.e., $u=H(x,\dot{y},\ldots,y^{(\rho^{k^*})},\delta)$ for some function $H$. By denoting $\tilde{\xi}=(y^{(\rho^1)},x)$ and $\bar{\xi}=(\dot{y},\ldots,y^{(\rho^1-1)},y^{(\rho^1+1)},\ldots,y^{(\rho^{k^*})})$, we can rewrite $H=H(\tilde{\xi},\bar{\xi},\delta)$. Notice that there always exists [24] a unimodular matrix $\tilde{Q}^1$ such that $\tilde{Q}^1H_{\bar{\xi}}=\left[\begin{smallmatrix}0\\ \hat{H}\end{smallmatrix}\right]$ with $\hat{H}$ being of full row rank over $\mathcal{K}(\delta]$. Then by defining new inputs $\left[\begin{smallmatrix}\tilde{u}_1\\ \bar{u}_1\end{smallmatrix}\right]=\tilde{Q}^1\bar{u}_0$, we have $\tilde{u}_1=\tilde{H}(\tilde{\xi},\delta)=\tilde{H}(x,y^{(\rho^1)},\delta)$ and $\bar{u}_1=\bar{H}(\tilde{\xi},\bar{\xi},\delta)$ with $\bar{H}_{\bar{\xi}}$ being of full row rank, for some functions $\tilde{H}$ and $\bar{H}$. Consider the differential of (5), i.e.,

$$\mathrm{d}y^{(\rho^1)}=(a_x^1+\tilde{b}_x^1\tilde{u}_1+\bar{b}_x^1\bar{u}_1)\mathrm{d}x+\tilde{b}^1\mathrm{d}\tilde{u}_1+\bar{b}^1\mathrm{d}\bar{u}_1,$$

where $[\tilde{b}^1,\bar{b}^1]=b^1(\tilde{Q}^1)^{-1}$. It follows that $\bar{b}^1\equiv 0$ since if $\bar{b}^1\not\equiv 0$, then the right-hand side of the above differential depends on $\mathrm{d}\bar{\xi}$ (because $\mathrm{d}\bar{u}_1=\mathrm{d}\bar{H}$ depends on $\mathrm{d}\bar{\xi}$ and $\bar{H}_{\bar{\xi}}$ is of full row rank). Moreover, by comparing the above differential with

$$\mathrm{d}\tilde{u}_1=\tilde{H}_x\mathrm{d}x+\tilde{H}_{y^{(\rho_1)}}\mathrm{d}y^{(\rho_1)},$$

we have $\tilde{H}_{y^{(\rho_1)}}\tilde{b}^1=I$, which implies that $\mathcal{U}_1=\mathrm{span}_{\mathcal{K}(\delta]}\left\{b^1\mathrm{d}u\right\}=\mathrm{span}_{\mathcal{K}(\delta]}\left\{\tilde{b}^1\mathrm{d}\tilde{u}_1\right\}$ is closed and its right-annihilator is causal. Now we use an induction method, suppose that for a certain $\hat{k}>1$ that $\mathcal{U}_1,\ldots,\mathcal{U}_{\hat{k}-1}$ are closed and their right-annihilators are causal, then we perform Algorithm 6 until Step $\hat{k}-1$. It can be deduced from $u=H(\dot{y},\ldots,y^{(\rho^{k^*})},x,\delta)$ and (7) for $k=1,2,\ldots,\hat{k}-1$ that $\tilde{u}_{\hat{k}-1}=L(\dot{y},\ldots,y^{(\rho^{k^*})},x,\delta)$ for some function $L$. Denote $\tilde{\eta}=(\dot{y},\ldots,y^{(\rho^{\hat{k}})},x)$ and $\bar{\eta}=(y^{(\rho^{\hat{k}}+1)},\ldots,y^{(\rho^{k^*})})$ to have $L=L(\tilde{\eta},\bar{\eta},\delta)$. Let $\tilde{Q}^{\hat{k}}$ be a unimodular matrix such that $\tilde{Q}^{\hat{k}}L_{\bar{\eta}}=\left[\begin{smallmatrix}0\\ \hat{L}\end{smallmatrix}\right]$, where $\hat{L}$ is of full row rank over $\mathcal{K}(\delta]$. Consider the matrix $b^{\hat{k}}$ in Step $\hat{k}$, define $[\tilde{b}^{\hat{k}},\bar{b}^{\hat{k}}]:=b^{\hat{k}}(\tilde{Q}^{\hat{k}})^{-1}$. By a similar argument above, it can be deduced that $\bar{b}^{\hat{k}}\equiv 0$ and $\tilde{L}_{y^{(\rho^{\hat{k}})}}\tilde{b}^{\hat{k}}=I$, which proves that $\mathcal{U}_{\hat{k}}=\mathcal{U}_{\hat{k}-1}+\mathrm{span}_{\mathcal{K}(\delta]}\left\{b^{\hat{k}}\mathrm{d}\bar{u}_{\hat{k}-1}\right\}$ is closed and its right-annihilator is causal, thus the latter condition holds for all $\mathcal{U}_k$, $k\ge 1$. Moreover, by Remark 7 (i) and that $\mathcal{U}_k$ are closed, there exists a smallest integer $k^*\le m$ such that $\dim\mathcal{U}_{k^*}=\dim\mathcal{U}_{k^*+j}$ and $\mathcal{U}_{k^*}=\mathcal{U}_{k^*+j}$ for $j\ge 1$. Then we suppose that $\mathcal{U}_{k^*+j}\neq\mathcal{U}$ for $j\ge 0$, it means that by differentiating $y$, it is not possible to reach all inputs $u$, i.e., $\mathcal{U}\nsubseteq\mathcal{X}+\tilde{\mathcal{Y}}$, which is a contradiction, hence we have $\mathcal{U}=\mathcal{U}_{k^*}$.

$(iii)\Rightarrow(iv)$: Under the conditions of $(iii)$, we perform Algorithm 6 until Step $k^*\le m$. It follows by $\mathcal{U}_{k^*}=\mathcal{U}$ that $\dim\bar{u}_{k^*}=0$ and $\dim\mathcal{U}_{k^*}=\dim\tilde{u}_1+\cdots+\dim\tilde{u}_{k^*}=r_1+\cdots+r_{k^*}=m$. Thus by the formulae (7) and (8), we have $\bar{u}_{k^*-1}=Q^{k^*}\psi_{k^*}$, $\bar{u}_{k^*-2}=Q^{k^*-1}\left[\begin{smallmatrix}\psi_{k^*-1}\\ \bar{u}_{k^*-1}\end{smallmatrix}\right]$, …, $u=\bar{u}_0=Q^1\left[\begin{smallmatrix}\psi_1\\ \bar{u}_1\end{smallmatrix}\right]$. So $u=H(x,\tilde{Y}_{k^*},\delta)$ for some map $H$. A full-order, i.e., $\dim z=\dim x$, input reconstructor of the form (9) can always be constructed by setting $z=x$ and $F(z,\tilde{Y}_{k^*},\delta)=f(z,\delta)+G(z,\delta)H(z,\tilde{Y}_{k^*},\delta)$.

$(iv)\Rightarrow(ii)$ is clear and thus its proof is omitted. $\square$

Now we use two observability submodules $\mathcal{X}\cap\tilde{\mathcal{Y}}$ and $\mathcal{X}\cap(\tilde{\mathcal{U}}+\tilde{\mathcal{Y}})$ (see e.g., [10] for delay-free systems and [1], [18] for time-delay systems) to characterize the strong input-observability.

**Corollary 10.** *A time-delay system $\Sigma$ is strongly input-observable if and only if $\Sigma$ is strongly left-invertible and*

$$\mathcal{X}\cap\tilde{\mathcal{Y}}=\mathcal{X}\cap(\tilde{\mathcal{U}}+\tilde{\mathcal{Y}}).$$

*Proof.* By Definition 4 and Theorem 9, we need to prove

$$\left(\mathcal{U}\subseteq\mathcal{X}+\tilde{\mathcal{Y}}\text{ and }\mathcal{X}\cap\tilde{\mathcal{Y}}=\mathcal{X}\cap(\tilde{\mathcal{U}}+\tilde{\mathcal{Y}})\right)\Leftrightarrow\mathcal{U}\subseteq\tilde{\mathcal{Y}}.$$

"$\Leftarrow$" is clear because $\mathcal{U}\subseteq\tilde{\mathcal{Y}}$ implies $\mathcal{U}\subseteq\mathcal{X}+\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{U}}\subseteq\tilde{\mathcal{Y}}$. Now we prove "$\Rightarrow$". Suppose that $\mathcal{U}\subseteq\mathcal{X}+\tilde{\mathcal{Y}}$, then we have $u=H(x,\dot{y},\ldots,y^k,\delta)$ for some function $H$ and integer $k\ge 0$. Then assume $\mathcal{X}\cap\tilde{\mathcal{Y}}=\mathcal{X}\cap(\tilde{\mathcal{U}}+\tilde{\mathcal{Y}})$ to get $H_x(\cdot,\delta)\mathrm{d}x=\mathrm{d}u-H_{\dot{y}}(\cdot,\delta)\mathrm{d}\dot{y}+\cdots+H_{y^{(k)}}(\cdot,\delta)\mathrm{d}y^{(k)}\subseteq\mathcal{X}\cap(\tilde{\mathcal{U}}+\tilde{\mathcal{Y}})=\mathcal{X}\cap\tilde{\mathcal{Y}}$. It follows that $H_x(\cdot,\delta)\mathrm{d}x=\sum_{i=0}^{k}H_i(\cdot,\delta)\mathrm{d}y^{(i)}$ for some functions $H_i$, which is only possible when $H$ can be expressed as $H=H(y,\ldots,y^{(k)},\delta)$. Hence $\mathcal{U}\subseteq\tilde{\mathcal{Y}}$. $\square$

## IV. Illustration Examples

**Example 11.** Consider the following time-delay system

$$\Sigma:\quad\begin{cases}\dot{x}_1=-\delta x_1+(\delta x_4)u_1, & \dot{x}_2=-\delta x_3+x_4,\\ \dot{x}_3=x_2-(\delta^2x_4)\delta u_1, & \dot{x}_4=u_2.\end{cases}$$

Let the outputs $y_1=x_1+\delta x_4$ and $y_2=x_3+x_2-\delta^3x_4$. We have $\rho_1^1=\rho_2^1=1$ and

$$\begin{bmatrix}y_1^{(\rho_1^1)}\\ y_2^{(\rho_2^1)}\end{bmatrix}=\begin{bmatrix}-\delta x_1\\ \delta x_2-\delta x_3+x_4\end{bmatrix}+\begin{bmatrix}\delta x_4 & \delta\\ -(\delta^3x_4)\delta^2 & -\delta^3\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}.$$

The submodule $\mathcal{U}_1$ is closed and its right-annihilator is causal because there exist two unimodular matrices $P^1=\left[\begin{smallmatrix}\frac{1}{\delta x_4} & 0\\ \delta^2 & 1\end{smallmatrix}\right]$ and $Q^1=\left[\begin{smallmatrix}1 & -\frac{1}{\delta x_4}\delta\\ 0 & 1\end{smallmatrix}\right]$ such that $P^1b^1Q^1=\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$. Define new inputs $\left[\begin{smallmatrix}\tilde{u}_1\\ \bar{u}_2\end{smallmatrix}\right]=(Q^1)^{-1}u=\left[\begin{smallmatrix}u_1+\frac{1}{\delta x_4}\delta u_2\\ u_2\end{smallmatrix}\right]$. We have $\tilde{u}_1=\frac{\dot{y}_1+\delta y_1}{\delta x_4}$. It follows that $\left[\begin{smallmatrix}y_1^{(1)}\\ y_2^{(1)}\end{smallmatrix}\right]=\left[\begin{smallmatrix}y_1^{(1)}\\ \delta x_2-\delta x_3+x_4-\delta^2\dot{y}_1-\delta^3y_1\end{smallmatrix}\right]$. Then we have $\rho_1^2=\infty$, $\rho_2^2=2$ and

$$\begin{bmatrix}y_1^{(\infty)}\\ y_2^{(2)}\end{bmatrix}=\begin{bmatrix}y_1^{(\infty)}\\ -\delta^2x_3+\delta x_4-\delta x_2+\delta^2\dot{y}_1+\delta^3y_1-\delta^2\ddot{y}_1-\delta^3\dot{y}_1\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}\bar{u}_2.$$

It is clear that $\mathcal{U}_2=\mathrm{span}_{\mathcal{K}(\delta]}\left\{\mathrm{d}\tilde{u}_2,\mathrm{d}\tilde{u}_1\right\}$ is closed and its right-annihilator is causal. Moreover, $\tilde{u}_2=\delta^2x_3-\delta x_4+\delta x_2+\gamma(y,\dot{y},\ddot{y},\delta)$, where $\gamma=\ddot{y}_2-\delta^2\dot{y}_1-\delta^3y_1+\delta^2\ddot{y}_1+\delta^3\dot{y}_1$. So the system $\Sigma$ is strongly left-invertible and an input reconstructor is given by

$$\begin{cases}\begin{bmatrix}\dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{bmatrix}=\begin{bmatrix}-\delta z_3+z_4\\ z_2-\delta\dot{y}_1-\delta^2y_1+\delta^2(\delta^2z_3-\delta z_4+\delta z_2+\gamma(y,\dot{y},\ddot{y},\delta))\\ \delta^2z_3-\delta z_4+\delta z_2+\gamma(y,\dot{y},\ddot{y},\delta)\end{bmatrix},\\ \begin{bmatrix}\hat{u}_1\\ \hat{u}_2\end{bmatrix}=\begin{bmatrix}\frac{\dot{y}_1+\delta y_1-\delta(\delta^2z_3-\delta z_4+\delta z_2+\gamma(y,\dot{y},\ddot{y},\delta))}{\delta x_4}\\ \delta^2z_3-\delta z_4+\delta z_2+\gamma(y,\dot{y},\ddot{y},\delta)\end{bmatrix},\end{cases}$$

providing some initial value functions $\left[\begin{smallmatrix}\dot{z}_2(s)\\ \dot{z}_3(s)\\ \dot{z}_4(s)\end{smallmatrix}\right]=\varphi_z(s)$, $s\in[-2,0]$ (thus $\bar{i}_z=2$). However, $\Sigma$ is not strongly input-observable because $\mathcal{X}\cap\tilde{\mathcal{Y}}$ does *not* coincide with

$$\mathcal{X}\cap(\tilde{\mathcal{U}}+\tilde{\mathcal{Y}})=\mathcal{X}\cap\tilde{\mathcal{Y}}+\mathrm{span}_{\mathcal{K}(\delta]}\left\{\delta\mathrm{d}x_4,\delta^2\mathrm{d}x_3+\delta\mathrm{d}x_2\right\}.$$

**Example 12.** We now apply our results to a unicycle system subjected to some actuators delays (see e.g., Example 3.20 of [10] for its delay-free model):

$$\dot{x} = \cos\theta(u_1 + \delta u_1), \quad \dot{z} = \sin\theta(u_1 - \delta u_1), \quad \dot{\theta} = u_2,$$

where $(x, z)$ are the coordinates of the unicycle positions and $\theta$ is the angle with respect to the $x$-axis. The purpose is to recover the inputs $u = (u_1, u_2)$.

*Case 1:* Let the outputs $y = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} x \\ z \end{bmatrix}$. In Step 1 of Algorithm 6, we have $\rho_1^1 = \rho_2^1 = 1$ and $\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = b^1(\theta, \delta)u = \begin{bmatrix} \cos\theta + \cos\theta\delta \\ \sin\theta - \sin\theta\delta \end{bmatrix} u_1$. The submodule $\mathcal{U}_1 = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}u_1\}$ is closed and its right-annihilator is causal because $P^1 b^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, where $P^1 = \begin{bmatrix} \frac{1}{2}\sec\theta & \frac{1}{2}\csc\theta \\ -\sec\theta + \sec(\delta\theta)\delta & \csc\theta + \csc(\delta\theta)\delta \end{bmatrix}$ is unimodular. Thus we solve $u_1$ as

$$u_1 = \frac{1}{2}\dot{y}_1 \sec\theta + \frac{1}{2}\dot{y}_2 \csc\theta.$$

As a consequence, $\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \psi^1(\theta, \dot{y}, \delta) \\ \psi^2(\theta, \dot{y}, \delta) \end{bmatrix}$, where $\psi^1 = \frac{1}{2}\dot{y}_1 + \frac{1}{2}\dot{y}_2 \cot\theta + \frac{1}{2}(\delta\dot{y}_1)\cos(\theta)\sec(\delta\theta) + \frac{1}{2}(\delta\dot{y}_2)\cos(\theta)\csc(\delta\theta)$ and $\psi^2 = \frac{1}{2}\dot{y}_2 + \frac{1}{2}\dot{y}_1 \tan\theta - \frac{1}{2}(\delta\dot{y}_1)\sin(\theta)\sec(\delta\theta) - \frac{1}{2}(\delta\dot{y}_2)\sin(\theta)\csc(\delta\theta)$.

Then in Step 2, we have $\rho_1^2 = \rho_2^2 = 2$ and

$$\begin{bmatrix} y_1^{(\rho_1^2)} \\ y_2^{(\rho_2^2)} \end{bmatrix} = \begin{bmatrix} \alpha_1(\dot{y}, \ddot{y}, \theta, \delta) \\ \alpha_2(\dot{y}, \ddot{y}, \theta, \delta) \end{bmatrix} + \begin{bmatrix} \psi_\theta^1 + \psi_{\delta\theta}^1 \delta \\ \psi_\theta^2 + \psi_{\delta\theta}^2 \delta \end{bmatrix} u_2.$$

where $\alpha_1 = \psi^1_{\dot{y}_1}\ddot{y}_1 + \psi^1_{\dot{y}_2}\ddot{y}_2 + \psi^1_{\delta\dot{y}_1}\delta\ddot{y}_1 + \psi^1_{\delta\dot{y}_2}\delta\ddot{y}_2$ and $\alpha_2 = \psi^2_{\dot{y}_1}\ddot{y}_1 + \psi^2_{\dot{y}_2}\ddot{y}_2 + \psi^2_{\delta\dot{y}_1}\delta\ddot{y}_1 + \psi^2_{\delta\dot{y}_2}\delta\ddot{y}_2$. It follows that

$$u_2 = \frac{\psi^2_{\delta\theta}(\ddot{y}_1 - \alpha_1) - \psi^1_{\delta\theta}(\ddot{y}_2 - \alpha_2)}{\psi^1_\theta \psi^2_{\delta\theta} - \psi^2_\theta \psi^1_{\delta\theta}}.$$

The system is strongly left-invertible. Moreover, by

$$\begin{bmatrix} \mathrm{d}\dot{y}_1 \\ \mathrm{d}\dot{y}_2 \end{bmatrix} = \begin{bmatrix} (\psi^1_{\dot{y}_1} + \psi^1_{\delta\dot{y}_1}\delta)\mathrm{d}\dot{y}_1 + (\psi^1_{\dot{y}_2} + \psi^1_{\delta\dot{y}_2}\delta)\mathrm{d}\dot{y}_2 \\ (\psi^2_{\dot{y}_1} + \psi^2_{\delta\dot{y}_1}\delta)\mathrm{d}\dot{y}_1 + (\psi^2_{\dot{y}_2} + \psi^2_{\delta\dot{y}_2}\delta)\mathrm{d}\dot{y}_2 \end{bmatrix} + \begin{bmatrix} \psi_\theta^1 + \psi_{\delta\theta}^1 \delta \\ \psi_\theta^2 + \psi_{\delta\theta}^2 \delta \end{bmatrix} \mathrm{d}\theta,$$

it is seen that $\mathrm{d}\theta \in \tilde{\mathcal{Y}}$. Hence $\mathcal{X} \cap \tilde{\mathcal{Y}} = \mathcal{X} \cap (\tilde{\mathcal{Y}} + \tilde{\mathcal{U}}) = \mathcal{X} = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}x, \mathrm{d}z, \mathrm{d}\theta\}$ and the system is also strongly input-observable.

*Case 2*: Let $\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} x + \delta\theta \\ \delta x + \delta^2\theta \end{bmatrix}$, then $\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = b^1(\theta, \delta)u = \begin{bmatrix} \cos\theta + \cos\theta\delta & \delta \\ \cos(\delta\theta) + \cos(\delta\theta)\delta^2 & \delta^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Then $\mathcal{U}_1 = \mathrm{span}_{\mathcal{K}(\delta]}\{(\cos\theta + \cos\theta\delta)\mathrm{d}u_1 + \delta\mathrm{d}u_2\}$ is closed but does *not* have a causal right-annihilator. Hence the system is not strongly left-invertible.

## V. Conclusions and perspectives

In this paper, necessary and sufficient conditions of strong left-invertibility and strong input-observability are proposed for nonlinear time-delay control systems. A generalized structure algorithm is given to calculate the unknown inputs by causal outputs and initial value functions of states for strong left-invertibility and by only outputs for strong input-observability. We show that the causality for recovering $u(t)$ can be interpreted using a sequences of inputs submodules $\mathcal{U}_k$. For future works, we will study the case where the inputs are not strongly left-invertible or strongly input-observable but only with extra knowledge of inputs initial value functions.

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