On impulse-free solutions and stability of switched nonlinear differential-algebraic equations

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Abstract

In this paper, we study solutions and stability for switched nonlinear differential-algebraic equations (DAEs). A novel notion of solutions, called the impulse-free (jump-flow) solution, is proposed and a geometric characterization for its existence and uniqueness is given as a nonlinear version of the impulse-free condition used in, e.g., [27, 28], for linear DAEs. Then we show that the common Lyapunov functions stability conditions proposed in our previous work [16] (which differ from the ones in [28]) can be applied to switched nonlinear DAEs with high-index models which are not equivalent to the nonlinear Weierstrass form. Moreover, we generalize the commutativity stability conditions [32] for switched nonlinear ordinary differential equations to the switched nonlinear DAEs case. Finally, some simulation results of switching electrical circuits and numerical examples are given to illustrate the usefulness of the proposed stability conditions.

Key words: switched systems; nonlinear differential-algebraic equations; impulse-freeness; stability; common Lyapunov functions; commutativity condition; electrical circuits

1 Introduction

We consider a switched nonlinear differential-algebraic equation (DAE) of the form

$$\Xi_\sigma: \quad E_\sigma(x)\dot{x} = F_\sigma(x), \tag{1}$$

where $x \in X$ is called the generalized state and $(x, \dot{x}) \in TX$, where TX is the tangent bundle of an open subset X of $\mathbb{R}^n$ (or more general, X is an n-dimensional manifold), the function $\sigma : \mathbb{R} \to \mathcal{N}$ is a switching signal and we assume throughout that σ is right continuous with a locally finite number of jumps and $\mathcal{N} := \{1, \ldots, N\}$, where $N \in \mathbb{N}$ is the number of DAE models. For each $p \in \mathcal{N}$, the maps $E_p : TX \to \mathbb{R}^n$ and $F_p : X \to \mathbb{R}^n$ are $\mathcal{C}^\infty$-smooth. The non-switching case of (1), i.e., equation (4) below, is also called an implicit, singular or descriptor system, which, due to its special features, is useful for modeling e.g., constrained mechanics [39], chemical processes [22], power systems [49, 37]. In particular, the DAEs are conventional tools to model electrical circuits [45, 43] as the use of Kirchhoff's laws results in constraints that are algebraic equations. As a consequence, switched DAEs of the form (1) emerge naturally in modeling electrical circuits with switching devices. Note that the switching devices which we consider in the paper are ideal switches but not ideal diodes, the latter lead to complementarity systems [6, 7].

Note that for each $x \in X$, the map $E_p(x) : T_xX \to \mathbb{R}^n$ of each model Ξ_p is a linear map. If $E_p(x)$ is invertible for all $x \in X$, then the switched DAE (1) can be seen as a switched ordinary differential equation (ODE) $\dot{x} = f_\sigma(x)$, where $f_p := E_p^{-1}F_p$ is a vector field. Switched linear and nonlinear ODEs and more specifically, the stability analysis of such systems, have drawn attentions from researchers for decades, there is a rich literature devoted to them, see e.g. the book by Liberzon [24], the reviews [26, 46, 30] and the references therein. In this paper, we will be particularly interested in generalizing classical switched ODE results like common Lyapunov functions stability conditions [24], commutativity and Lie-algebraic conditions [32, 33, 25] as well as converse Lyapunov theorems [19, 31, 56].

A special case of (1) is a switched linear DAE of the form

$$\Delta_\sigma: \quad E_\sigma\dot{x} = H_\sigma x, \tag{2}$$

⋆ This work was supported by Vidi-grant 639.032.733.

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where $E_p : \mathbb{R}^n \to \mathbb{R}^n$ and $H_p : \mathbb{R}^n \to \mathbb{R}^n$ are linear maps, which received increased interests in the recent past, see e.g., [27, 59, 61, 55, 44] for its stability analysis using Lyapunov method and dwell time technique, and [29, 60, 41] for commutativity and Lie-algebraic conditions, and [34, 35] for averaging methods. Compared to the linear case, much less results on switched nonlinear DAEs can be found. The first comprehensive paper to discuss the nonlinear case is [28], in which both common Lyapunov function conditions and average dwell time conditions for checking the stability of switched nonlinear DAEs are proposed, such results are inspirations for the present paper, but we will take a different approach to define solutions and to obtain our stability conditions.

One main challenge of studying (switched) DAEs is their discontinuous behavior, i.e., jumps and impulses. Unlike ODEs, the $\mathcal{C}^1$-solutions of a DAE (see section 2.1) exist only on a subset of the generalized state space X, which we will call the *consistency space* $\mathfrak{C}$ of the DAE. Even for a non-switching DAE, it is possible that a given initial point $x_0^- \in X$ is not consistent, i.e., $x_0^- \notin \mathfrak{C}$. The problem of finding a consistent point $x_0^+ \in \mathfrak{C}$ from x_0^- is called the consistent initialization of DAEs. In assumption A4 of [28], the consistent point x_0^+ is given by the following jump rule (a similar jump rule can be found in [38] for linear time-varying DAEs)

$$x_0^+ - x_0^- \in \ker E(x_0^+). \tag{3}$$

However, we have shown in our recent works [13, 15] that nonlinear coordinate transformations do not preserve the jump rule (3), namely, we may get different consistent points x_0^+ from (3) depending on which coordinates are chosen for the DAE Ξ (see also Remark 2.6 below). To have a coordinates-free jump rule, the notion of impulse-free jump solution is proposed in [15] (see also Definition 2.4 below). Because inconsistent initialization can be frequently triggered by switching behaviors in switched DAEs, the main purpose of the present paper is to extend the impulse-free jump rule to switched nonlinear DAEs and to discuss their solutions and stability. Some other works related to inconsistent initial value problems can be found in [48] discussing nonsmooth DAEs and their applications on chemical processes [47], in [7] for linear complementarity DAEs (in particular, state-dependent switching DAEs) of semi-explicit form, and in [50] for impact mechanics.

There are three main contributions of this paper: Firstly, we define the notion of impulse-free jump-flow solution for (switched) nonlinear DAEs (see Definition 3.1); a geometric characterization of the impulse-free consistent space, i.e., the space on which impulse-free (jump-flow) solutions exist (see Definition 3.2), is given for non-switching DAEs in Theorem 3.3; the extension of such a characterisation to the case of switched nonlinear DAEs results in an existence and uniqueness condition (see Corollary 3.5), which generalizes the known impulse-free condition of switched linear DAEs (see [27, 28] or Remark 3.6 below) to the nonlinear case. Secondly, with the help of a notion called the jump-flow explicitation of DAEs, we give novel common Lyapunov functions conditions for checking the asymptotic stability of switched nonlinear DAEs (Theorem 4.5), these condition are different from the corresponding results in [53]. Finally, we give a nonlinear version of the commutativity conditions for switched linear DAEs (see [29, 60]), we will show in Theorem 4.10 that in order to guarantee the asymptotic stability of switched nonlinear DAEs with all models being asymptotically stable, not only the commutativity of the flow vector fields but also some extra invariant distributions conditions are needed.

Some preliminary results on impulse-freeness and common Lyapunov function conditions of switched nonlinear DAEs can be found in our recent conference publication [16], in which we assume that all models of the switched DAE are globally equivalent to a nonlinear Weierstrass form **(NWF)** (see Corollary 3.4). In the present paper, both the impulse-freeness condition in Corollary 3.5 and the common Lyapunov functions conditions in Theorems 2.7 can be applied to high-index DAEs which are *not necessarily* equivalent to the **(NWF)**. Additionally, we give a practical Example 4.7 of a switched electric circuit to verify our stability conditions and to show the construction of the common Lyapunov function.

This paper is organized as follows: We review the existence and uniqueness of $\mathcal{C}^1$-solutions and impulse-free jumps of non-switching DAEs in Sections 2.1 and 2.2, respectively. The results on impulse-free consistency space, and the existence and uniqueness of impulse-free solutions of switched DAEs are given in Section 3. In Sections 4.1 and 4.2, respectively, we discuss the stability of nonlinear switched DAEs using common Lyapunov function conditions and commutativity conditions. The proofs are put into Section 5. The conclusions and perspectives of the paper are given in Section 6.

Notations: We denote by $T_xM \subseteq \mathbb{R}^n$ the tangent space of a submanifold M of $\mathbb{R}^n$ at $x \in M$ and by TM the corresponding tangent bundle. By $\mathcal{C}^k$ the class of k-times differentiable functions is denoted. For a smooth map $f : X \to \mathbb{R}$, we denote its differential by $\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\mathrm{d}x_i = [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}]$ and for a vector-valued map $f : X \to \mathbb{R}^m$, where $f = [f_1, \ldots, f_m]^T$, we denote its differential by $\mathrm{d}f = \begin{bmatrix} \mathrm{d}f_1 \\ \vdots \\ \mathrm{d}f_m \end{bmatrix}$. For a vector filed $g : X \to TX$, we denote its flow map by Φ_t^g, i.e., $g(x) = \frac{\mathrm{d}\Phi_\tau^g(x)}{\mathrm{d}\tau}\,|_{\tau=0}$. For a map $A : X \to \mathbb{R}^{n\times n}$, $\ker A(x)$, $\operatorname{Im} A(x)$ and $\operatorname{rank} A(x)$ are the kernel, the image and the rank of A at x, respectively. We use $GL(n, \mathbb{R})$ to denote the general linear group of degree n (or in other words, the set of invertible linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$).

For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$. Let $U \subseteq \mathbb{R}^n$ be a neighborhood of $x = 0$, a continuous function $V : U \to \mathbb{R}$ is positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0 \in U$. A function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t > 0$ and $\lim_{t\to\infty} \beta(r, t) = 0$ for each fixed $r > 0$. We assume familiarity with basic notions from differential geometry [23] and nonlinear geometric control theory [20, 36], e.g., submanifolds, distributions, involutivity, zero dynamics.

2 $\mathcal{C}^1$-solutions and impulse-free jumps of non-switching DAEs

In this section, we review some notions related to $\mathcal{C}^1$-solutions and jumps of the non-switching case of (1), i.e., a nonlinear DAE of the form

$$\Xi : \quad E(x)\dot{x} = F(x), \tag{4}$$

where $E : TX \to \mathbb{R}^n$ and $F : X \to \mathbb{R}^n$ are $\mathcal{C}^\infty$-smooth maps, we denote a nonlinear DAE of the form (4) by $\Xi = (E, F)$.

2.1 $\mathcal{C}^1$-solutions of non-switching DAEs

A $\mathcal{C}^1$-curve $x : \mathcal{I} \to X$ for some open interval $\mathcal{I} \subseteq \mathbb{R}$ is called a *$\mathcal{C}^1$-solution* of Ξ if $E(x(t))\dot{x}(t) = F(x(t))$ for all $t \in I$. We call a $\mathcal{C}^1$-solution $x : \mathcal{I} \to (U \subseteq)X$ *maximal* (in U) if there is no other solution $\widetilde{x} : \widetilde{\mathcal{I}} \to (U \subseteq)X$ with $\mathcal{I} \subsetneq \widetilde{\mathcal{I}}$ and $x(t) = \widetilde{x}(t)$ for all $t \in \mathcal{I}$.

Definition 2.1 (consistency space and internal regularity)**.** A point $x_c \in X$ is called *consistent* (or *admissible* [8, 12]) if there exist a $\mathcal{C}^1$-solution $x : \mathcal{I} \to X$ and $t_c \in \mathcal{I}$ such that $x(t_c) = x_c$. The *consistency space* $\mathfrak{C} \subseteq X$ is the set of all consistent points. A nonlinear DAE Ξ is called *internally regular* (or *autonomous*) around a point $x_p \in \mathfrak{C}$ if there exists a neighborhood $U \subseteq X$ of x_p such that for any point $x_0 \in \mathfrak{C} \cap U$, there exists *only one* maximal solution $x : \mathcal{I} \to \mathfrak{C} \cap U$ satisfying $x(t_0) = x_0$ for a certain $t_0 \in \mathcal{I}$.

The above two notions of consistency space and internal regularity characterize the existence and the uniqueness of $\mathcal{C}^1$-solutions, respectively. In the following definition, we recall the *geometric reduction method* [42, 40, 43, 12], which is a recursive procedure to construct a sequence of submanifolds M_k^c whose limit M^* coincides locally with the consistency space $\mathfrak{C}$ (see Proposition 2.3 below).

Definition 2.2 (geometric reduction method [2, 12, 17])**.** Consider a DAE Ξ and fix a point $x_p \in X$. Let $U_0 \subseteq X$ be a connected neighborhood of x_p. Step 0: Set $M_0^c = U_0$. Step k ($k \geq 1$): Suppose that a sequence of smooth connected embedded submanifolds $M_{k-1}^c \subsetneq \cdots \subsetneq M_0^c$ of U_{k-1} for a certain $k-1$, have been constructed. Define recursively

$$M_k := \left\{ x \in M_{k-1}^c \,|\, F(x) \in E(x) T_x M_{k-1}^c \right\}. \tag{5}$$

As long as $x_p \in M_k$, let $M_k^c = M_k \cap U_k$ be a smooth embedded connected submanifold for some neighborhood $U_k \subseteq U_{k-1}$. The (local) *geometric index*, or shortly, the *index* [1], of Ξ is defined by

$$\nu_g := \min \left\{ k \geq 0 \,|\, M_{k+1}^c = M_k^c \right\}.$$

Proposition 2.3 ([12])**.** *In the above geometric reduction method, there always exists a smallest k such that either $x_p \notin M_k$ or $M_{k+1}^c = M_k^c$ in U_{k+1}. In the latter case denote $k^* = k$ (thus the geometric index $\nu_g = k^*$) and $M^* = M_{k^*+1}^c$ and assume that there exists an open neighborhood $U \subseteq U_{k^*+1}$ of x_p such that $\dim E(x)T_xM^*$ is constant for $x \in M^* \cap U$, then*

(i) x_p is a consistent point and $M^* \cap U = \mathfrak{C} \cap U$.
(ii) Ξ is internally regular around x_p if and only if $\dim E(x)T_xM^* = \dim M^*$ for all $x \in M^* \cap U$.

Note that M^* is called a *locally maximal invariant submanifold* [2, 12] and the word "invariant" means that the $\mathcal{C}^1$-solutions starting from any point $x_0^+ \in M^*$ exist and stay in M^* for all $t \in \mathcal{I}$. So any point $x_0^- \in U \setminus M^*$ is inconsistent and there exist no $\mathcal{C}^1$-solutions starting from x_0^-.

2.2 Impulse-free jumps of non-switching DAEs

In our recent contributions [13, 15], we studied impulse-free jumps for DAEs with inconsistent initial values. The main idea behind the following definition of impulse-free jump (solutions) is that we view a jump not only as an instant change between an inconsistent point and a consistent one but also as a parametrized curve $J(\tau)$ [2] whose derivatives with respect to τ satisfy a certain rule, i.e., staying in $\ker E$, such a rule ensures that the jump does not cause any impulse.

Definition 2.4 (impulse-free jump [15])**.** Consider a DAE $\Xi = (E, F)$, let $\mathfrak{C}$ be the consistency space of Ξ, fix an initial point $x_0^- \in X$. An impulse-free jump solution (trajectory), shortly, an IFJ solution, of Ξ starting

[1] Another commonly used DAE index is the so called differentiation index, which is the smallest integer ν_d such that the combination of the ν_d-times differentiation of the DAE uniquely determines $\dot{x}$ as a function of x. Actually, the two notions of index coincide when the forthcoming assumptions **(RE)** and **(CR)** are satisfied, see [14].

[2] Note that τ is a parametrization variable which is *not* necessarily related to the time t.

from x_0^- is a $\mathcal{C}^1$-curve $J : [0, a] \to X$, $a \geq 0$, satisfying $J(0) = x_0^- \in X$, $J(a) = x_0^+ \in \mathfrak{C}$ and

$$\forall \tau \in [0, a]: \ E(J(\tau))\frac{dJ(\tau)}{d\tau} = 0.$$

A jump $x_0^- \to x_0^+$ associated with an IFJ trajectory $J(\cdot)$ is called an *impulse-free jump* IFJ of Ξ.

Definition 2.5. (external equivalence) Two DAEs $\Xi = (E, F)$ and $\tilde{\Xi} = (\tilde{E}, \tilde{F})$ are called externally equivalent, shortly ex-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$ and a smooth map $Q : X \to GL(n, \mathbb{R})$ such that $\tilde{E}(\psi(x)) = Q(x)E(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}$ and $\tilde{F}(\psi(x)) = Q(x)F(x)$. Fix a point $x_p \in X$, if ψ and Q are defined locally around x_p, we will speak about local ex-equivalence.

Remark 2.6. It is important to note that the ex-equivalence preserves both $\mathcal{C}^1$-solutions and IFJ solutions (and thus IFJs) of DAEs [12, 15]. Note that the jump rule (3) shown in [28] is *not* invariant under the ex-equivalence, i.e., given a jump $x_0^- \to x_0^+$ of Ξ defined by (3) then, in general, the jump $\tilde{x}_0^- = \psi(x_0^-) \to \tilde{x}_0^+ = \psi(x_0^+)$ of $\tilde{\Xi}$ does *not* satisfy $\tilde{x}_0^+ - \tilde{x}_0^- \in \ker \tilde{E}(\tilde{x}_0^+)$.

We recall the results on existence and uniqueness of IFJs for *index-1* nonlinear DAEs from [15]. For a DAE $\Xi = (E, F)$ and a consistent point $x_c \in X$, define $F_2 := Q_2F$, where $Q_2 : U \to \mathbb{R}^{(n-r)\times n}$ is of full row rank and $Q_2E = 0$, and recall $M_1^c := \{x \in U \,|\, F(x) \in \operatorname{Im} E(x)\}$ by (5). We now introduce the following regularity and constant rank conditions: there exists a neighborhood U of x_c such that

- **(RE)** the locally maximal invariant submanifold M^* around x_c exists and Ξ is internally regular;
- **(CR)** $\operatorname{rank} E(x) = \text{const.} = r$ for $x \in U$; $\dim \mathrm{d}F_2(x)$ and $\dim E(x)T_xM_1^c$ are constant for $x \in M_1^c \cap U$.

Theorem 2.7 (Thm. 4.6 and Cor. 4.9 of [15]). *Consider a DAE $\Xi = (E, F)$ and a consistent point $x_c \in X$. Assume that* **(RE)** *and* **(CR)** *hold in an open neighborhood U of x_c. Then there exists a neighborhood $U_c \subseteq U$ of x_c such that the the following statements are equivalent:*

(i) The DAE Ξ is index-1 and the distribution $\ker E$ *is involutive* [3].

(ii) The DAE Ξ is locally on U_c, via an invertible matrix-valued function Q and a diffeomorphism ψ, ex-equivalent to the following index-1 nonlinear Weierstrass form

$$(\mathbf{INWF}) : \begin{cases} \dot{\xi}_1 = f^*(\xi_1), \\ 0 = \xi_2, \end{cases} \tag{6}$$

where $(\xi_1, \xi_2) = \psi(x) \in \tilde{U}_1 \times \tilde{U}_2 \subseteq \mathbb{R}^r \times \mathbb{R}^m$ and $m = n - r = \dim \ker E$.

(iii) For any point $x_0^- \in U_c$ such that $M^* \cap N_{x_0^-} \neq \emptyset$, there exists a unique IFJ $x_0^- \to x_0^+$, where $N_{x_0^-} \subseteq U_c$ is the integral submanifold of the distribution $\ker E$ *on U_c passing through x_0^-.*

If one of (i),(ii),(iii) holds, then the unique IFJ from x_0^- is given by $x_0^- \to x_0^+ = \Omega_{E,F}(x_0^-) \in M^* \cap N_{x_0^-}$, where $\Omega_{E,F} : X \to M^*$ is the nonlinear consistency projector defined by

$$\Omega_{E,F} := \psi^{-1} \circ \pi \circ \psi, \tag{7}$$

where π is the canonical projection $(\xi_1, \xi_2) \mapsto (\xi_1, 0)$ and ψ is the diffeomorphism in (ii).

The submanifold $N_{x_0^-}$ in Theorem 2.7(iii) can be seen as a local reachable space of IFJ solutions [15]. Note that if (and only if) the set $\tilde{U}_2$ of item (ii) above is a star field (i.e., $\lambda\xi_2 \in \tilde{U}_2$, $\forall \xi_2 \in \tilde{U}_2$ and $\forall \lambda \in [0, 1]$), then we always have $N_{x_0^-} \subseteq U_c$ and $M^* \cap N_{x_0^-} \neq \emptyset$; thus by Theorem 2.7(iii) we have that for any point $x_0^- \in U_c$, there exists a unique IFJ starting from x_0^-. If for some point $x_0^- \in U_c$, the set $M^* \cap N_{x_0^-}$ is empty, then in order to have a well-defined IFJ for any $x_0^- \in U_c$, we need to take a smaller U_c to exclude those points such that $\tilde{U}_2$ is a star field. The results shown above on $\mathcal{C}^1$-solutions and IFJs of nonlinear DAEs have their linear counterparts which we will discuss in the following remark.

Remark 2.8 ($\mathcal{C}^1$-solutions and jumps of linear DAEs). For a linear DAE $\Delta = (E, H)$, its consistency space $\mathfrak{C}$ coincides with the limit $\mathscr{V}^* = \mathscr{V}_n$ of the Wong sequence [58] $\mathscr{V}_k$ defined by

$$\mathscr{V}_0 = \mathbb{R}^n, \quad \mathscr{V}_{k+1} = H^{-1}E\mathscr{V}_k, \ k \geq 1. \tag{8}$$

It is clear that the sequence of subspaces $\mathscr{V}_k$ is a linear version of the submanifolds sequence M_k^c. The DAE Δ is called *regular* if $\det(sE - H)$ is not identically zero. Note that the notions of internal regularity and regularity are equivalent [4] for (square) linear DAEs. A linear *regular* DAE $\Delta = (E, H)$ is always ex-equivalent, via two constant invertible matrices Q and P, to the Weierstrass form [57, 3] $\tilde{\Delta} = (QEP^{-1}, QHP^{-1})$, given by

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{9}$$

[3] A distribution $\mathcal{D}$ is called involutive if for any two vector fields $f_1, f_2 \in \mathcal{D}$, we have $[f_1, f_2] \in \mathcal{D}$.

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ and $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent matrix with nilpotency index ν, i.e. $N^{\nu-1} \neq 0$ and $N^\nu = 0$. The index of Δ is defined to be the nilpotency index ν of N, which coincides with its geometric index ν_g (i.e., the least integer such that $\mathscr{V}_{\nu_g+1} = \mathscr{V}_{\nu_g}$). The consistency projector [27, 28] of Δ is defined by

$$\Pi_{E,H} := P^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} P. \tag{10}$$

For a given inconsistent point $x_0^- \in \mathbb{R}^n \setminus \mathscr{V}^*$, the consistent point $x_0^+ \in \mathscr{V}^*$ jumping from x_0^- is *unique* and is defined by $x_0^+ = \Pi_{E,H}(x_0^-)$. A jump $x_0^- \to x_0^+$ is called impulse-free if $x_0^+ - x_0^- \in \ker E$. It follows that all the jumps from any point $x_0^- \in \mathbb{R}^n$ are impulse-free if and only if $E\Pi_{E,H} = 0$, the latter condition is also equivalent to $\nu = 1$ (i.e., Δ is index-1) or $\mathscr{V}^* + \ker E = \mathbb{R}^n$. It should be pointed out that the involutivity of $\ker E$ and condition **(CR)** above are always satisfied for any linear DAE.

3 Impulse-free solutions of switched nonlinear DAEs

Definition 3.1 (impulse-free solutions)**.** Consider a switched DAE Ξ_σ, given by (1). Let σ be a switching signal with k switches at $t_1, \ldots, t_k \in \mathcal{I}$, respectively, where $\mathcal{I} = [t_0, t_{k+1})$ is the time interval of interest. An impulse-free jump-flow solution, shortly, an impulse-free solution, of Ξ_σ is a piecewise $\mathcal{C}^1$-curve $x : \mathcal{I} \to X$ such that for all $0 \leq i \leq k$, the jump $x(t_i^-) \to x(t_i^+)$ is an impulse-free jump of $\Xi_{\sigma(t_i^+)}$ in the sense of Definition 2.4 and the curve $x(\cdot)$ is a $\mathcal{C}^1$-solution of $\Xi_{\sigma(t_i^+)}$ on $[t_i, t_{i+1})$ such that $x(t_i) = x(t_i^+)$.

In this section, we will study the following problem: given a switched nonlinear DAE under an arbitrary switching signal $\sigma : \mathcal{I} \to \mathcal{N}$, where $\mathcal{I}$ is an interval on which all $\mathcal{C}^1$-solutions of each model are well-defined, when does there exist a unique impulse-free solution defined on $\mathcal{I}$? A simple solution to the latter problem is to assume that all models Ξ_p of the switched DAE Ξ_σ are index-1 and that all distributions $\ker E_p$ are involutive, because the latter conditions imply that each model Ξ_p is ex-equivalent to its **(INWF)** and there exists a unique IFJ at each switching time by Theorem 2.7. Recall that being index-1 is *not* a necessary condition to have IFJs, it is possible that IFJs exist for high-index nonlinear DAEs (see Remark 4.7(iii) of [15]). We will show in Corollary 3.5 below that a switched nonlinear DAE with high-index models can have uniquely defined impulse-free solution under a sufficient condition, which can be regarded as a nonlinear generalization of the impulse-free condition for linear DAEs shown e.g. in [51, 27].

3.1 *Impulse-free consistency space for non-switching DAEs*

We start from the definition of *impulse-free consistent space* for non-switching DAEs.

Definition 3.2 (impulse-free consistency space)**.** For a nonlinear DAE $\Xi = (E, F)$, a point $x_0 \in X$ is called an *impulse-free consistent point* if there exists an impulse-free solution from x_0. The set of all impulse-free consistent points is called the *impulse-free consistency space* of Ξ, denoted by $\mathfrak{C}_{IF}$.

From Definitions 3.1 and 3.2, it is clear to see that the consistency space $\mathfrak{C} \subseteq \mathfrak{C}_{IF}$. For a linear regular DAE $\Delta = (E, H)$, the impulse-free consistency space coincides with the *consistent initial differential variables space* (see Chapter 3.1 of [1]), i.e., the set of points x_0 such that there exists a $\mathcal{C}^1$-solution $x(t)$ of Δ satisfying $Ex(0) = Ex_0$, which can be characterized by

$$\mathfrak{C}_{IF} = \mathscr{V}^* + \ker E, \tag{11}$$

where $\mathscr{V}^* = \mathscr{V}_\nu$ is the limit of the Wong sequences $\mathscr{V}_k$, given by (8). For a nonlinear DAE $\Xi = (E, F)$ with $\ker E(x)$ being involutive, the set $\mathfrak{C}_{IF}$ is, roughly speaking, the union of the integral manifolds $N_{x_0^+}$ of $\ker E(x)$ for all $x_0^+ \in M^*$, which is in general *not* a smooth submanifold. We show below that under certain constant rank and involutivity conditions, the set $\mathfrak{C}_{IF}$ coincides locally with a smooth submanifold M_{IF}^*, which can be parametrized as the zero level set of certain functions.

Theorem 3.3. *Consider a DAE $\Xi = (E, F)$ and a consistent point $x_c \in X$, let M^* be the locally maximal invariant submanifold of Ξ around x_c, assume that there exists a neighborhood U of x_c such that condition* **(RE)** *is satisfied and there exists a distribution $\mathcal{D}(x)$ such that on U:*

(D1) *$\mathcal{D}(x)$, $\ker E(x)$ and $\mathcal{D}(x) + \ker E(x)$ are of constant dimensions and involutive.*
(D2) *$\mathcal{D}(x) = T_x M^*$, $\forall x \in M^* \cap U$.*

Let $M_{IF}^* \subseteq U$ be the integral submanifold of the distribution $\mathcal{D}(x) + \ker E(x)$ passing through x_c, then there exists a neighborhood $U_c \subseteq U$ such that the impulse-free consistency space $\mathfrak{C}_{IF}$ satisfies

$$\mathfrak{C}_{IF} \cap U_c = M_{IF}^* \cap U_c = \{x \in U_c \mid \xi_2(x) = 0\},$$

where $\xi_2 = (\xi_2^1, \ldots, \xi_2^{n_2})$ and $\xi_2(x_c) = 0$, the codistribution $\operatorname{span}\{\mathrm{d}\xi_2^1, \ldots, \mathrm{d}\xi_2^{n_2}\}$ *annihilates the distribution $\mathcal{D}(x) + \ker E(x)$. Moreover, the IFJ from any initial point $x_0^- \in M_{IF}^* \cap U_c$ is uniquely defined.*

The proof is given in Section 5.The following corollary says that if a DAE is ex-equivalent to the nonlinear

Weierstrass form [12, 16], then it is straightforward to obtain M^* and M^*_{IF}.

Corollary 3.4. *Consider a nonlinear DAE $\Xi = (E, F)$ and a consistent point x_c. Assume that on a neighborhood U_c of x_c, the DAE Ξ is ex-equivalent, via a diffeomorphism $\psi = (\psi_1, \psi_2) = (\xi_1, \xi_2) : U_c \to \tilde{U}_1 \times \tilde{U}_2$ and an invertible map Q defined on a neighborhood U_c of x_c, to the following nonlinear Weierstrass form*

$$(\mathbf{NWF}) : \begin{cases} \dot{\xi}_1 = f^*(\xi_1), \\ N\dot{\xi}_2 = \xi_2, \end{cases}$$

where $f^* : \tilde{U}_1 \to T\tilde{U}_1$ is a vector field on $\tilde{U}_1 \subseteq \mathbb{R}^{n_1}$ and N is a constant nilpotent matrix. Then condition **(RE)** *holds and the distributions* $\ker E$ *and* $\mathcal{D} = \operatorname{span}\left\{\frac{\partial}{\partial \xi_1^1}, \ldots, \frac{\partial}{\partial \xi_1^{n_1}}\right\}$ *satisfy* **(D1)** *and* **(D2)** *of Theorem 3.3. Moreover, we have*

$$M^* \cap U_c = \mathfrak{C} \cap U_c = \{x \in U_c \,|\, \psi_2(x) = 0\},$$
$$M^*_{IF} \cap U_c = \mathfrak{C}_{IF} \cap U_c = \{x \in U_c \,|\, N\psi_2(x) = 0\}.$$

3.2 Existence and uniqueness of impulse-free solutions for switched nonlinear DAEs

We extend the results of Theorem 3.3 to the switched case as a corollary shown below.

Corollary 3.5 (impulse-free solution)**.** *Consider a switched DAE Ξ_σ under an arbitrary switching signal $\sigma : \mathcal{I} \to \mathcal{N}$ and let x_{cp} be a consistent point of the model Ξ_p, i.e., $x_{cp} \in \mathfrak{C}(\Xi_p)$, for $p \in \mathcal{N}$. Assume that each DAE model Ξ_p satisfies* **(RE)**, **(D1)** *and* **(D2)** *around x_{cp}. By Theorem 3.3, for each model Ξ_p, there exists a neighborhood U_{cp} of x_{cp} such that $M^*_{IF}(\Xi_p) \cap U_{cp} = \mathfrak{C}_{IF}(\Xi_p) \cap U_{cp}$. Suppose that all $\mathcal{C}^1$-solutions of each model Ξ_p defined on $\mathfrak{C}(\Xi_p) \cap U_{cp}$ can be extended on the interval $\mathcal{I}$. Then, given any initial point $x_0 \in M^*_{IF}(\Xi_{\sigma(t_0)}) \cap U_{c\sigma(t_0)}$, there exists a unique impulse-free solution $x : \mathcal{I} \to \bigcup_{p=1}^{N} U_{cp}$ of Ξ_σ if*

$$\forall p, q \in \mathcal{N} : \quad M^*(\Xi_p) \cap U_{cp} \subseteq M^*_{IF}(\Xi_q) \cap U_{cq}. \quad (12)$$

Remark 3.6. For a switched linear DAE Δ_σ with all models $\Delta_p = (E_p, H_p)$ being regular, the distributional solution [4] of Δ_σ is impulse-free [27, 28] if

$$\forall p, q \in \mathcal{N} : \quad E_q(I - \Pi_{E_q,H_q})\Pi_{E_p,H_p} = 0, \quad (13)$$

the latter condition holds if and only if $\operatorname{Im} \Pi_{E_p,H_p} \subseteq \ker E_q(I - \Pi_{E_q,H_q})$, or, equivalently,

$$\forall p, q \in \mathcal{N} : \quad \mathscr{V}^*(\Delta_p) \subseteq \mathscr{V}^*(\Delta_q) + \ker E_q,$$

where $\mathscr{V}^*$ is the limit of the Wong sequence $\mathscr{V}_i$ of (8). Because $\mathfrak{C}(\Delta_p) = \mathscr{V}^*(\Delta_p)$ and $\mathfrak{C}_{IF}(\Delta_q) = \mathscr{V}^*(\Delta_q) + \ker E_q$ (see (11)), it is seen that condition (12) is a nonlinear generalization of the linear impulse-free condition (13).

Example 3.7. Consider a switched nonlinear DAE Ξ_σ with the generalized states $x = (x_1, x_2, x_3) \in X = \mathbb{R}^3$, and two models $\Xi_1 = (E_1, F_1)$ and $\Xi_2 = (E_2, F_2)$, where

$$E_1(x) = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 0 & 0 \\ x_3 & 1 & x_1 \end{bmatrix}, \quad F_1(x) = \begin{bmatrix} x_2 - x_1 \\ x_2 + x_1 x_3 \\ x_3(x_1 + x_3 + 1) \end{bmatrix},$$
$$E_2(x) = \begin{bmatrix} x_1+1 & 0 & 0 \\ x_1+1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} x_1 \\ x_2 + x_1(x_3+1) \\ x_1 + x_3 \end{bmatrix}.$$

By (5), $M_1(\Xi_1) = \{x \in \mathbb{R}^3 \,|\, x_2 + x_1 x_3 = 0\}$, $M^*(\Xi_1) = M_2(\Xi_1) = \{x \in \mathbb{R}^3 \,|\, x_2 + x_1 x_3 = x_3(x_1 + x_3 + 1) = 0\}$ and

$$M^*(\Xi_2) = M_1(\Xi_2) = \{x \in \mathbb{R}^3 \,|\, x_2 + x_1 x_3 = x_1 + x_3 = 0\}.$$

The point $x_c = (0, 0, 0)$ is a consistent point for both Ξ_1 and Ξ_2, we consider Ξ_1 on the neighborhood $U_1 = \{x \in \mathbb{R}^3 \,|\, x_1 + x_3 > -1\}$ of x_c such that $M^*(\Xi_1) \cap U_1 = \{x \in \mathbb{R}^3 \,|\, x_2 = x_3 = 0,\ x_1 > 0\}$ is a smooth embedded connected submanifold and is locally invariant; we examine Ξ_2 on the neighborhood $U_2 = \{x \in \mathbb{R}^3 \,|\, x_1 + 1 > 0\}$ in order that $\operatorname{rank} E_2(x) = const.$ on U_2.

Observe that Ξ_1 is index-2 and satisfies **(RE)** by $\dim E_1(x) T_x M^*(\Xi_1) = \dim M^* = 1$ and Proposition 2.3(ii). The distributions $\mathcal{D}_1 = \operatorname{span}\left\{\frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2}\right\}$ and

$$\ker E_1 = \operatorname{span}\left\{-x_1 \frac{\partial}{\partial x_1} + (x_1 x_3 - x_3)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right\}$$

satisfy conditions **(D1)** and **(D2)** of Theorem 3.3 on U_1. Choose $\psi_{12}(x) = x_2 + x_1 x_3$ such that $\operatorname{span}\{\mathrm{d}\psi_{12}\} = (\mathcal{D}_1 + \ker E_1)^\perp$. It follows that

$$M^*_{IF}(\Xi_1) \cap U_1 = \{x \in \mathbb{R}^3 \,|\, x_2 + x_1 x_3 = 0, x_1 + x_3 > -1\}.$$

Actually, the DAE Ξ_1 is locally on U_1 ex-equivalent, via the diffeomorphism

$$\psi_1(x) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (e^{x_3} x_1, x_2 + x_1 x_3, x_3)$$

and $Q_1(x) = \begin{bmatrix} e^{x_3} & -e^{x_3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, to

$$\tilde{\Xi}_1 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} -\tilde{x}_1 - \tilde{x}_1 \tilde{x}_3 \\ \tilde{x}_2 \\ \tilde{x}_3(e^{-\tilde{x}_3}\tilde{x}_1 + \tilde{x}_3 + 1) \end{bmatrix}, \quad (14)$$

[4] For distributional solutions theory of linear DAEs, see e.g., [18, 54, 52]

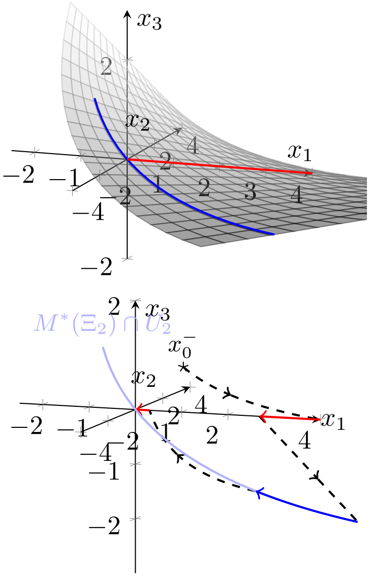

Fig. 1. Above: red line: $M^*(\Xi_1) \cap U_1$, mesh surface: $M^*_{IF}(\Xi_1) \cap U_1$, blue curve: $M^*(\Xi_2) \cap U_2$, the set $M^*_{IF}(\Xi_2) \cap U_2 = U_2 = \{x \in \mathbb{R}^3 \mid x_1 > -1\}$ is clear to see and thus is not shown; Below: red curve with arrows: $\mathcal{C}^1$-solutions of Ξ_1, blue curve with arrows: $\mathcal{C}^1$-solutions of Ξ_2, dashed lines: IFJ solutions.

which is in the form (29) but not in the **(NWF)** of Corollary 3.4. The DAE Ξ_2 is index-1 and locally on U_2 ex-equivalent to

$$\bar{\Xi}_2 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} \frac{-\bar{x}_1}{\bar{x}_1+1} \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}, \tag{15}$$

via the diffeomorphism $\psi_2(x) = (\bar{x}_1, \bar{x}_2, \bar{x}_3) = (x_1, x_2 + x_1x_3, x_1 + x_3)$ and $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Observe that $\bar{\Xi}_2$ is in the **(NWF)** (more precisely, it is in the **(INWF)** of (6)). It follows that

$$M^*_{IF}(\Xi_2) \cap U_2 = X \cap U_2 = U_2.$$

It is seen that $M^*(\Xi_1) \cap U_1 \subsetneq M^*_{IF}(\Xi_2) \cap U_2$ and $M^*(\Xi_2) \cap U_2 \subsetneq M^*_{IF}(\Xi_1) \cap U_1$. We draw those submanifolds on the above subfigure of Figure 1. By Corollary 3.5, for any switching signal $\sigma : \mathcal{I} \to \mathcal{N}$ such that $\mathcal{C}^1$-solutions of Ξ_1 and Ξ_2 are well-defined on $\mathcal{I}$, there exists a unique impulse-free solution $x : \mathcal{I} \to U_1 \cup U_2$ for any initial point $x_0 \in M^*_{IF}(\Xi_{\sigma(t_0)}) \cap U_{\sigma(t_0)}$. For example, we fix a switching signal $\sigma : [0, \infty) \to \mathcal{N}$ with $\sigma(0) = 1$ and two switches at $t_1 = 0.4$ and $t_2 = 1.4$, respectively, choose an initial point $x_0^- = (4/e, -4/e, 1) \in M^*_{IF}(\Xi_1) \cap U_1$, the impulse-free solution of Ξ_σ starting from x_0^- is shown on the below subfigure of Figure 1. Observe that the dashed curves are IFJ solutions which satisfy the jump rule in Definition 2.4. Moreover, it is seen that the impulse-free solution of Ξ_σ converges to 0, we will discuss its asymptotic stability in the next section, see Example 4.9 below.

4 Stability analysis of switched DAEs under arbitrary switching signal

Throughout the remaining parts of the paper, we focus on switched nonlinear DAEs Ξ_σ with all models Ξ_p being *index-1*. More specifically, we will make the following assumptions **(S1)** and **(S2)**. If a model Ξ_p has an index higher than one, it is possible (see Example 4.9 below) to use the results in Proposition 4.8 to replace Ξ_p with an index-1 DAE $\hat{\Xi}_p$, which has the same impulse-free solution as Ξ_p for any initial point $x_0 \in \mathfrak{C}_{IF}(\Xi_p)$.

(S1) There exists a neighborhood U_c of $x_c = 0$ such that each DAE model Ξ_p, $p \in \mathcal{N}$, is locally on U_c ex-equivalent to its **(INWF)**, given by (6), via a smooth map $Q_p : U_c \to GL(n, \mathbb{R})$ and a diffeomorphism $\psi_p = (\psi_{1p}, \psi_{2p}) = (\xi_{1p}, \xi_{2p}) : U_c \to \tilde{U}_{cp}$. Moreover, all points $(\xi_{1p}, \lambda\xi_{2p}) \in \tilde{U}_{cp}$, $\forall \lambda \in [0, 1]$ and $\forall (\xi_{1p}, \xi_{2p}) \in \tilde{U}_{cp}$.

(S2) All $\mathcal{C}^1$-solutions of Ξ_p on $U_c \cap \mathfrak{C}(\Xi_p)$ can be extended on $\mathcal{I} = [0, +\infty)$.

Remark 4.1. Note that **(S1)** implies **(CR)** and **(RE)** and by Theorem 2.7, **(S1)** is equivalent to

(S1)' there exists a neighborhood U_c of $x_c = 0$ such that for any initial point $x_0^- \in U_c$, there exists a well-defined IFJ $x_0^- \to x_0^+$ and its associated IFJ trajectory $J(\tau) \in U_c$, $\forall 0 \leq \tau \leq a$ for the model Ξ_p.

It is seen that under condition **(S1)** (or **(S1)'**), condition (12) is always satisfied because **(S1)** implies $M^*_{IF}(\Xi_q) \cap U_c = U_c$, $\forall q \in \mathcal{N}$. Hence if **(S1)** and **(S2)** are both satisfied, by Corollaries 3.4 and 3.5, there exists a unique impulse-free solution $x : [0, +\infty) \to U_c$ for any initial point $x_0 \in U_c$. Note that for the case that **(S1)** holds only on $U_c \setminus \{0\}$, the latter conclusion is still true if $x(t) = 0$ is the unique solution for $x_c = 0$.

To both linear and nonlinear DAEs, one can attach a class of control systems, called the explicitation of DAEs, which is a general framework to use control theory to solve DAE problems, see e.g., [8, 11, 12, 10, 9] for details. Now we recall the following notion of jump-flow explicitation [16], which is a control system, associated with any DAE being ex-equivalent to the **(INWF)**.

Definition 4.2 (jump-flow explicitation of DAEs)**.** Consider a DAE $\Xi = (E, F)$, assume that Ξ is ex-equivalent to the **(INWF)** of (6) via an invertible matrix $Q(x)$ and a diffeomorphism $\psi = (\psi_1, \psi_2) = (\xi_1, \xi_2)$ defined on X, the *jump-flow explicitation* of Ξ is the following nonlinear control system

$$\Sigma^e : \begin{cases} \dot{x} = f^e(x) + \sum_{i=1}^{m} g_i^e(x)v_i = f^e(x) + g^e(x)v, \\ y = h^e(x), \end{cases} \tag{16}$$

denoted by $\Sigma^e = (f^e, g^e, h^e)$, where $v \in \mathbb{R}^m$ is a vector of control inputs, $m = n - r = \dim \ker E$. The vector field $f^e : X \to TX$, the matrix valued-function $g^e : X \to \mathbb{R}^{n\times m}$ (whose columns $g_i^e : X \to TX$, $1 \le i \le m$ are vector fields) and $h^e : X \to \mathbb{R}^m$ are defined by

$$f^e := \left(\frac{\partial \psi}{\partial x}\right)^{-1} \left[\begin{smallmatrix} f^* \circ \psi_1 \\ 0 \end{smallmatrix}\right], \; g^e := \left(\frac{\partial \psi}{\partial x}\right)^{-1} \left[\begin{smallmatrix} 0 \\ I_m \end{smallmatrix}\right], \; h^e := \psi_2.$$

Remark 4.3. The vector f^e plays a similar role as the flow matrix $A^{\text{diff}} = P^{-1} \left[\begin{smallmatrix} A_1 & 0 \\ 0 & 0 \end{smallmatrix}\right] P$ for a linear DAE Δ, see e.g., [27], [53]. The ODE $\dot{x} = f^e(x)$, which is the zero dynamics of the control system Σ^e, has the same $\mathcal{C}^1$-solutions with the DAE Ξ. Moreover, because of $\operatorname{Im} g^e = \ker E$, any IFJ solution $J : [0, a] \to X$ of Ξ by Definition 2.4 can be seen as a solution of the control system $\frac{dJ(\tau)}{d\tau} = g^e(J(\tau))v(J(\tau))$ for a certain choice of input v which renders the solution $J(\tau)$ from $J(0) = x_0^- \in X$ to $x_0^+ = J(a) \in \mathfrak{C}$. It follows that the nonlinear consistency projector $\Omega_{E,F}$, given by (7), coincides with the flow map $\Phi_\tau^{v^e}$ of the vector field $v^e = g^e v$, i.e.,

$$x_0^+ = \Omega_{E,F}(x_0^-) = \Phi_a^{v^e}(x_0^-).$$

A particular choice of v is $v(x) = -h^e(x)$, i.e., $v^e = -g^e h^e$, then we have $a = \infty$ because the solution $J : [0, +\infty) \to X$ of $\frac{dJ}{d\tau} = -g^e h^e(J)$ (the latter is $\frac{d\xi_1}{d\tau} = 0$, $\frac{d\xi_2}{d\tau} = -\xi_2$ in (ξ_1, ξ_2)-coordinates) is an IFJ solution of Ξ. The impulse-free solution of Ξ for any initial point x_0 can be expressed as $x(t) = \Phi_t^{f^e} \circ \Omega_{E,F} \circ x_0$, where $\Phi_t^{f^e}$ is the flow map of the vector field f^e. Furthermore, the following properties hold for the jump-flow explicitation

$$\begin{aligned} &f^e \in \ker \mathrm{d}h^e, && \operatorname{Im} g^e \cap \ker \mathrm{d}h^e = 0, \\ &\mathrm{d}h^e \cdot g^e = I_m, && \dim(\operatorname{Im} g^e \oplus \ker \mathrm{d}h^e) = n. \end{aligned} \tag{17}$$

4.1 Stability analysis of switched DAEs via common Lyapunov functions

Given any internally regular DAE $\Xi = (E, F)$, if $F(0) = 0$, then $x_c = 0$ is clearly consistent and is also an *equilibrium* of Ξ, because $x(t) = 0$ is the only $\mathcal{C}^1$-solution passing through $x_c = 0$. For a switched DAE Ξ_σ, we make the following assumption to guarantee that $x_c = 0$ is a common equilibrium for all models $\Xi_p = (E_p, F_p)$:

(S3) the vector-valued functions $F_p(x)$ satisfy $F_p(0) = 0$, $\forall p \in \mathcal{N}$.

Consider a switched DAE Ξ_σ satisfying **(S3)** and a domain $\mathbb{D} \subseteq \mathbb{R}^n$ containing $x_c = 0$, fix a switching signal σ, suppose that for any initial point $x_0 \in \mathbb{D}$, the *impulse-free solution* $x : [0, +\infty) \to \mathbb{D}$ of Ξ_σ is well-defined.

Definition 4.4 (stability)**.** The equilibrium $x_c = 0$ is called *stable* if for any $\epsilon > 0$, there exists $\delta > 0$ such that the implication $||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \forall t > 0$ is true for all impulse-free solutions x of Ξ_σ; the DAE Ξ_σ is called *asymptotically stable* over $\mathbb{D}$ if $x_c = 0$ is stable and all impulse-free solutions on $\mathbb{D}$ converge to zero, or equivalently, if there exists $\beta : [0, \infty) \times [0, \infty) \to \mathcal{KL}$ such that $||x(t)|| \le \beta(||x_0||, t)$, $\forall t \ge 0$, $\forall x_0 \in \mathbb{D}$.

The following theorem is the "index-1" and "local" case of Theorem 15 in [16], the latter was given under the assumption that each DAE model Ξ_p is ex-equivalent to its **(NWF)** (see Corollary 3.4) on the whole generalized state space X. We will show in Example 4.9 below that with the help of the novel results in Proposition 4.8 below, the results of Theorem 4.5 can be also applied to switched DAEs with high-index models which are not necessarily ex-equivalent to the **(NWF)**.

Theorem 4.5. *For a switched nonlinear DAE Ξ_σ, given by (1), assume that there exists a neighborhood U_c of $x_c = 0$ such that* **(S1)**-**(S3)** *are satisfied on U_c. Let a control system $\Sigma_p^e = (f_p^e, g_p^e, h_p^e)$ be the jump-flow explicitation of the model Ξ_p for each $p \in \mathcal{N}$. Then the switched DAE Ξ_σ is asymptotically stable over U_c, uniformly for arbitrary switching signal σ if there exists a common $\mathcal{C}^1$-positive definite (Lyapunov) function $V : U_c \to [0, \infty)$ such that the level set $\mathcal{L}_a := \{x \in U_c \,|\, V(x) \le a\}$ is compact for every $a \in V(U_c)$ and $\forall p, q \in \mathcal{N}$:*

$$\frac{\partial V(x)}{\partial x} f_p^e(x) < 0, \quad \forall x \in (M^*(\Xi_p) \cap U_c) \setminus \{0\}, \tag{18}$$

$$\frac{\partial V(x)}{\partial x} v_p^e(x) \le 0, \quad \forall x \in M^*(\Xi_q) \cap U_c, \tag{19}$$

where $v_p^e := -g_p^e h_p^e$ is a vector field on U_c and $M^*(\Xi_p) \cap U_c = \left\{x \in U_c \,\middle|\, h_p^e(x) = 0\right\}$.

Conditions (18) and (19) mean that the Lyapunov function $V(x)$ decreases along the flow dynamics ($\mathcal{C}^1$-solutions) and the jump dynamics (IFJ solutions) of the model Ξ_p, respectively. It was shown in Lemma 16 of [16] that condition (19) is equivalent to condition (14) in Theorem 4.1 of [28]), i.e.,

$$V(\Omega_{E_p,F_p}(x)) - V(x) \le 0, \quad \forall x \in M^*(\Xi_q) \cap U_c, \tag{20}$$

where Ω_{E_p,F_p} is the nonlinear consistency projector of Ξ_p. The differences between Theorem 4.5 and Theorem 4.1 of [28], and the advantages of using jump-flow explicitation are explained in Remark 17 of [16]. We give the full proof of Theorem 4.5 in Section 5, which was absent in [16].

Any linear regular *index-1* DAE $\Delta = (E, H)$ is ex-equivalent (via two invertible constant matrices Q and P) to the Weierstrass form (9) with $N = 0$. The jump-flow explicitation of the linear DAE Δ is a linear control system $\Lambda^e = (A^e, B^e, C^e) : \dot{x} = A^e x + B^e u$, $y = C^e x$, where

$$A^e = P^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} P,\ B^e = P^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix},\ C^e = \begin{bmatrix} 0 & I_m \end{bmatrix} P. \tag{21}$$

By choosing a common Lyapunov function in the quadratic form $V(x) = x^T L x$, we can straightforwardly formulate the linear version of Theorem 4.5 as a linear matrices inequalities (LMIs) problem:

Corollary 4.6 (linear case)**.** *Consider a switched linear DAE Δ_σ of the form (2) with all models $\Delta_p = (E_p, H_p)$ being index-1 regular linear DAEs. For each $p \in \mathcal{N}$, let $\Lambda_p^e = (A_p^e, B_p^e, C_p^e)$ be the jump-flow explicitation of the model $\Delta_p = (E_p, H_p)$. Then Δ_σ is asymptotically stable under arbitrary switching signal σ if there exists a positive-definite matrix $L = L^T > 0$ such that*

$$\forall p, q \in \mathcal{N}: \quad \begin{cases} (\mathbf{C}_p^e)^T((A_p^e)^T L + L A_p^e)\mathbf{C}_p^e < 0 \\ (\mathbf{C}_q^e)^T(L B_p^e C_p^e + (B_p^e C_p^e)^T L)\mathbf{C}_q^e \geq 0, \end{cases}$$

where $\mathbf{C}_p^e$ is a full column rank matrix satisfying $\operatorname{Im} \mathbf{C}_p^e = \ker C_p^e$.

Example 4.7. Consider a switched electrical circuit shown in Figure 2 below. The circuit consists of a nonlinear resistor N, a nonlinear capacitor with voltage-related capacitance $C(v_c)$, an inductor with constant inductance L and a switching device S. Let

$$\xi = (x, y, z) = (i, v, v_c) \in X = \mathbb{R}^3$$

be the generalized states, where $i = x$ is the current and $v_C = z$ is the voltage of the capacitor and $v = y$ denotes the voltage between the nodes 1 and 2. The capacitance $C(v_c)$ and the characteristic of the nonlinear resistor $a(i_N, v_N) = 0$ are given by

$$C(v_C) = v_C^2 + 1, \quad a(i_N, v_N) = i_N - v_N^3 = 0.$$

Notice that we have $i - v^3 = x - y^3 = 0$ when S is open and $i_L = i - i_N = i - v^3 = x - y^3$ when S is closed. Using the Kirchhoff's laws, the circuit can be modeled by a switched nonlinear DAE Ξ_σ with two models Ξ_1 (representing that S is open) and Ξ_2 (representing that S is closed), where

$$\Xi_1 : \begin{bmatrix} 0 & 0 & C(z) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} x \\ x-y^3 \\ y+z \end{bmatrix}$$

and

$$\Xi_2 : \begin{bmatrix} 0 & 0 & C(z) \\ L & -3Ly^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} x \\ -R(x-y^3)+y \\ y+z \end{bmatrix}.$$

The two models are ex-equivalent on $U_c = X$ to their

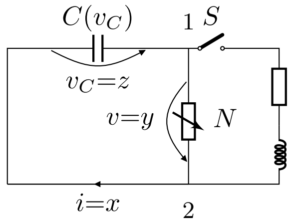

Fig. 2. A nonlinear switching electric circuit

(INWF), given by, respectively,

$$\tilde{\Xi}_1 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} \frac{-\tilde{z}^3}{\tilde{z}^2+1} \\ \tilde{x} \\ \tilde{y} \end{bmatrix}$$

and

$$\tilde{\Xi}_2 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} \frac{\tilde{x}-\tilde{z}^3}{\tilde{z}^2+1} \\ -L^{-1}R\tilde{x} - L^{-1}\tilde{z} \\ \tilde{y} \end{bmatrix}$$

via suitable invertible matrix-valued functions Q_1 and Q_2, and the following coordinates transformations

$$\psi_1 = (\tilde{z}, \tilde{x}, \tilde{y}) = (z, x - y^3, y + z) \quad \text{and} \quad \psi_2 = \psi_1.$$

Both Ξ_1 and Ξ_2 are ex-equivalent to their **(INWF)** on $U_c = X$ and satisfy conditions **(S1)**-**(S3)** on U_c. Then by Definition 4.2, we construct the jump-flow explicitation $\Sigma_1^e = (f_1^e, g_1^e, h_1^e)$ and $\Sigma_2^e = (f_2^e, g_2^e, h_2^e)$ of Ξ_1 and Ξ_2, respectively, where

$$f_1^e = \begin{bmatrix} -3y^2 \\ -1 \\ 1 \end{bmatrix} \cdot \frac{-z^3}{z^2+1},\ g_1^e = \begin{bmatrix} 1 & 3y^2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ h_1^e = \begin{bmatrix} x-y^3 \\ y+z \end{bmatrix},$$
$$f_2^e = \begin{bmatrix} -3y^2 \\ -1 \\ 1 \end{bmatrix} \cdot \frac{x-y^3-z^3}{z^2+1} + \begin{bmatrix} \frac{-R(x-y^3)-z}{L} \\ 0 \\ 0 \end{bmatrix},$$
$$g_2^e = \begin{bmatrix} 3y^2 \\ 1 \\ 0 \end{bmatrix},\ h_2^e = y+z.$$

Consider the following common Lyapunov function candidate defined on $U_c = X$:

$$V(\xi) = V(x, y, z) = \frac{R}{4}z^4 + \frac{R}{2}z^2 + \frac{L}{2}(x-y^3)^2 + \frac{1}{2}(y+z)^2.$$

Define $v_1^e := -g_1^e h_1^e$ and $v_2^e := -g_2^e h_2^e$, it follows that $L_{f_1^e}V(\xi) = \frac{\partial V(\xi)}{\partial \xi} f_1^e(\xi) = -Rz^4$, $L_{v_1^e}V(\xi) = \frac{\partial V(\xi)}{\partial \xi} v_1^e(\xi) = -L(x - y^3)^2 - (y + z)^2$, $L_{f_2^e}V(\xi) = \frac{\partial V(\xi)}{\partial \xi} f_2^e(\xi) = -R(x-y^3)^2 - Rz^4$, $L_{v_2^e}V(\xi) = -(y+z)^2$. Thus by $M^*(\Xi_1) = \left\{\xi \in X \,\middle|\, x - y^3 = y + z = 0\right\}$ and $M^*(\Xi_2) = \{\xi \in X \mid y + z = 0\}$, we get

$$L_{f_1^e}V(\xi)|_{M^*(\Xi_1)} = -Rz^4 < 0,\ \forall \xi \in M^*(\Xi_1) \setminus \{0\},$$
$$L_{v_1^e}V(\xi)|_{M^*(\Xi_2)} = -L(x - y^3)^2 \leq 0,\ \forall \xi \in M^*(\Xi_2),$$
$$L_{f_2^e}V(\xi)|_{M^*(\Xi_2)} < 0,\ \forall \xi \in M^*(\Xi_2) \setminus \{0\},$$
$$L_{v_2^e}V(\xi)|_{M^*(\Xi_1)} = 0,\ \forall \xi \in M^*(\Xi_1).$$

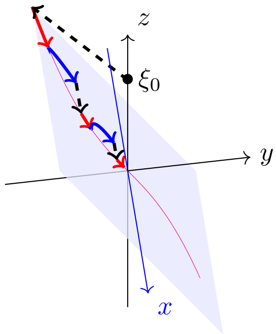

Fig. 3. Magenta curve: $M^*(\Xi_1)$, light blue surface: $M^*(\Xi_2)$, dark red curve: $\mathcal{C}^1$-solutions of Ξ_1, dark blue curve: $\mathcal{C}^1$-solutions of Ξ_2, dashed lines: IFJ solutions.

It follows that conditions (18) and (19) of Theorem 4.5 are satisfied on $U_c = X$. Hence, the switched DAE Ξ_σ is globally asymptotically stable, uniformly for arbitrary switching signal σ. For example, let $L = R = 1$, we take an initial point $\xi_0 = (0, 0, 1)$ (which is not consistent for both Ξ_1 and Ξ_2) and choose a periodical switched signal σ with the period $T = 0.4$ and $\sigma(0) = 1$, the impulse-free solution of Ξ_σ starting from ξ_0 is drawn in Figure 3.

Now we show how to use the results in Section 3 and Theorem 4.5 to check the stability of high-index DAEs which may not be ex-equivalent to the **(NWF)**.

Proposition 4.8 (index-reduction)**.** *Consider the switched DAE Ξ_σ in Corollary 3.5. Assume additionally that $F_p(0) = 0$ and $\mathcal{I} = [0, \infty)$. Then, there exists another switched DAE $\hat{\Xi}_\sigma$ defined on the neighborhood U_c of $x_c = 0$ such that each model $\hat{\Xi}_p$ of $\hat{\Xi}_\sigma$ is in the* **(INWF)** *and the two switched DAEs $\hat{\Xi}_\sigma$ and Ξ_σ have the same impulse-free solution $x(\cdot)$ for any initial point $x_0 \in M^*_{IF}(\Xi_{\sigma(0)}) \cap U_c$. Moreover, we have that $\hat{\Xi}_\sigma$ satisfies conditions **(S1)-(S3)** and the solution $x(\cdot)$ is asymptotically stable over U_c if conditions (18) and (19) are satisfied for the jump-flow explicitations of the models of $\hat{\Xi}_\sigma$.*

Proof. Consider the DAE Ξ in Theorem 3.3 and the following index-1 DAE $\hat{\Xi}$ defined on U_c, given by

$$\bar{\Xi} : \begin{cases} \dot{\xi}_1 = \tilde{F}_1(\xi_1,0,0) \\ 0 = \xi_2 \\ 0 = \xi_3 \end{cases} \overset{\psi(x)=\xi}{\Rightarrow} \hat{\Xi} : \begin{cases} \frac{\partial \psi_1(x)}{\partial x}\dot{x} = \tilde{F}(\psi_1(x),0,0) \\ 0 = \psi_2(x) \\ 0 = \psi_3(x), \end{cases}$$

where $\psi = (\psi_1, \psi_2, \psi_3) = (\xi_1, \xi_2, \xi_3)$ and $\bar{\Xi}$ is constructed from (29) and is in the **(INWF)**. Then Ξ and $\hat{\Xi}$ have the same impulse-free solution for any initial point $x_0 \in M^*_{IF} \cap U_c = \mathfrak{C}_{IF} \cap U_c$. Indeed, recall from the proof of Theorem 3.3 that Ξ is ex-equivalent to $\tilde{\Xi}$, given by (29). Notice that $\tilde{\Xi}$ and $\bar{\Xi}$ have the same $\mathcal{C}^1$-solutions $\xi(t) = (\xi_1(t), 0, 0)$ for any initial point $(\xi_{10}^+, 0, 0) \in \psi(M^* \cap U_c)$, where $\xi_1(t)$ is a solution of the ODE $\dot{\xi}_1 = \tilde{F}_1(\xi_1, 0, 0)$, and the same IFJ: $(\xi_{10}^-, 0, \xi_{30}^-) \to (\xi_{10}^-, 0, 0)$ for any initial point $(\xi_{10}^-, 0, \xi_{30}^-) \in \psi(M^*_{IF} \cap U_c)$, so $\tilde{\Xi}$ and $\bar{\Xi}$ have the same impulse-free solution for any initial point $\xi_0 \in \psi(M^*_{IF} \cap U_c)$. The ex-equivalence preserves both $\mathcal{C}^1$-solutions and impulse-free jumps (see Remark 2.6), so the ex-equivalent DAEs Ξ and $\tilde{\Xi}$, and also $\bar{\Xi}$ and $\hat{\Xi}$, have corresponding impulse-free solutions. Therefore, Ξ and $\hat{\Xi}$, which are both represented in x-coordinates, have the same impulse-free solutions for any initial point $x_0 \in M^*_{IF} \cap U_c$.

Using the method above, for each models Ξ_p of Ξ_σ, we can construct a DAE $\hat{\Xi}_p$ which has the same impulse-free solutions with Ξ_p. Let $\hat{\Xi}_\sigma$ be a switched DAE with models $\hat{\Xi}_p$ and with the same switching signal $\sigma(t)$ as Ξ_σ. Then $x(\cdot)$ is the impulse-free solution of Ξ_σ starting from $x_0 \in M^*_{IF}(\Xi_{\sigma(0)}) \cap U_c$ if and only if it is that of $\hat{\Xi}_\sigma$. Clearly, $\hat{\Xi}_\sigma$ satisfies **(S1)-(S3)**, we can check its asymptotically stability by (18) and (19) of Theorem 4.5, which would imply the asymptotically stability of any solution $x(\cdot)$ of Ξ_σ. □

Example 4.9 (continuation of Example 3.7)**.** Consider the switched DAE Ξ_σ in Example 3.7. The DAE Ξ_1 is index-2 and is not ex-equivalent to the **(NWF)**. Using the method in Proposition 4.8, we construct a DAE $\bar{\Xi}_1$ from (14) and transform it into $\hat{\Xi}_1$:

$$\bar{\Xi}_1 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} -\tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} \overset{x=\psi_1^{-1}(\tilde{x})}{\Rightarrow}$$

$$\hat{\Xi}_1 : \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2+x_1x_3 \\ x_3 \end{bmatrix}.$$

The DAE $\hat{\Xi}_1$ has the same impulse-free solution with Ξ_1 for any initial point $x_0 \in M^*_{IF}(\Xi_1) \cap U_1$. Now $\hat{\Xi}_1$ and Ξ_2 are ex-equivalent to $\bar{\Xi}_1$ and $\bar{\Xi}_2$ (see (15)), respectively, on $U_c = U_1 \cap U_2 = \{x \in \mathbb{R}^3 \mid x_1 > -1,\ x_1 + x_3 > -1\}$, and $\bar{\Xi}_1$ and $\bar{\Xi}_2$ are both in **(INWF)**. It can be seen that conditions **(S1)-(S3)** are satisfied on U_c for the switched system $\hat{\Xi}_\sigma$ with models $\hat{\Xi}_1$ and $\hat{\Xi}_2 = \Xi_2$. By Definition 4.2, we construct the jump-flow explicitation systems $\Sigma_1^e = (f_1^e, g_1^e, h_1^e)$ and $\Sigma_2^e = (f_2^e, g_2^e, h_2^e)$ for $\hat{\Xi}_1$ and Ξ_2, respectively, where $f_1^e(x) = \begin{bmatrix} -x_1 \\ x_1x_3 \\ 0 \end{bmatrix}$, $g_1^e(x) = \begin{bmatrix} 0 & -x_1 \\ 1 & x_1x_3-x_1 \\ 0 & 1 \end{bmatrix}$, $h_1^e(x) = \begin{bmatrix} x_2+x_1x_3 \\ x_3 \end{bmatrix}$, $f_2^e(x) = \frac{-x_1}{x_1+1}\begin{bmatrix} 1 \\ x_1-x_3 \\ -1 \end{bmatrix}$, $g_2^e(x) = \begin{bmatrix} 0 & 0 \\ 1 & -x_1 \\ 0 & 1 \end{bmatrix}$, $h_2^e(x) = \begin{bmatrix} x_2+x_1x_3 \\ x_1+x_3 \end{bmatrix}$. Thus $v_1^e := -g_1^eh_1^e = \begin{bmatrix} x_1x_3 \\ -x_2-x_1(x_3)^2 \\ -x_3 \end{bmatrix}$ and $v_2^e := -g_2^eh_2^e = \begin{bmatrix} 0 \\ (x_1)^2-x_2 \\ -x_1-x_3 \end{bmatrix}$. Choose the following Lyapunov function candidate

$$V(x) = \frac{1}{2}(x_1 + x_3)^2 + \frac{1}{2}(x_2 + x_1x_3)^2 + \frac{1}{2}(x_3)^2.$$

It follows that $L_{f_1^e}V(x)|_{M^*(\Xi_1)} = -x_1^2 < 0$, $\forall x \in (M^*(\Xi_1) \cap U_c) \setminus \{0\}$; $L_{v_1^e}V(x)|_{M^*(\Xi_2)} = -(x_3)^2 \leq 0$, $\forall x \in M^*(\Xi_2) \cap U_c$; $L_{f_2^e}V(x)|_{M^*(\Xi_2)} = -\frac{(x_1)^2}{x_1+1} < 0$, $\forall x \in (M^*(\Xi_2) \cap U_c) \setminus \{0\}$; $L_{v_2^e}V(x)|_{M^*(\Xi_1)} = -(x_1+x_3)^2 \leq 0$, $\forall x \in M^*(\Xi_1) \cap U_c$. Hence (18) and (19) hold, we have that $\hat{\Xi}_\sigma$ and thus Ξ_σ are asymptotically stable over U_c under arbitrary switching signals for any initial point $x_0 \in M^*_{IF}(\Xi_{\sigma(0)}) \cap U_c$.

4.2 Commutativity and invariance conditions for switched nonlinear DAEs

It is well-known (see [32, 24]) that for a switched nonlinear ODE $\dot{x} = f_\sigma(x)$ with all models being asymptotically stable, if

$$\forall p, q \in \mathcal{N} : [f_p, f_q] := \frac{\partial f_q}{\partial x} f_p - \frac{\partial f_p}{\partial x} f_q = 0,$$

then the switched ODE is asymptotically stable for arbitrary switching signal σ. In this section, we discuss how to generalize the above commutativity condition to switched nonlinear DAEs. The results in [29] show that for a switched linear DAE Δ_σ, given by (2), with all models being regular and asymptotically stable, the commutativity of the flow matrices A^{diff} (i.e., A^e of (21)) for each model, i.e.,

$$\forall p, q \in \mathcal{N} : [A_p^e, A_q^e] = A_q^e A_p^e - A_p^e A_q^e = 0, \tag{22}$$

implies the asymptotical stability of Δ_σ under arbitrary switching signal σ. We will show in the following theorem that for a switched nonlinear DAE Ξ_σ, not only commutativity conditions (i.e., (23)) but also certain invariant distributions conditions (i.e., (24)-(25)) are required to guarantee the asymptotically stability of Ξ_σ under arbitrary switching signal.

Theorem 4.10 (commutativity and invariance conditions)**.** *Consider a switched nonlinear DAE Ξ_σ, given by (1). Assume that there exists a neighborhood U_c of $x_c = 0$ such that* **(S1)**-**(S3)** *are satisfied on U_c. Suppose that each model Ξ_p of Ξ_σ is asymptotically stable over U_c. Then Ξ_σ is asymptotically stable, uniformly for arbitrary switching signal σ, over U_c, if $\forall p, q \in \mathcal{N}$:*

$$[f_p^e, f_q^e] = 0, \tag{23}$$

$$[f_p^e, \mathcal{G}_q^e] \subseteq \mathcal{G}_q^e, \quad [f_p^e, \mathcal{H}_q^e] \subseteq \mathcal{H}_q^e, \tag{24}$$

$$(g_p^e \cdot \mathrm{d}h_p^e) \cdot \mathcal{G}_q^e \subseteq \mathcal{G}_q^e, \quad (g_p^e \cdot \mathrm{d}h_p^e) \cdot \mathcal{H}_q^e \subseteq \mathcal{H}_q^e, \tag{25}$$

where $f_p^e : U_c \to TU_c$, $g_p^e : U_c \to \mathbb{R}^{n \times m_p}$ and $h_p^e : U_c \to \mathbb{R}^{m_p \times n}$ are from the jump-flow explicitation $\Sigma_p^e = (f_p^e, g_p^e, h_p^e)$, given by (16), of the model Ξ_p, and where $\mathcal{G}_p^e = \operatorname{Im} g_p^e = \ker E_p$ and $\mathcal{H}_p^e = \ker \mathrm{d}h_p^e$ are distributions.

The following lemma shows that (25) can be replaced by condition (26) below, the latter is crucial for proving Theorem 4.10.

Lemma 4.11. *Condition (25) is equivalent to*

$$\begin{aligned} (\mathcal{G}_p^e \cap \mathcal{G}_q^e) \oplus (\mathcal{H}_p^e \cap \mathcal{G}_q^e) = \mathcal{G}_q^e, \\ (\mathcal{G}_p^e \cap \mathcal{H}_q^e) \oplus (\mathcal{H}_p^e \cap \mathcal{H}_q^e) = \mathcal{H}_q^e. \end{aligned} \tag{26}$$

The proofs of Lemma 4.11 and Theorem 4.10 are given in Section 5.

Remark 4.12. For a switched linear DAE Δ_σ with all models $\Delta_p = (E_p, H_p)$, $p \in \mathcal{N}$, being index-1, regular, and asymptotically stable, the linear commutativity condition (22) implies the linear version of the invariance conditions (24)-(25), i.e., $\forall p, q \in \mathcal{N} : A_p^e \cdot \mathcal{B}_q^e \subseteq \mathcal{B}_q^e$, $A_p^e \cdot \mathcal{C}_q^e \subseteq \mathcal{C}_q^e$, $B_p^e C_p^e \cdot \mathcal{B}_q^e \subseteq \mathcal{B}_q^e$, $B_p^e C_p^e \cdot \mathcal{C}_q^e \subseteq \mathcal{C}_q^e$, where A_p^e, B_p^e, C_p^e are system matrices of the jump-flow explicitation of Δ_p, given by (21), the subspaces $\mathcal{B}_p^e = \operatorname{Im} B_p^e$ and $\mathcal{C}_p^e = \ker C_p^e$. Indeed, we know from Lemma 9 of [29] that (22) implies $\forall p, q \in \mathcal{N} : [A_p^e, \Pi_{E_q, H_q}] = [\Pi_{E_p, H_p}, \Pi_{E_q, H_q}] = 0$. Moreover, we have $\forall p \in \mathcal{N}$: $\Pi_{E_p, H_p} = I_n - B_p^e C_p^e$ by (10) and (21). Then by a direct calculation, we get

$$\forall p, q \in \mathcal{N} : \quad [A_p^e, B_q^e C_q^e] = [B_p^e C_p^e, B_q^e C_q^e] = 0.$$

Recall by constructions that $\mathcal{B}_p^e = \operatorname{Im} B_p^e C_p^e$ and $\mathcal{C}_p^e = \ker B_p^e C_p^e$. So by $A_p^e \cdot B_q^e C_q^e = B_q^e C_q^e \cdot A_p^e$, we have $A_p^e \cdot \mathcal{B}_q^e = \operatorname{Im} B_q^e C_q^e \cdot A_p^e \subseteq \mathcal{B}_q^e$ and $\{0\} = B_q^e C_q^e \cdot A_p^e \cdot \mathcal{C}_q^e \Rightarrow A_p^e \cdot \mathcal{C}_q^e \subseteq \mathcal{C}_q^e$. Similarly, the condition $[B_p^e C_p^e, B_q^e C_q^e] = 0$ implies $B_p^e C_p^e \cdot \mathcal{B}_q^c \subseteq \mathcal{B}_q^c$ and $B_p^e C_p^e \cdot \mathcal{C}_q^c \subseteq \mathcal{C}_q^c$.

It is known (see e.g., [32, 56]) that for pairwise commuting asymptotically stable nonlinear ODEs

$$\dot{x} = f_p(x), \quad p \in \mathcal{N}, \tag{27}$$

it is possible to find a common Lyapunov function. In particular, assume that the family of systems in (27) is defined on a ball $B_r := \{x \in \mathbb{R}^n \,|\, ||x|| \leq r\}$. Then there exist $r_0 \in (0, r)$ and a positive-definite $\mathcal{C}^1$-(Lyapunov) function $V(x)$ such that $\mathcal{L}_a := \{x \in B_{r_0} \,|\, V(x) \leq a\}$ is compact for every $a \in V(B_{r_0})$ and $\frac{\partial V(x)}{\partial x} f_p(x) < 0$, $\forall p \in \mathcal{N}$, $\forall x \in B_{r_0}/\{0\}$ (see Theorem 4 of [56]). We now use the latter result to construct Lyapunov functions for asymptotically stable switched nonlinear DAEs satisfying the commutativity and invariance conditions of Theorem 4.10.

Corollary 4.13 (converse Lyapunov theorem)**.** *Consider the switched DAE Ξ_σ satisfying* **(S1)**-**(S3)** *on a neighborhood U_c of $x_c = 0$. Suppose that the jump-flow explicitation $\Sigma_p = (f_p^e, g_p^e, h_p^e)$ of each model Ξ_p satisfies the commutativity and invariance conditions (23)-(25) on U_c. Assume that all models $\Xi_p = (E_p, H_p)$ are* *asymptotically stable on a ball $B_r \subseteq U_c$. Then there exist $r_0 \in (0, r)$ and a positive-definite $\mathcal{C}^1$-(Lyapunov) function $V(x)$ such that $\mathcal{L}_a := \{x \in B_{r_0} \,|\, V(x) \le a\}$ is compact for every $a \in V(U_c)$ and satisfying (18)-(19) of Theorem 4.5 on B_{r_0}.*

The proof is given in Section 5. We now illustrate the above results by two examples.

Example 4.14. Consider a switched DAE Ξ_σ defined on $X = \mathbb{R}^2$ with two models $\Xi_1 = (E_1, F_1)$ and $\Xi_2 = (E_2, F_2)$, where $\Xi_1 : \left[\begin{smallmatrix} 0 & C \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} \dot{x} \\ \dot{y} \end{smallmatrix}\right] = \left[\begin{smallmatrix} -x \\ x-y^3 \end{smallmatrix}\right]$ and $\Xi_2 : \left[\begin{smallmatrix} L & -3Ly^2 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} \dot{x} \\ \dot{y} \end{smallmatrix}\right] = \left[\begin{smallmatrix} y-R(x-y^2) \\ y \end{smallmatrix}\right]$, where C, L and R are all positive constant scalars. Clearly, assumptions **(S1)-(S3)** are satisfied globally, in fact, Ξ_1 and Ξ_2 are ex-equivalent to, respectively, the following two DAEs $\tilde{\Xi}_1$ and $\tilde{\Xi}_2$ represented in the **(INWF)**, via the same coordinates transformation $(\tilde{x}, \tilde{y}) = \psi = (x - y^3, y)$,

$$\tilde{\Xi}_1 : \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\begin{bmatrix} \dot{\tilde{y}} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} -\tilde{y}^3/C \\ \tilde{x} \end{bmatrix}, \quad \tilde{\Xi}_2 : \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} -R\tilde{x}/L \\ \tilde{y} \end{bmatrix}.$$

The jump-flow explicitations of Ξ_1 and Ξ_2 are, respectively, $\Sigma_1^e = (f_1^e, g_1^e, h_1^e)$ and $\Sigma_2^e = (f_2^e, g_2^e, h_2^e)$, given by $f_1^e = \left(\frac{\partial\psi}{\partial x}\right)^{-1}\begin{bmatrix} 0 \\ -y^3/C \end{bmatrix}$, $g_1^e = \left(\frac{\partial\psi}{\partial x}\right)^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $h_1^e = x - y^3$ $f_2^e = \left(\frac{\partial\psi}{\partial x}\right)^{-1}\begin{bmatrix} -R(x-y^3)/L \\ 0 \end{bmatrix}$, $g_2^e = \left(\frac{\partial\psi}{\partial x}\right)^{-1}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $h_2^e = y$. Observe that for our systems, $\mathcal{G}_1^e = \operatorname{Im} g_1^e$ coincides with $\mathcal{H}_2^e = \ker \mathrm{d}h_2^e$ and $\mathcal{H}_1^e = \ker \mathrm{d}h_1^e$ coincides with $\mathcal{G}_2^e = \operatorname{Im} g_2^e$. Then it is easy to verify that conditions (23)-(25) are all satisfied. Since both Ξ_1 and Ξ_2 are asymptotically stable, we conclude by Theorem 4.10 that Ξ is asymptotically stable under arbitrary switching signal. Moreover, we can choose the common Lyapunov function $V(x, y) = \frac{1}{2}y^2 + \frac{1}{2}(x - y^3)^2$. It can be checked that $V(x, y)$ satisfies conditions (18) and (19) of Theorem 4.5.

Note that the above switched DAE Ξ_σ is an academic example, we show below that it can be easily realized by slightly modifying the electrical circuit shown in Example 4.7, we change the nonlinear capacitance $C(y)$ to a constant one C and add an additional switching devices S_1 parallel to the capacitor (although to short-circuit the capacitor may not have a strong practical meaning for real electrical circuits). The switches S and S_1 are required to be simultaneously open or closed.

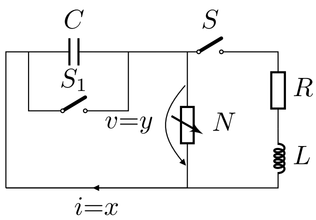

Fig. 4. The modified nonlinear switching electric circuit

The second example is to show the importance of the invariance conditions (24)-(25), a nonlinear switched DAE satisfies (23) but not (24)-(25) could be unstable (which is different from the linear case, see Remark 4.12).

Example 4.15. Consider a switched DAE Ξ_σ defined on $\mathbb{R}^2$ with two models $\Xi_1 = (E_1, F_1) : \left[\begin{smallmatrix} \dot{x} \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -\sqrt{x} \\ y \end{smallmatrix}\right]$ and $\Xi_2 = (E_2, F_2) : \begin{bmatrix} \dot{\psi}_1(x,y) \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{\psi_1(x,y)} \\ \psi_2(x,y) \end{bmatrix}$, where $\psi_1(x, y) = 2e^{2\sqrt{x}} - e^{-2y} - 1$ and $\psi_2(x, y) = \sqrt{x} + y$. Consider Ξ_σ on the set $U_c = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x \ge 0, y \le 0\right\}$. The mode Ξ_1 is already in **(INWF)** and Ξ_2 can transformed into $\left[\begin{smallmatrix} \dot{\tilde{x}} \\ 0 \end{smallmatrix}\right] = \begin{bmatrix} -\sqrt{\tilde{x}} \\ \tilde{y} \end{bmatrix}$ via $(\tilde{x}, \tilde{y}) = \psi = (\psi_1, \psi_2)$ on $U_c \setminus \{0\}$ (so $x_c = 0$ is a singular point for the ex-equivalence but the impulse-free solution exist for all $x \in U_c$, see Remark 4.1). The jump-flow explications are given by $\Sigma_1^e = (f_1^e, g_1^e, h_1^e)$ and $\Sigma_2^e = (f_2^e, g_2^e, h_2^e)$, where $f_1^e = \begin{bmatrix} -\sqrt{x} \\ 0 \end{bmatrix}$, $g_1^e = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $h_1^e = y$, $f_2^e = \begin{bmatrix} -\sqrt{x} \\ 0.5 \end{bmatrix}$, $g_2^e = \frac{1}{e^{2\sqrt{x}} - e^{-2y}} \cdot \begin{bmatrix} -\sqrt{x}e^{-2y} \\ e^{2\sqrt{x}} \end{bmatrix}$, $h_2^e = \psi_2$. It can checked that condition (23) is satisfied but (24)-(25) do not hold. Both Ξ_1 and Ξ_2 are asymptotically stable on U_c but it can be seen from the following figure that Ξ_σ can be unstabilized via impulse-free jumps (note that Ξ_σ remains unstable for switching signals with small enough dwell-time).

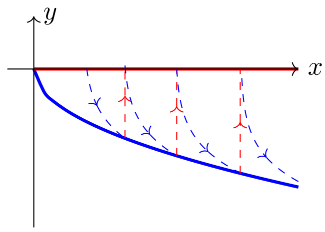

Fig. 5. x-axis: $M^*(\Xi_1)$, blue curve: $M^*(\Xi_2)$, dashed blue curve: jumps of Ξ_1, dashed blue curve: jumps of Ξ_2.

5 Proof of the results

Proof of Theorem 3.3. Since the distributions $\mathcal{D}(x)$, $\ker E(x)$ and $\mathcal{D}(x) + \ker E(x)$ are all of constant dimension on U by **(D1)**, we have $\dim(\mathcal{D}(x) \cap \ker E(x)) = \dim \mathcal{D}(x) + \dim \ker E(x) - \dim(\mathcal{D}(x) + \ker E(x)) = const.$ (by e.g., Theorem 2.3.1 of [5]) and thus $\dim E(x)\mathcal{D}(x) = const.$, for all $x \in U$. Then by **(D2)**, we have that $\dim E(x)T_xM^* = \dim E(x)D(x) = const.$ for all $x \in M^* \cap U$. Observe that $\dim E(x)T_xM^* = \dim M^*$ by **(RE)** and Proposition 2.3(ii). Thus we have $\dim E(x)D(x) = \dim D(x)$ on U, which implies that $\ker E(x) \cap \mathcal{D}(x) = 0$ for all $x \in U$. Since the distributions $\mathcal{D}(x)$, $\ker E(x)$ and $\mathcal{D}(x) + \ker E(x)$ are all involutive, by Frobenius theorem (see e.g., [23]), there exist a neighborhood $U_c \subseteq U$ and smooth maps $\xi_1 : U_c \to U_{c1} \subseteq \mathbb{R}^{n_1}$,

$\xi_2 : U_c \to U_{c2} \subseteq \mathbb{R}^{n_2}$ and $\xi_3 : U_c \to U_{c3} \subseteq \mathbb{R}^{n_3}$ such that

$$\begin{aligned} \operatorname{span}\left\{\mathrm{d}\xi_2^1, \ldots, \mathrm{d}\xi_2^{n_2}\right\} &= (\mathcal{D} + \ker E)^\perp, \\ \operatorname{span}\left\{\mathrm{d}\xi_2^1, \ldots, \mathrm{d}\xi_2^{n_2}, \mathrm{d}\xi_3^1, \ldots, \mathrm{d}\xi_3^{n_3}\right\} &= \mathcal{D}^\perp, \\ \operatorname{span}\left\{\mathrm{d}\xi_1^1, \ldots, \mathrm{d}\xi_1^{n_1}, \mathrm{d}\xi_2^1, \ldots, \mathrm{d}\xi_2^{n_2}\right\} &= (\ker E)^\perp, \end{aligned} \tag{28}$$

and $\xi_2(x_c) = 0$, $\xi_3(x_c) = 0$, where $\perp$ denotes the left annihilator of a distribution, the functions ξ_i^j, $1 \le i \le 3$, $1 \le j \le n_i$, are the rows of the vector ξ_i, where $n_1 = \dim \mathcal{D}$, $n_3 = \dim \ker E$ and $n_2 = n - (n_1 + n_3)$. Now by $\ker E \cap \mathcal{D} = 0$, we have

$$\operatorname{span}\left\{\mathrm{d}\xi_i^j,\ 1 \le i \le 3,\ 1 \le j \le n_i\right\} = T^* U_c,$$

where T^*U_c denotes the cotangent bundle of U_c, thus $\xi = (\xi_1, \xi_2, \xi_3)$ are local coordinates and $\psi = \xi$ is a local diffeomorphism on U_c. Then via ψ, the DAE Ξ is locally on U_c ex-equivalent to

$$\begin{bmatrix} \tilde{E}_1(\xi) & \tilde{E}_2(\xi) & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \tilde{F}(\xi),$$

where $\begin{bmatrix} \tilde{E}_1 \circ \psi & \tilde{E}_2 \circ \psi & \tilde{E}_3 \circ \psi \end{bmatrix} = E \left(\frac{\partial \psi}{\partial x}\right)^{-1}$ with $\tilde{E}_3 \circ \psi \equiv 0$ and $\tilde{F} \circ \psi = F$. Note that $\tilde{E}_3 \circ \psi \equiv 0$ because $\operatorname{Im} \tilde{E}_3 = E \ker \begin{bmatrix} \mathrm{d}\xi_1 \\ \mathrm{d}\xi_2 \end{bmatrix} = 0$ by (28). Now since $\operatorname{rank} E(x) = const. = n - n_3$, there exists $Q : \psi(U_c) \to GL(n, \mathbb{R})$ such that

$$\begin{aligned} & Q(\xi) \begin{bmatrix} \tilde{E}_1(\xi) & \tilde{E}_2(\xi) & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = Q(\xi) \tilde{F}(\xi) \\ \Leftrightarrow\ & \tilde{\Xi} : \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1(\xi_1, \xi_2, \xi_3) \\ \tilde{F}_2(\xi_1, \xi_2, \xi_3) \\ \tilde{F}_3(\xi_1, \xi_2, \xi_3) \end{bmatrix}. \end{aligned} \tag{29}$$

Notice that by **(D2)**, we have

$$\psi(M^* \cap U_c) = \{\xi \in \psi(U_c) \,|\, \xi_2 = 0,\ \xi_3 = 0\}.$$

Now by taking a smaller U_c if necessary [5], given any initial point $\xi_0^- = (\xi_{10}^-, \xi_{20}^-, \xi_{30}^-) \in \psi(U_c)$, there exists an IFJ of $\tilde{\Xi}$ of (29) starting from ξ_0^- if and only if $\xi_{20}^- = 0$. The latter conclusion comes from Definition 2.4, since by which the direction of the IFJs of $\tilde{\Xi}$ should stay in $\ker \tilde{E} = \operatorname{span}\left\{\frac{\partial}{\partial \xi_3^1}, \ldots, \frac{\partial}{\xi_3^{n_3}}\right\}$, i.e., only ξ_3-variables are allowed to jump. Moreover, from any initial point $\xi_0^- = (\xi_{10}^-, 0, \xi_{30}^-)$, there exists a unique IFJ $\xi_0^- \to \xi_0^+ = (\xi_{10}^+, 0, 0) \in \psi(M^* \cap U_c)$ with $\xi_{10}^+ = \xi_{10}^-$. Thus by Definition 3.2, the impulse-free consistency set $\mathfrak{C}_{IF}(\tilde{\Xi}) = \{\xi \in \psi(U_c) \,|\, \xi_2 = 0\}$. Since the ex-equivalence preserves both $\mathcal{C}^1$-solutions and IFJs (see Remark 2.6), for the original DAE Ξ, we have

$$\mathfrak{C}_{IF} \cap U_c = \{x \in U_c \,|\, \xi_2(x) = 0\} = M_{IF}^* \cap U_c.$$

Clearly, M_{IF}^* is the integral submanifold of $\mathcal{D}(x) + \ker E(x)$ passing through x_c because by construction (notice that $\xi_2(x_c) = 0$) there exists a unique IFJ $x_0^- = \psi^{-1}(\xi_0^-) \to x_0^+ = \psi^{-1}(\xi_0^+)$ for any initial point $x_0^- = \psi^{-1}(\xi_0^-) \in M_{IF}^* \cap U_c$. $\square$

Proof of Theorem 4.5. We show that $t \mapsto V(x(t))$ is monotonically decreasing for any jump-flow solution $x : [0, \infty) \to U_c$ of Ξ_σ. Let $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ be the switching times of the switching signal σ and let $\mathcal{I}_i := (t_i, t_{i+1})$. On each interval $\mathcal{I}_i \subseteq [0, \infty)$, $x(\cdot)$ is a $\mathcal{C}^1$-solution of the model Ξ_p, where $p = \sigma(t)$ for any $t \in \mathcal{I}_i$, so $x(\cdot)$ is also a solution of the ODE $\dot{x} = f_p^e(x)$ defined on $M^*(\Xi_p)$ (see Remark 4.3). By (18), we have $\dot{V}(x(t)) = \frac{\partial V}{\partial x} f_p^e(x(t)) < 0$, $\forall t \in \mathcal{I}_i$. For any switching time t_i, denote $q = \sigma(t_i^-)$ and $p = \sigma(t_i^+)$, then $x(t_i^-) \in M^*(\Xi_q) \cap U_c$ and $x(t_i^+) = \Omega_{E_p, F_p}(x(t_i^-)) \in M^*(\Xi_p) \cap U_c$, thus by (20), we have $V(x(t_i^+)) - V(x(t_i^-)) \le 0$. Hence $t \mapsto V(x(t))$ is decreasing on the whole interval $[0, \infty)$.

Step 2: We show $x_c = 0$ is stable. Fix $\epsilon > 0$, choose $r \in (0, \epsilon]$ and let $B_r := \{x \in U_c \,|\, ||x|| \le r\}$. Then we prove that there exists $\beta_r > 0$ depending on r such that the set $\mathcal{L}_{\beta_r} := \{x \in U_c \,|\, V(x) \le \beta_r\}$ is strictly contained in B_r, i.e., $\mathcal{L}_{\beta_r} \subsetneq B_r$. Assume the contrary, i.e., for all $\beta_r > 0$, there exists $x \in \mathcal{L}_{\beta_r}$ satisfying $||x|| \ge r$, which implies that there exists a sequence $(x_n)_{n \in \mathbb{N}} \in \mathcal{L}_{\frac{1}{n}}$ such that $||x_n|| \ge r$. By construction, $\lim\limits_{n \to \infty} V(x_n) = 0$. Moreover, since $\mathcal{L}_{\frac{1}{n}}$ is compact by assumption (for sufficiently large n), there exists a subsequence of (x_n) whose limit x^* exists and satisfies $||x^*|| \ge r$. Then we get $\lim\limits_{n \to \infty} V(x_n) = V(x^*) > 0$, which is a contradiction, so we can choose $\beta_r > 0$ such that $\mathcal{L}_{\beta_r} \subsetneq B_r$. Recall from Step 1 that $t \mapsto V(x(t))$ is decreasing, it follows that $\mathcal{L}_{\beta_r}$ is an invariant set for any jump-flow solution $x(t)$ starting from $x(0) = x_0 \in \mathcal{L}_{\beta_r}$ because $V(x(t)) \le V(x(0)) \le \beta_r$ implies that $x(t) \in \mathcal{L}_{\beta_r}$, $\forall t \ge 0$. Since $V(x)$ is continuous and $V(0) = 0$, there exists $\delta > 0$ such that $B_\delta \subsetneq \mathcal{L}_{\beta_r}$. We thus have $x(0) \in B_\delta \subsetneq \mathcal{L}_{\beta_r} \Rightarrow x(t) \in \mathcal{L}_{\beta_r} \subsetneq B_r$, which implies that $||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon$, hence $x_c = 0$ is stable.

Step 3: We prove that all jump-flow solutions $x(t)$ converge to zero. Seeking a contradiction, assume that $x(t)$ does not converge to zero. Then, since $V(x(t))$ is nonnegative and decreasing, we have $\lim\limits_{t \to \infty} V(x(t)) = c > 0$. Notice that the set $\mathcal{L}_{c,d} = \{x \in U_c \,|\, c \le V(x) \le V(x(0)) = d\}$ is compact by assumption, it follows that, for each $p \in \mathcal{N}$, the con-

[5] we may need to take a smaller U_c to guarantee U_{c3} is a star field such that the jump $\xi_{30}^- \to 0$ exists on U_{c3}

tinuous function $\frac{\partial V(x)}{\partial x} f_p^e(x)$ attains its maximum $s_p < 0$ within $\mathcal{L}_{c,d}$. Then with $s = \max_{p\in\mathcal{N}} s_p$, we have that $\dot V(x(t)) \le s < 0$ for all $t \in \mathcal{I}_i$ for any interval $\mathcal{I}_i = (t_i, t_{i+1}) \subseteq [0, +\infty)$ without switching times. Consequently, for any $k \ge 1$,

$$V(x(t_k^-)) = V(x(t_0^-)) + \sum_{i=0}^{k-1} \int_{t_i^+}^{t_{i+1}^-} \dot V(x(t))dt + \sum_{i=0}^{k-1} \left(V(x(t_i^+)) - V(x(t_i^-))\right) \le V(x(t_0^-)) + st_k.$$

So for $t_k > -\frac{V(x(0^-))}{s}$, the above relation results in $V(x(t_k^-)) < 0$, which is a contradiction. Hence all jump-flow solutions $x(t)$ converge to zero. □

Proof of Lemma 4.11. Since $\Sigma_q^e = (f_q^e, g_q^e, h_q^e)$ is the jump-flow explicitation of Ξ_q, we have $\frac{\partial\psi_q}{\partial x} g_q^e = \left[\begin{smallmatrix} 0 \\ I_{m_q}\end{smallmatrix}\right]$ and $h_q^e = \psi_{2q}$, where $\psi_q = (\psi_{1q}, \psi_{2q}) = (\xi_{1q}, \xi_{2q})$ is the diffeomorphism transforming Ξ_q into its **(INWF)**. Thus condition (25) is equivalent to $\frac{\partial\psi_q}{\partial x} \cdot (g_p^e \cdot \mathrm{d}h_p^e) \cdot \left(\frac{\partial\psi_q}{\partial x}\right)^{-1} \left(\frac{\partial\psi_q}{\partial x}\right) g_q^e \subseteq \mathrm{Im} \left(\frac{\partial\psi_q}{\partial x}\right) g_q^e$, $\frac{\partial\psi_q}{\partial x} \cdot (g_p^e \cdot \mathrm{d}h_p^e) \cdot \left(\frac{\partial\psi_q}{\partial x}\right)^{-1} \left(\frac{\partial\psi_q}{\partial x}\right) \ker \mathrm{d}h_q^e \subseteq \frac{\partial\psi_q}{\partial x} \ker \mathrm{d}h_q^e$, i.e.,

$$\begin{aligned} &\mathrm{Im}\,\Gamma^e \cdot \begin{bmatrix} 0 \\ I_{m_q}\end{bmatrix} \subseteq \mathrm{Im} \begin{bmatrix} 0 \\ I_{m_q}\end{bmatrix}, \\ &\Gamma^e \ker \begin{bmatrix} 0 & I_{m_q}\end{bmatrix} \subseteq \ker \begin{bmatrix} 0 & I_{m_q}\end{bmatrix}, \end{aligned} \tag{30}$$

where $\Gamma^e = \frac{\partial\psi_q}{\partial x} \cdot (g_p^e \cdot \mathrm{d}h_p^e) \left(\frac{\partial\psi_q}{\partial x}\right)^{-1} : \psi_q(U_c) \to \mathbb{R}^{n\times n}$. Notice that by $\mathrm{d}h_p^e \cdot g_p^e = I_{m_p}$ of (17), we have $\mathrm{Im}\,(g_p^e \cdot \mathrm{d}h_p^e) = \mathrm{Im}\, g_p^e$ and $\ker(g_p^e \cdot \mathrm{d}h_p^e) = \ker \mathrm{d}h_p^e$. It follows that $\mathrm{Im}\,\Gamma^e = \frac{\partial\psi_q}{\partial x}\mathcal{G}_p^e$ and $\ker \Gamma^e = \frac{\partial\psi_q}{\partial x}\mathcal{H}_p^e$. Recall that $\frac{\partial\psi_q}{\partial x}\mathcal{G}_q^e = \mathrm{Im} \left[\begin{smallmatrix} 0 \\ I_{m_q}\end{smallmatrix}\right]$ and $\frac{\partial\psi_q}{\partial x}\mathcal{H}_q^e = \ker \left[\begin{smallmatrix} 0 & I_{m_q}\end{smallmatrix}\right]$. So by expressing condition (26) in $\xi_q = \psi_q$-coordinate, we get $\frac{\partial\psi_q}{\partial x}(\mathcal{G}_p^e \cap \mathcal{G}_q^e) \oplus \frac{\partial\psi_q}{\partial x}(\mathcal{H}_p^e \cap \mathcal{G}_q^e) = \frac{\partial\psi_q}{\partial x}\mathcal{G}_q^e \Leftrightarrow$

$$(\mathrm{Im}\,\Gamma^e \cap \mathrm{Im}\, \left[\begin{smallmatrix} 0 \\ I\end{smallmatrix}\right]) \oplus (\ker \Gamma^e \cap \mathrm{Im}\, \left[\begin{smallmatrix} 0 \\ I\end{smallmatrix}\right]) = \mathrm{Im}\, \left[\begin{smallmatrix} 0 \\ I\end{smallmatrix}\right] \tag{31}$$

and $\frac{\partial\psi_q}{\partial x}(\mathcal{G}_p^e \cap \mathcal{H}_q^e) \oplus \frac{\partial\psi_q}{\partial x}(\mathcal{H}_p^e \cap \mathcal{H}_q^e) = \frac{\partial\psi_q}{\partial x}\mathcal{H}_q^e \Leftrightarrow$

$$(\mathrm{Im}\,\Gamma^e \cap \ker \left[\begin{smallmatrix} 0 & I\end{smallmatrix}\right]) \oplus (\ker \Gamma^e \cap \ker \left[\begin{smallmatrix} 0 & I\end{smallmatrix}\right]) = \ker \left[\begin{smallmatrix} 0 & I\end{smallmatrix}\right]. \tag{32}$$

Now, assume (25) holds, then the matrix-valued function Γ^e is block diagonal by (30), i.e., $\Gamma^e = \left[\begin{smallmatrix} \Gamma_1^e & 0 \\ 0 & \Gamma_2^e\end{smallmatrix}\right]$, where $\Gamma_1^e : \psi_q(U_c) \to \mathbb{R}^{r_q\times r_q}$ and $\Gamma_2^e : \psi_q(U_c) \to \mathbb{R}^{m_q\times m_q}$. Thus $\mathrm{Im}\,\Gamma_1^e \oplus \ker \Gamma_1^e \simeq \mathbb{R}^{r_q}$ and $\mathrm{Im}\,\Gamma_2^e \oplus \ker \Gamma_2^e \simeq \mathbb{R}^{m_q}$ because $\mathrm{Im}\,\Gamma^e \oplus \ker \Gamma^e = \frac{\partial\psi_q}{\partial x}\mathcal{G}_p^e \oplus \frac{\partial\psi_q}{\partial x}\mathcal{H}_p^e \simeq \mathbb{R}^n$ by (17). By a direct calculation, it follows that both (31) and (32) hold. Conversely, if (31) holds, then the left-multiplication of (31) by Γ^e yields

$$\Gamma^e \left(\mathrm{Im}\,\Gamma^e \cap \mathrm{Im}\, \begin{bmatrix} 0 \\ I_{m_q}\end{bmatrix}\right) = \Gamma^e \mathrm{Im}\, \begin{bmatrix} 0 \\ I_{m_q}\end{bmatrix}.$$

Observe that $\Gamma^e = \frac{\partial\psi_q}{\partial x} \cdot (g_p^e \cdot \mathrm{d}h_p^e) \left(\frac{\partial\psi_q}{\partial x}\right)^{-1} = \frac{\partial\psi_q}{\partial x} \left(\frac{\partial\psi_p}{\partial x}\right)^{-1} \left[\begin{smallmatrix} 0 & 0 \\ 0 & I_p\end{smallmatrix}\right] \frac{\partial\psi_p}{\partial x} \left(\frac{\partial\psi_q}{\partial x}\right)^{-1}$ has the property that $\Gamma^e \cdot \Gamma^e = \Gamma^e$. It follows that $\Gamma^e\,(\mathrm{Im}\,\Gamma^e \cap \mathrm{Im}\, \left[\begin{smallmatrix} 0 \\ I\end{smallmatrix}\right]) = (\mathrm{Im}\,\Gamma^e \cap \mathrm{Im}\, \left[\begin{smallmatrix} 0 \\ I\end{smallmatrix}\right])$, so

$$\mathrm{Im}\,\Gamma^e \cdot \begin{bmatrix} 0 \\ I_{m_q}\end{bmatrix} = \left(\mathrm{Im}\,\Gamma^e \cap \mathrm{Im}\, \begin{bmatrix} 0 \\ I_{m_q}\end{bmatrix}\right) \subseteq \mathrm{Im}\, \begin{bmatrix} 0 \\ I_{m_q}\end{bmatrix}.$$

Similarly, it can be shown that (32) indicates the inclusion $\Gamma^e \ker \left[\begin{smallmatrix} 0 & I_{m_q}\end{smallmatrix}\right] \subseteq \ker \left[\begin{smallmatrix} 0 & I_{m_q}\end{smallmatrix}\right]$. Hence (31) and (32) imply (30) and the latter is equivalent to (25). □

Proof of Theorem 4.10. Step 1: By $\mathcal{H}_q^e \oplus \mathcal{G}_q^e = TU_c$ of (17) and (26) (which is equivalent to (25) by Lemma 4.11), we have

$$(\mathcal{H}_p^e \cap \mathcal{H}_q^e) \oplus (\mathcal{G}_p^e \cap \mathcal{H}_q^e) \oplus (\mathcal{H}_p^e \cap \mathcal{G}_q^e) \oplus (\mathcal{G}_p^e \cap \mathcal{G}_q^e) = TU_c.$$

Recall that the distributions $\mathcal{G}_p$ and $\mathcal{H}_p$ for all $p \in \mathcal{N}$ are of constant dimension and involutive by constructions. It follows that the intersections $\mathcal{G}_p^e \cap \mathcal{G}_q^e$, $\mathcal{H}_p^e \cap \mathcal{G}_q^e$, $\mathcal{G}_p^e \cap \mathcal{H}_q^e$, $\mathcal{H}_p^e \cap \mathcal{H}_q^e$ are all of constant dimension and involutive as well. Thus by Frobenius theorem, we can choose local coordinates $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) = \psi_{pq}(x)$, where $\psi_{pq} : U_c \to \mathbb{R}^n$ is a local diffeomorphism, such that

$$\begin{aligned} \mathrm{span}\left\{\frac{\partial}{\partial\xi_1^1}, \dots, \frac{\partial}{\partial\xi_1^{n_1}}\right\} &= \frac{\partial\psi_{pq}}{\partial x}(\mathcal{H}_p^e \cap \mathcal{H}_q^e), \\ \mathrm{span}\left\{\frac{\partial}{\partial\xi_2^1}, \dots, \frac{\partial}{\partial\xi_2^{n_2}}\right\} &= \frac{\partial\psi_{pq}}{\partial x}(\mathcal{G}_p^e \cap \mathcal{H}_q^e), \\ \mathrm{span}\left\{\frac{\partial}{\partial\xi_3^1}, \dots, \frac{\partial}{\partial\xi_3^{n_3}}\right\} &= \frac{\partial\psi_{pq}}{\partial x}(\mathcal{H}_p^e \cap \mathcal{G}_q^e), \\ \mathrm{span}\left\{\frac{\partial}{\partial\xi_4^1}, \dots, \frac{\partial}{\partial\xi_4^{n_4}}\right\} &= \frac{\partial\psi_{pq}}{\partial x}(\mathcal{G}_p^e \cap \mathcal{G}_q^e), \end{aligned} \tag{33}$$

where $n_1 = \dim \mathcal{H}_p^e \cap \mathcal{H}_q^e$, $n_2 = \dim \mathcal{G}_p^e \cap \mathcal{H}_q^e$, $n_3 = \dim \mathcal{H}_p^e \cap \mathcal{G}_q^e$, $n_4 = \dim \mathcal{G}_p^e \cap \mathcal{G}_q^e$ and $n_1 + n_2 + n_3 + n_4 = n$. Since $f_p \in \mathcal{H}_p$ and $\mathcal{H}_p$ is involutive, we have $[f_p, \mathcal{H}_p] \subseteq \mathcal{H}_p$. Notice that $[f_p, \mathcal{G}_p] = 0 \subseteq \mathcal{G}_p$ by construction. Thus by (24), we get $\forall p, q \in \mathcal{N}$:

$$\begin{aligned} &[f_p^e, \mathcal{H}_p^e \cap \mathcal{H}_q^e] \subseteq [f_p^e, \mathcal{H}_p^e] \cap [f_p^e, \mathcal{H}_q^e] \subseteq \mathcal{H}_p^e \cap \mathcal{H}_q^e, \\ &[f_p^e, \mathcal{G}_p^e \cap \mathcal{H}_q^e] \subseteq [f_p^e, \mathcal{G}_p^e] \cap [f_p^e, \mathcal{H}_q^e] \subseteq \mathcal{G}_p^e \cap \mathcal{H}_q^e, \\ &[f_p^e, \mathcal{H}_p^e \cap \mathcal{G}_q^e] \subseteq [f_p^e, \mathcal{H}_p^e] \cap [f_p^e, \mathcal{G}_q^e] \subseteq \mathcal{H}_p^e \cap \mathcal{G}_q^e, \\ &[f_p^e, \mathcal{G}_p^e \cap \mathcal{G}_q^e] \subseteq [f_p^e, \mathcal{G}_p^e] \cap [f_p^e, \mathcal{G}_q^e] \subseteq \mathcal{G}_p^e \cap \mathcal{G}_q^e. \end{aligned} \tag{34}$$

Then by (33) and (34), the vector fields f_p^e and f_q^e are of the following form in $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$-coordinates

$$\tilde{f}_p^e = \frac{\partial \psi_{pq}}{\partial x} f_p^e = \begin{bmatrix} \tilde{f}_p^1(\xi_1) \\ \tilde{f}_p^2(\xi_2) \\ \tilde{f}_p^3(\xi_3) \\ \tilde{f}_p^4(\xi_4) \end{bmatrix}, \quad \tilde{f}_q^e = \frac{\partial \psi_{pq}}{\partial x} f_q^e = \begin{bmatrix} \tilde{f}_q^1(\xi_1) \\ \tilde{f}_q^2(\xi_2) \\ \tilde{f}_q^3(\xi_3) \\ \tilde{f}_q^4(\xi_4) \end{bmatrix}. \tag{35}$$

Since $f_p^e \in \mathcal{H}_p^e$ and $f_q^e \in \mathcal{H}_q^e$ by (17), it can be deduced from (33) and (34) that

$$\tilde{f}_p^2(\xi_2) \equiv 0, \ \ \tilde{f}_p^4(\xi_4) \equiv 0, \ \ \tilde{f}_q^3(\xi_3) \equiv 0, \ \ \tilde{f}_q^4(\xi_4) \equiv 0. \tag{36}$$

Note that the nonlinear consistency projectors (see (7)) of the models Ξ_p and Ξ_q are, respectively,

$$\Omega_{E_p,F_p} = \psi_{pq}^{-1} \circ \pi_p \circ \psi_{pq}, \ \ \Omega_{E_q,F_q} = \psi_{pq}^{-1} \circ \pi_q \circ \psi_{pq}, \tag{37}$$

where $\pi_p : (\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_1, 0, \xi_3, 0)$ and $\pi_q : (\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_1, \xi_2, 0, 0)$.

Step 2: We show that $\forall p, q \in \mathcal{N}$:

$$\begin{aligned} &\Phi_t^{f_p^e} \circ \Omega_{E_p,F_p} \circ \Phi_s^{f_q^e} \circ \Omega_{E_q,F_q} = \\ &\quad \Phi_s^{f_q^e} \circ \Omega_{E_q,F_q} \circ \Phi_t^{f_p^e} \circ \Omega_{E_p,F_p}, \end{aligned} \tag{38}$$

where $\Phi_t^{f_p^e}$ and $\Phi_s^{f_q^e}$ are the flow map of f_p^e and f_q^e, respectively. Indeed, first it can be seen from (35) and (36) that

$$\Phi_t^{\tilde{f}_p^e} \circ \pi_p \circ \Phi_s^{\tilde{f}_q^e} \circ \pi_q = \begin{bmatrix} \Phi_t^{\tilde{f}_p^1} \circ \Phi_s^{\tilde{f}_q^1} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\Phi_s^{\tilde{f}_q^e} \circ \pi_q \circ \Phi_t^{\tilde{f}_p^e} \circ \pi_p = \begin{bmatrix} \Phi_s^{\tilde{f}_q^1} \circ \Phi_t^{\tilde{f}_p^1} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Observe that (23) implies $[\tilde{f}_p^e, \tilde{f}_q^e] = 0$, we thus have $[\tilde{f}_p^1, \tilde{f}_q^1] = 0$, which is equivalent to (see Proposition 1.7 of [21]) $\Phi_t^{\tilde{f}_p^1} \circ \Phi_s^{\tilde{f}_q^1} = \Phi_s^{\tilde{f}_q^1} \circ \Phi_t^{\tilde{f}_p^1}$. It follows that

$$\Phi_t^{\tilde{f}_p^e} \circ \pi_p \circ \Phi_s^{\tilde{f}_q^e} \circ \pi_q = \Phi_s^{\tilde{f}_q^e} \circ \pi_q \circ \Phi_t^{\tilde{f}_p^e} \circ \pi_p. \tag{39}$$

It is well-known (see Proposition 1.11 of [21]) that $\tilde{f}_p^e = \frac{\partial \psi_{pq}}{\partial x} f_p^e$ implies $\Phi_t^{\tilde{f}_p^e} = \psi_{pq} \circ \Phi_t^{f_p^e} \circ \psi_{pq}^{-1}$. Then by (39) and (37), we have

$$\begin{aligned} &\psi_{pq} \circ \Phi_t^{f_p^e} \circ \Omega_{E_p,F_p} \circ \Phi_s^{f_q^e} \circ \Omega_{E_q,F_q} \circ \psi_{pq}^{-1} = \\ &\quad \psi_{pq} \circ \Phi_s^{f_q^e} \circ \Omega_{E_q,F_q} \circ \Phi_t^{f_p^e} \circ \Omega_{E_p,F_p} \circ \psi_{pq}^{-1}. \end{aligned}$$

Hence the commutativity condition (38) holds.

Step 3: We prove that Ξ_σ is asymptotically stable. Recall that all models Ξ_p of Ξ_σ are asymptotically stable, which means (see Definition 4.4) that for each $p \in \mathcal{N}$, there exists $\beta_p : ||U_c|| \times [0, +\infty) \to \mathcal{KL}$ such that for any initial value $x_0 \in U_c$, the impulse-free solution $x_p(t)$ of Ξ_p satisfies $\forall t \geq 0$ and $\forall x_0 \in U_c$:

$$||x_p(t)|| = ||\Phi_t^{f_p^e} \circ \Omega_{E_p,F_p} \circ x_0|| \leq \beta_p(x_0, t).$$

Because $\mathcal{N}$ is finite, there exists a function $\beta : ||U_c|| \times [0, +\infty) \to \mathcal{KL}$ such that $\beta_p(x_0, t) \leq \beta(x_0, t)$, $\forall p \in \mathcal{N}$, $\forall x_0 \in U_c$, $\forall t \geq 0$. Let $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ be the switching time of σ, then given an initial point $x_0 \in U_c$, the impulse-free solution $x(t)$ of Ξ_σ can be expressed as

$$\begin{aligned} x(t) = \Phi_{t-t_k}^{f_{p_k}^e} \circ \Omega_{E_{p_k},F_{p_k}} \circ \cdots \circ \Phi_{t_2-t_1}^{f_{p_1}^e} \circ \Omega_{E_{p_1},F_{p_1}} \circ \\ \Phi_{t_1-t_0}^{f_{p_0}^e} \circ \Omega_{E_{p_0},F_{p_0}} \circ x_0, \end{aligned}$$

where $t \in [t_k, t_{k+1})$ and $p_i = \sigma(t_i^+)$ for $0 \leq i \leq k$. Then by the commutativity condition (38), we have

$$x(t) = \Phi_{\Delta t_1}^{f_1^e} \circ \Omega_{E_1,F_1} \circ \Phi_{\Delta t_2}^{f_2^e} \circ \Omega_{E_2,F_2} \circ \cdots \Phi_{\Delta t_N}^{f_N^e} \circ \Omega_{E_N,F_N} \circ x_0,$$

where Δt_p is the total amount time of activation of the p-th model in $[0, t)$. Note that $\Delta t_p = 0$ if the p-th models is not activated and $\sum_{p=1}^N \Delta t_p = t$. Since $||\Phi_t^{f_p^e} \circ \Omega_{E_p,F_p} \circ x_0|| \leq \beta(x_0, t)$, $\forall t \geq 0$, $\forall x_0 \in U_c$, $\forall p \in \mathcal{N}$, we have $x(t) \leq \beta(\cdot, \Delta t_1) \circ \cdots \circ \beta(||x_0||, \Delta t_N)$. By Lemma 2.2 of [32], there exists a function $\tilde{\beta} : ||U_c|| \to \mathcal{KL}$ such that $\beta(\cdot, \Delta t_1) \circ \cdots \circ \beta(||x_0||, \Delta t_N) \leq \tilde{\beta}(||x_0||, \Delta t_1 + \cdots + \Delta t_N)$. It follows that $x(t) \leq \tilde{\beta}(||x_0||, t)$, hence Ξ_σ is asymptotically stable. $\square$

Proof of Corollary 4.13. Define $\kappa = 2^N$ distributions $\mathcal{D}_i$, $1 \leq i \leq \kappa$, by $\mathcal{D}_1 := \bigcap\limits_{i=1}^N \mathcal{H}_i$, $\mathcal{D}_2 := \left(\bigcap\limits_{i=1}^{N-1} \mathcal{H}_i\right) \cap \mathcal{G}_N$, $\ldots$, $\mathcal{D}_{\kappa-1} := \left(\bigcap\limits_{i=1}^{N-1} \mathcal{G}_i\right) \cap \mathcal{H}_N$, $\mathcal{D}_\kappa := \bigcap\limits_{i=1}^N \mathcal{G}_i$. Similarly as Step 1 in the proof of Theorem 4.10 above, it is possible to show $\mathcal{D}_i \cap \mathcal{D}_j = 0$, $\forall i \neq j$ and

$$\mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \cdots \oplus \mathcal{D}_{\kappa-1} \oplus \mathcal{D}_\kappa = TU_c.$$

By the involutivity of $\mathcal{D}_i$, we can choose new coordinates $\xi = (\xi_1, \xi_2, \ldots, \xi_{\kappa-1}, \xi_\kappa)$ to rectify the distributions $\mathcal{D}_i$, $1 \leq i \leq \kappa$ as $\tilde{\mathcal{D}}_i = \operatorname{span}\left\{\frac{\partial}{\partial \xi_i^1}, \ldots, \frac{\partial}{\partial \xi_i^{n_i}}\right\} = \frac{\partial \psi}{\partial x} \mathcal{D}_i$, where $n_i = \dim \mathcal{D}_i$. It follows from (24) that $[f_p^e, \mathcal{D}_i] \in \mathcal{D}_i$ (equivalently, $[\tilde{f}_p^e, \tilde{\mathcal{D}}_i] \in \tilde{\mathcal{D}}_i$), $1 \leq i \leq \kappa$ and $p \in \mathcal{N}$. Thus we have

$$\tilde{f}_p^e(\xi) = \frac{\partial \psi}{\partial x} f_p^e(\psi^{-1}(\xi)) = \begin{bmatrix} \tilde{f}_p^1(\xi_1) \\ \tilde{f}_p^2(\xi_2) \\ \vdots \\ \tilde{f}_p^{\kappa-1}(\xi_{\kappa-1}) \\ \tilde{f}_p^\kappa(\xi_\kappa) \end{bmatrix}, \quad p \in \mathcal{N}.$$

Since $f_p^e \in \mathcal{H}_p$, $\forall p \in \mathcal{N}$, we have

$$\tilde{f}_p^i(\xi_i) \equiv 0, \quad \forall p \in \mathcal{N},\ \forall i : \mathcal{D}_i \cap \mathcal{H}_p = 0.$$

It follows that $\tilde{f}_p^i(\xi_i)$ is either zero or a vector field defined on $\mathcal{D}_i$ with asymptotically stable flow dynamics. Moreover, by (23), we have $[\tilde{f}_p^i, \tilde{f}_q^i] = 0$, $\forall p, q \in \mathcal{N}$, $\forall 1 \le i \le \kappa$. It is known from Theorem 4 of [56] that for each i, there exist $r_{0i} \in (0, r)$ and a positive definite $\mathcal{C}^1$-function $V_i(\xi_i) = V_i(\psi_i(x))$ such that $\mathcal{L}_{a_i} := \{x \in B_{r_{0i}} \,|\, V_i(\psi_i(x)) \le a_i\}$ is compact and $\forall \xi_i \in \tilde{B}_{r_{0i}}^{\xi_i} \setminus \{0\}$:

$$\frac{\partial V_i(\xi_i)}{\partial \xi_i} \tilde{f}_p^i(\xi_i) < 0, \quad \forall p \in \mathcal{N},\ \forall i : \tilde{f}_p^i \neq 0. \tag{40}$$

Notice that $\tilde{f}_p^\kappa \equiv 0$, $\forall p \in \mathcal{N}$, so we define $V_\kappa(\xi_\kappa) := \frac{1}{2}\xi_\kappa^T \xi_\kappa$ (for the other $\tilde{f}_p^i$, $i \neq \kappa$, there exists at least one $p^* \in \mathcal{N}$ such that $\tilde{f}_{p^*}^i \neq 0$). Then we claim that

$$V(\psi(x)) = V(\xi) := \sum_{i=1}^{\kappa} V_i(\xi_i)$$

is a common Lyapunov function satisfying (18) and (20) (and thus satisfying (19)). Indeed, there exists a positive scalar $r_0 \le r_{0i}$, $\forall 1 \le i \le \kappa$ such that $\forall p \in \mathcal{N}$ and $\forall x \in B_{r_0} \setminus \{0\}$, we have

$$\frac{\partial V(\psi(x))}{\partial x} f_p^e(x) = \frac{\partial V(\xi)}{\partial \xi} \tilde{f}_p^e(\xi) = \sum_{i=1}^{\kappa} \frac{\partial V_i(\xi_i)}{\partial \xi_i} \tilde{f}_p^i(\xi_i) < 0$$

and $\forall x \in B_{r_0}$, we have

$$\begin{aligned} V(\psi(x)) - V(\psi \circ \Omega_{E_p, F_p}(x)) &= V(\xi) - V(\pi_p(\xi)) \\ &= \sum_{i : \mathcal{D}_i \cap \mathcal{G}_p = 0} V_i(\xi_i) \ge 0, \end{aligned}$$

where π_p is the canonical projection $\psi(U_c) \to \psi(U_c)$, attaching $\xi_p^i \mapsto \xi_p^i$, $\forall i : \mathcal{D}_i \cap \mathcal{G}_p = 0$ and attaching $\xi_p^i \mapsto 0$, $\forall i : \mathcal{D}_i \cap \mathcal{H}_p = 0$. Note that $\mathcal{L}_a := \{x \in B_{r_0} \,|\, V(\psi(x)) \le a\}$ is compact, hence the corollary holds. $\square$

6 Conclusions and perspectives

We define the notion of (jump-flow) impulse-free solution for switched nonlinear DAEs, which is different from the distributional solution framework used in [27, 28]. Existence and uniqueness conditions of such solutions are given using geometric methods. Then we show that several notions and results for switched linear DAEs, such as the consistency projector, the impulse-free condition, and the stability analysis using common Lyapunov functions and using commutativity conditions, can be generalized to the nonlinear case. To study the stability, we use a novel notion called the jump-flow explicitation, which is constructed based on a nonlinear Weierstrass form. The jump-flow explicitation facilitates the constructions of common Lyapunov functions and plays an important role for deriving commutativity and invariances conditions. For future works, we will concentrate on stability studies of impulse-free solutions of switched nonlinear DAEs with unstable models using the jump-flow explicitation, the impulse-freeness and stability of state-dependent switched nonlinear DAE are also interesting topics.

Acknowledgments

The authors would like to thank the anonymous reviewers whose suggestions and remarks have improved the final presentation of the paper.

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