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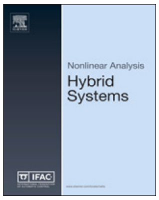

Impulse-free jump solutions of nonlinear differential–algebraic equations

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ABSTRACT

In this paper, we propose a novel notion called impulse-free jump solution for nonlinear differential–algebraic equations (DAEs) of the form $E(x)\dot{x} = F(x)$ with inconsistent initial values. The term "impulse-free" means that there are no Dirac impulses caused by jumps from inconsistent initial values, i.e., the directions of the jumps stay in $\ker E(x)$. We show that our proposed impulse-free jump rule is a coordinate-free concept, meaning that the calculation of the impulse-free jump does not depend on the coordinates of the DAE, which is a main advantage compared to some existing jump rules for nonlinear DAEs. We find that the existence and uniqueness of impulse-free jumps are closely related to the notion of geometric index-1 and the involutivity of the distribution defined by $\ker E(x)$. Moreover, a singular perturbed system approximation is proposed for nonlinear DAEs; we show that solutions of the perturbed system approximate both impulse-free jump solutions and $\mathcal{C}^1$-solutions of nonlinear DAEs. Finally, we show by some examples that our results of impulse-free jumps are useful for the problems like consistent initialization of nonlinear DAEs and transient behavior simulations of electric circuits.

1. Introduction

Consider a nonlinear differential–algebraic equation (DAE) in quasi-linear form

$$\Xi : \quad E(x)\dot{x} = F(x), \tag{1}$$

where $x \in X$ is a vector of the generalized states and $(x, \dot{x}) \in TX$, where TX is the tangent bundle of the open subset X in $\mathbb{R}^n$ (or an n-dimensional smooth manifold). The maps $E : TX \to \mathbb{R}^l$ (attaching $(x, \dot{x}) \mapsto E(x)\dot{x}$) and $F : X \to \mathbb{R}^l$ are $\mathcal{C}^\infty$-smooth, and for each $x \in X$, we have that $E(x) : \mathbb{R}^n \to \mathbb{R}^l$ is a linear map. We will denote a DAE of the form (1) by $\Xi_{l,n} = (E, F)$ or, simply, Ξ. A linear DAE of the form

$$\Delta : \quad E\dot{x} = Hx \tag{2}$$

will be denoted by $\Delta_{l,n} = (E, H)$ or, simply, Δ, where $E \in \mathbb{R}^{l\times n}$ and $H \in \mathbb{R}^{l\times n}$. A linear DAE is called *regular* if $l = n$ and $\det(sE - H) \in \mathbb{R}[s] \setminus \{0\}$.

Definition 1.1 ($\mathcal{C}^1$*-solutions and Consistency Space*)**.** The trajectory $x : \mathcal{I} \to X$ for some open interval $\mathcal{I} \subseteq \mathbb{R}$ is called a $\mathcal{C}^1$-solution of the DAE $\Xi_{l,n} = (E, F)$ if x is continuously differentiable and satisfies $E(x(t))\dot{x}(t) = F(x(t))$ for all $t \in \mathcal{I}$.

A point $x_c \in X$ is called *consistent* (or *admissible* [1]) if there exists a $\mathcal{C}^1$-solution $x : \mathcal{I} \to X$ and $t_c \in \mathcal{I}$ such that $x(t_c) = x_c$. The *consistency space* $S_c \subseteq X$ is the set of all consistent points.

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By re-parameterizing the time variable t, we can always assume $\mathcal{I} = (0, T)$ for some $T > 0$. For a nonlinear DAE of the form (1), the initial point x_0^- is usually defined (see e.g. [2,3]) via the right limit of some past trajectory $x(t)$, $t < 0$ (which may be or may not be governed by (1)). In the present paper, we are interested in nonlinear DAEs with inconsistent initial points, i.e., when the initial points $x_0^- \notin S_c$. Assume that there exists one $\mathcal{C}^1$-solution $x(\cdot)$ of Ξ on $(0, T)$, then we have $x_0^+ = \lim_{t\to 0^+} x(t) = x(0^+) \in S_c$. Thus if x_0^- is not consistent, then there has to be an "instantaneous" change of values for $x(t)$ at $t = 0$, i.e., a jump $x_0^- \to x_0^+$ to steer the inconsistent point x_0^- towards a consistent one x_0^+.

The jump behaviors in practical DAE systems are not rare phenomenons, e.g., the inconsistent initial values of electric circuits caused by switching devices (see e.g., [4–6]), the discontinues transient dynamics in hybrid/switched systems as power systems [7], multi-body dynamics [8] and battery models [9]. Discontinues solutions of a more general class of system which includes linear differential–algebraic dynamics and complementarity conditions, called the linear complementary systems, were discussed in [10], where the problem of re-initialization (jump) rules plays also an important role for the definition of solutions. All the jumps which we consider in the present article, are called external/exogenous jumps [11], which are different from the jumps happened at the impasse or singular points discussed in [12–14]. More specifically, we suppose throughout that once the inconsistent initial point x_0^- jumps to a consistent point $x_0^+ \in S_c$, then we will consider only $\mathcal{C}^1$-solutions starting from x_0^+, that means, there are no jumps in $x(t)$ for $t \in (0, T)$.

For a linear DAE $\Delta = (E, H)$, given by (2), with an inconsistent initial value x_0^-, the jump behavior at $t = 0$ can be described by a vector $e_0 = x(0^+) - x(0^-) = x_0^+ - x_0^-$. To deal with the discontinuity introduced by the jump behavior at $t = 0$, the distributional (generalized function)[1] solutions theory for linear DAEs were established e.g. in [2,3,15,16]. The distributional derivative of the jump of x at $t = 0$ is $(x_0^+ - x_0^-)\delta_0$, where δ_0 is the Dirac impulse at $t = 0$, i.e., taking distributional derivative of a jump results in a Dirac impulse δ_0 whose amplitude is the jump vector e_0 [16]. The distributional restriction of Δ to $t = 0$ can be represented by $E\dot{x}[0] = Hx[0]$, where $x[0] = \sum_{i=0}^{k} \alpha_k \delta_0^{(i)}$ and $\dot{x}[0] = e_0\delta_0 + \sum_{i=0}^{k} \alpha_k \delta_0^{(i+1)}$ for some $k \geq 0$. It can be deduced that there are no Dirac impulses and their derivatives $\delta_0, \ldots, \delta_0^{(k)}$ caused by jumps at $t = 0$ if and only if $E \cdot e_0\delta_0 = 0$, i.e., $e_0 \in \ker E$, and we call a jump satisfies the latter condition an impulse-free jump of the linear DAE Δ.

The difficulty of studying jump behaviors for DAEs of the form (1) comes from the nonlinearity of the map E, which makes the distributional (generalized function) solution theory a possible non-suitable setting for our problems. As stated in Remark 46.2 of [17], "*This does not mean that discontinuous solutions of quasilinear problems cannot be investigated, but only that their treatment as distribution solutions is inadequate. In other words, discontinuous solutions of general quasilinear problems must, if possible at all, be introduced by a different process which remains to be determined.*" An extension of the notion of impulsive-free jump to nonlinear DAEs of the form (1) was made in Assumption A4 of [18], where it is assumed that a jump vector $e_0 = x_0^+ - x_0^-$ should satisfy the jump rule $e_0 \in \ker E(x_0^+)$. The problem of finding the consistent point x_0^+ for a given inconsistent point x_0^- is called the consistent initialization problem in the numerical analysis of nonlinear DAEs (see e.g., [19,20]). In particular, the consistent initialization of nonlinear DAEs can be solved by the function *decic* of MATLAB (see [21]). We will show below by examples that both the jumps defined by the rule of [18] and that calculated by *decic* are not invariant under nonlinear coordinates transformations, meaning that those two consistent initialization methods in [18,21] are not coordinate-free. A main contribution of this paper is the coordinate-free jump rule introduced in Definition 4.1, which allows to calculate the desired consistent points in any coordinates. In well-chosen coordinates, the DAE may be expressed as a simple form (normal form or canonical form), which can be easier for defining jumps, e.g., the linear consistency projectors [18,22] are constructed with the help of the Weierstrass form **(WF)** as the consistent points of the **(WF)** are straightforward to be found. Thus the coordinate-free property is an important feature for analyzing the jump behaviors of DAEs. Another main result concerns the existence and uniqueness of the impulse-free jumps, we use a notion of geometric index (see Definition 3.1) and show that the impulse-free jump always exists for any index-1 nonlinear DAE satisfying certain reachability conditions and the jump is uniquely defined if and only if the distribution $\ker E(x)$ is involutive.

Some other works of studying jump behaviors of DAEs (see e.g., [11–14,17]) mainly focused on semi-explicit (called also semi-linear) DAEs of the form

$$\Xi^{SE} : \begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ 0 = f_2(x_1, x_2), \end{cases} \tag{3}$$

i.e., $E(x) = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$; such DAEs are usually related to the models of electric circuits and singular perturbation theory (see e.g., [13], Chapter 11 of [23] and Chapter VIII of [17]). A preliminary result of using singular perturbation theory to study nonlinear DAEs of the form (1) is our conference publication [24], which considers only the case that Ξ is equivalent to a fully decoupled normal form (see the index-1 nonlinear Weierstrass form **(INWF)** in Theorem 4.6) without a formal definition of impulse-free jump. In this paper, we propose a singular perturbed system approximation for nonlinear DAEs (which are not necessarily equivalent to the **(INWF)**) and we show that the solutions of the perturbed systems approximate both the $\mathcal{C}^1$-solutions and the impulse-free jump solutions of the DAEs.

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[1] Note that there are two terminologies called distribution in our paper, one is a generalized function which helps to differentiate functions whose derivatives do not exist in the classical sense, the other is a subset of the tangent bundle of a manifold in differential geometry.

This paper is organized as follows: We give the notations of the paper and a brief review of the existence and uniqueness of $\mathcal{C}^1$-solutions for nonlinear DAEs in Section 2. We recall the notion of geometric index-1 and give some characterizations for that notion in Section 3. We introduce the definition of impulse-free jumps for nonlinear DAEs and study the existence and uniqueness of impulse-free jumps in Section 4. Singular perturbed system approximations of nonlinear DAEs are discussed in Section 5. The proofs of the main results are given in Section 6. The conclusions and perspectives of the paper are given in Section 7.

2. Notations and preliminaries on $\mathcal{C}^1$-solutions of DAEs

We denote by $T_xM \subseteq \mathbb{R}^n$ the tangent space at $x \in M$ of a submanifold M of $\mathbb{R}^n$ and by TM we denote the corresponding tangent bundle. By $\mathcal{C}^k$ the class of k-times differentiable functions is denoted. For a smooth map $f : X \to \mathbb{R}$, we denote its differentials by $\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mathrm{d}x_i = [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}]$ and for a vector-valued map $f : X \to \mathbb{R}^m$, where $f = [f_1, \ldots, f_m]^T$, we denote its differential by $\mathrm{D}f = \begin{bmatrix} \mathrm{d}f_1 \\ \vdots \\ \mathrm{d}f_m \end{bmatrix}$. For a map $A : X \to \mathbb{R}^{n\times n}$, $\ker A(x)$, $\operatorname{Im} A(x)$ and $\operatorname{rank} A(x)$ are the kernel, the image and the rank of A at x, respectively. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$. We assume familiarity with basic notions of differential geometry such as smooth embedded submanifolds, involutive distributions and refer the reader e.g. to the book [25] for the formal definitions of such notions.

We now recall some basic notions and results from the geometric analysis of the existence and uniqueness of $\mathcal{C}^1$-solutions for nonlinear DAEs (see e.g., [1,17,26–29]).

Definition 2.1 (*Invariant and Locally Invariant Submanifold*)**.** For a DAE $\Xi_{l,n} = (E, F)$, a smooth connected embedded submanifold M is called *invariant* if for any $x_0 \in M$, there exists a $\mathcal{C}^1$-solution $x : \mathcal{I} \to X$ such that $x(t_0) = x_0$ for some $t_0 \in \mathcal{I}$ and $x(t) \in M$, $\forall\, t \in \mathcal{I}$. Fix a point $x_p \in X$, a smooth embedded submanifold M containing x_p is called *locally invariant* if there exists a neighborhood U_{x_p} of x_p such that $M \cap U_{x_p}$ is invariant.

A locally invariant submanifold M^*, around a point x_p, is called locally *maximal*, if there exists a neighborhood U of x_p such that for any other locally invariant submanifold M, we have $M \cap U \subseteq M^* \cap U$. The following procedure is called the *geometric reduction method* [17,27,28], which is used to construct the locally maximal invariant submanifold M^* around a consistent point $x_p = x_c$ (see item (i) of Proposition 2.3).

Definition 2.2 (*Geometric Reduction Method*)**.** Consider a DAE $\Xi_{l,n}$ and fix a point $x_p \in X$. Let U_0 be a connected subset of X containing x_p. Step 0: $M_0^c = U_0$. Step k: Suppose that a sequence of smooth connected embedded submanifolds $M_{k-1}^c \subsetneq \cdots \subsetneq M_0^c$ of U_{k-1} for a certain $k-1$, have been constructed. Define recursively

$$M_k := \left\{x \in M_{k-1}^c \mid F(x) \in E(x)T_xM_{k-1}^c\right\}. \tag{4}$$

As long as $x_p \in M_k$ let $M_k^c = M_k \cap U_k$ be a smooth embedded connected submanifold for some neighborhood $U_k \subseteq U_{k-1}$.

Proposition 2.3 ([28,30])**.** *In the above geometric reduction method, there always exists a smallest k such that either $x_p \notin M_k$ or $M_{k+1}^c = M_k^c$ in U_{k+1}. In the latter case, denote $k^* = k$ and $M^* = M_{k^*+1}^c$ and assume that there exists an open neighborhood $U^* \subseteq U_{k^*+1}$ of x_p such that $\dim E(x)T_xM^* = const.$ for $x \in M^* \cap U^*$, then*

(i) x_p is a consistent point, i.e., $x_p = x_c$, and M^* is a locally maximal invariant submanifold around x_p;
(ii) M^* coincides locally with the consistency space S_c, i.e., $M^* \cap U^* = S_c \cap U^*$.

Notice that by item (ii) of Proposition 2.3, the consistency space S_c locally coincides with M^* on the neighborhood U^* of x_p. So any point $x_0^- \in U^* \backslash M^*$ is not consistent and there exist no $\mathcal{C}^1$-solutions starting from x_0^-. The uniqueness of $\mathcal{C}^1$-solutions is characterized via the following notion of local *internal regularity*. We call a $\mathcal{C}^1$-solution $x : \mathcal{I} \to (U \subseteq)X$ *maximal* (in U) if there is no other solution $\widetilde{x} : \widetilde{\mathcal{I}} \to (U \subseteq)X$ with $\mathcal{I} \subsetneq \widetilde{\mathcal{I}}$ and $x(t) = \widetilde{x}(t)$ for all $t \in \mathcal{I}$.

Definition 2.4 (*Local Internal Regularity*)**.** Consider a DAE Ξ and let M^* be the locally maximal invariant submanifold around a consistent point $x_c \in M^*$. Then Ξ is called locally *internally regular* (around x_c) if there exists neighborhood $U \subseteq X$ of x_c such that for any $t_0 \in \mathbb{R}$ and any point $x_0 \in M^* \cap U$, there exists only one maximal solution $x : \mathcal{I} \to U$ with $t_0 \in \mathcal{I}$ and $x(t_0) = x_0$.

Proposition 2.5 ([1,28])**.** *Given a DAE Ξ and its locally maximal invariant submanifold M^* around a consistent point $x_c \in X$, suppose that there exists an open neighborhood U of x_c such that $\dim E(x)T_xM^* = const.$ for $x \in M^* \cap U$. Then Ξ is locally internally regular around x_c if and only if*

$$\dim E(x)T_xM^* = \dim M^*, \quad \forall x \in M^* \cap U. \tag{5}$$

Two linear DAEs $\Delta = (E, H)$ and $\tilde{\Delta} = (\tilde{E}, \tilde{H})$ are called strictly equivalent or externally equivalent (see [31]) if there exist invertible matrices Q and P such that $\tilde{E} = QEP^{-1}$ and $\tilde{H} = QHP^{-1}$. The same notion can be extended to nonlinear DAEs.

Definition 2.6 (*External Equivalence*)**.** Two DAEs $\Xi_{l,n} = (E, F)$ and $\tilde{\Xi}_{l,n} = (\tilde{E}, \tilde{F})$ defined on X and $\tilde{X}$, respectively, are called externally equivalent, shortly ex-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$ and $Q : X \to GL(l, \mathbb{R})$ such that

$$\tilde{E}(\psi(x)) = Q(x)E(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1} \quad \text{and} \quad \tilde{F}(\psi(x)) = Q(x)F(x). \tag{6}$$

Fix a point $x_p \in X$, if ψ and Q is defined locally around x_p, we will speak about local ex-equivalence.

Remark 2.7. In the above definition of ex-equivalence, Q combines equations but does not change $\mathcal{C}^1$-solutions of the DAE; ψ defines new coordinates and maps $\mathcal{C}^1$-solutions to $\mathcal{C}^1$-solutions, i.e., a curve $x : \mathcal{I} \to X$ is a $\mathcal{C}^1$-solution of Ξ if and only if $\psi \circ x$ is a $\mathcal{C}^1$-solution of $\tilde{\Xi}$.

3. Geometric index-1 nonlinear DAEs

There are various notions of index for nonlinear DAEs, see our recent paper [30] and the references therein. In the present paper, we will use only the notion of geometric index, which is defined via the sequence of submanifolds $M_0^c \subsetneq \cdots \subsetneq M_k^c$ constructed by the geometric reduction method in Section 2.

Definition 3.1 (*Geometric Index [28,30]*)**.** Consider the sequence M_k^c constructed via Definition 2.2 around some consistent point $x_c \in S_c$, then the (local) geometric index, or shortly, the index, of a DAE Ξ is defined by

$$\nu_g := \min\left\{k \geq 0 \mid M_{k+1}^c = M_k^c\right\}.$$

Clearly, the geometric index ν_g is the least integer k such that the sequence of submanifolds M_k^c gets stabilized, which is also the smallest number of steps that has to be performed in order to construct the maximal invariant submanifold M^* and to solve the DAE.

Remark 3.2. A regular linear DAE $\Delta_{n,n} = (E, H)$ is always ex-equivalent, via two constant invertible matrices Q and P, to the Weierstrass form (**WF**)

$$\tilde{\Delta} = (QEP^{-1}, QHP^{-1}) : \begin{cases} \dot{x}_1 = A_1 x_1, \\ N\dot{x}_2 = x_2, \end{cases} \tag{7}$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ and N is a nilpotent matrix. The index ν of Δ is defined by the nilpotency of N, i.e., $N^{\nu-1} \neq 0$ and $N^{\nu} = 0$ (where $\nu = 0$ means that the x_2-variables vanish, see [32]). The geometric index ν_g is a nonlinear generalization of the index ν of linear DAEs [30]. Indeed, the index ν of Δ can be alternatively defined as: $\nu := \min\{k \geq 0 \mid \mathscr{V}_{k+1} = \mathscr{V}_k\}$, where the sequence $\mathscr{V}_k$ (called the Wong sequence [33]) is a linear counterpart of M_k^c and is given by

$$\mathscr{V}_0 = \mathbb{R}^n, \quad \mathscr{V}_{k+1} = H^{-1}E\mathscr{V}_k, \ \ k \geq 0. \tag{8}$$

Now for a DAE $\Xi_{l,n} = (E, F)$ and a consistent point $x_c \in X$, we introduce the following regularity and constant rank conditions:

(RE) $l = n$ and Ξ is locally internally regular;
(CR) there exists a neighborhood U of x_c such that $M_1^c = M_1 \cap U$ and the following ranks are constant: $\operatorname{rank} E(x) = const. = r$ for $x \in U$; $\dim E(x)T_x M_1^c = const.$ and $\dim \mathrm{D}F_2(x) = const.$ for $x \in M_1^c$, where $F_2 := F \backslash \operatorname{Im} E := Q_2 F$, where $Q_2 : U \to \mathbb{R}^{(n-r)\times n}$ is full row rank and $Q_2 E = 0$.

A linear DAE $\Delta_{l,n} = (E, H)$, given by (2), is regular if and only if $l = n$ and Δ is internally regular (see [31,32]). So condition **(RE)** is a nonlinear version of the regularity of linear DAEs. The condition $\operatorname{rank} E(x) = const. = r$ (throughout we denote this rank by r) ensures that there exists $Q : U \to GL(n, \mathbb{R})$ such that $E_1 : U \to \mathbb{R}^{r \times n}$ of $QE = \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right]$ is of full row rank r. The assumption $\operatorname{rank} \mathrm{D}F_2(x) = const.$ guarantees that the zero-level set $M_1^c = \{x \in U \mid F_2(x) = 0\}$ is a smooth embedded submanifold (by taking a smaller U, we can always assume M_1^c is connected) and the condition $\dim E(x)T_x M_1^c = const.$ excludes singular/impasses points (see [14]) and helps to view the DAE as a differential equation defined on its maximal invariant submanifold [1,28].

Proposition 3.3 (*Geometric Index-1*)**.** *Consider a DAE $\Xi = (E, F)$ and a consistent point $x_c \in S_c$. Assume that conditions* **(RE)**, **(CR)** *are satisfied in an open neighborhood U of x_c. Then the following statements are equivalent around x_c:*

(i) The DAE Ξ is of geometric index $\nu_g = 1$.
(ii) The locally maximal invariant submanifold satisfies $M^* = M_1^c$.
(iii) $\operatorname{rank} E(x) = \dim E(x)T_x M_1^c$ or, equivalently, $\ker E(x) \cap T_x M_1^c = 0$, $\forall x \in M_1^c$.
(iv) Let $Z : U \to \mathbb{R}^{n \times (n-r)}$ be any smooth map such that $\operatorname{Im} Z(x) = \ker E(x)$, $\forall x \in U$. Then $A(x) = \mathrm{D}F_2(x) \cdot Z(x)$ is invertible or, equivalently, $B(x) = \left[\begin{smallmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{smallmatrix}\right]$ is invertible, $\forall x \in M_1^c$.

(v) *There exists an open neighborhood* $V \subseteq U$ *of* x_c *such that* Ξ *is locally (on* V*) ex-equivalent to*

$$\begin{bmatrix} I_r & E_2(\xi_1, \xi_2) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} F^*(\xi_1, \xi_2) \\ \xi_2 \end{bmatrix}, \tag{9}$$

where $M^* \cap V = \{\xi \in V \mid \xi_2 = 0\}$, $\xi = (\xi_1, \xi_2)$ *and* ξ_1 *is a system of coordinates on* $M^* \cap V$.

The proof is given in Section 6.

Remark 3.4.

(i) By the constant rank assumption **(CR)**, we only need to check whether the item (iii) or (iv) of Proposition 3.3 holds at the point $x = x_c$ (or at any point x_0 of M_1^c) in order to conclude that Ξ is of geometric index-1 or not.

(ii) The map $F_2 = F \backslash \operatorname{Im} E$ in Proposition 3.3(iv) is not uniquely defined. More specifically, we may choose another invertible map $\tilde{Q} : U \to GL(n, \mathbb{R})$ such that $\tilde{E}_1$ of $\tilde{Q}E = \left[\begin{smallmatrix} \tilde{E}_1 \\ 0 \end{smallmatrix}\right]$ is of full row rank. Then $\tilde{F}_2$ of $\tilde{Q}F = \left[\begin{smallmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{smallmatrix}\right]$ is different from F_2, but there always exists $\bar{Q} : U \to GL(n - r, \mathbb{R})$ such that $\bar{Q}F_2 = \tilde{F}_2$. Then by $F_2(x) = 0$ on M_1^c, it is seen that $\mathrm{D}\tilde{F}_2(x) = \mathrm{D}(\bar{Q}F_2(x)) = \sum_{i=1}^{n-r} F_2^i(x)\mathrm{D}\bar{Q}_i(x) + \bar{Q}(x)\mathrm{D}F_2(x) = \bar{Q}(x)\mathrm{D}F_2(x)$, for all $x \in M_1^c$, where $\bar{Q}_i$ are the columns of $\bar{Q}$ and F_2^i are the rows of F_2. Therefore, item (iv) of Proposition 3.3 still holds even for any other choice of $\tilde{Q}$ since for all $x \in M_1^c$, $\tilde{A}(x) = \mathrm{D}\tilde{F}_2(x) \cdot Z(x) = \bar{Q}(x)\mathrm{D}F_2(x) \cdot Z(x) = \bar{Q}(x)A(x)$ is invertible if and only if $A(x)$ is invertible.

(iii) For a linear *regular* DAE $\Delta_{n,n} = (E, H)$, consider its index ν and the sequence $\mathscr{V}_i$ of (8). Then the following is equivalent: (i)′ $\nu = 1$; (ii)′ $\mathscr{V}_1$ is the largest subspace such that $A\mathscr{V}_1 \subseteq E\mathscr{V}_1$; (iii)′ $\operatorname{rank} E = \dim E\mathscr{V}_1$ or $\ker E \cap \mathscr{V}_1 = 0$; (iv)′ For any invertible matrices Q and P such that $QEP^{-1} = \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where $r = \operatorname{rank} E$, we have that A_4 of $QAP^{-1} = \left[\begin{smallmatrix} A_1 & A_2 \\ A3 & A_4 \end{smallmatrix}\right]$ is invertible; (v)′ Δ is ex-equivalent to the DAE $\dot{\xi}_1 = A^*\xi_1$, $0 = \xi_2$. Observe that item (iv)′ is also equivalent to $\operatorname{rank}[E, AZ] = n$, where Z is a full column rank matrix such that $\operatorname{Im} Z = \ker E$, or to $\operatorname{rank} \left[\begin{smallmatrix} E & 0 \\ A & E \end{smallmatrix}\right] = n + \dim \ker E$. The later two conditions are known (see e.g., [34]) to be characterizations of the impulse-freeness of linear DAEs.

4. Impulse-free jump solutions of nonlinear DAEs

We introduce the following definition of an impulse-free jump for a nonlinear DAE.

Definition 4.1 (*Impulse-free Jump*). Consider a DAE $\Xi = (E, F)$, let S_c be the consistency space of Ξ, fix an inconsistent initial point $x_0^- \in X/S_c$. An impulse-free jump solution (trajectory), shortly, an IFJ solution, of Ξ starting from x_0^- is a $\mathcal{C}^1$-curve $J : [0, a] \to X$ satisfying

$$J(0) = x_0^- \notin S_c, \quad J(a) = x_0^+ \in S_c, \quad \forall \tau \in [0, a] : \ E(J(\tau))\frac{dJ(\tau)}{d\tau} = 0. \tag{10}$$

A jump $x_0^- \to x_0^+$ associated with an IFJ trajectory $J(\cdot)$ is called an *impulse-free jump* of Ξ.

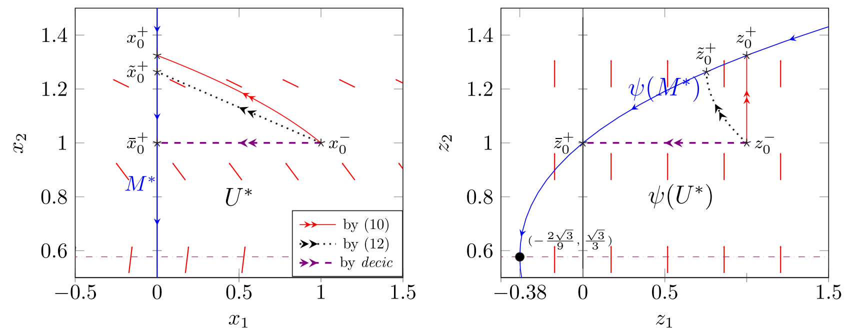
It is worth to remark that the parametrization variable τ of the differentiable curve $J(\tau)$ is, in general, *not* a time variable (unless we connect it with the time-variable t, see Section 4). In Definition 4.1, only the direction of the tangent vector $\frac{dJ(\tau)}{d\tau}$ is required to stay in $\ker E(J(\tau))$ while there are no other requirements on how fast the trajectory $J(\tau)$ should evolve with respect to τ (i.e, the magnitude of $\frac{dJ(\tau)}{d\tau}$). Moreover, even if the curve which we want to parameterize is possibly unique (indicating that there exists a unique impulse-free jump $x_0^- \to x_0^+$), the IFJ trajectory is always non-unique since there are infinitely many parameterizations of a curve. Indeed, by defining $\tilde{\tau} = \varphi(\tau)$ and $\tilde{J}(\varphi(\tau)) = J(\tau)$, where $\varphi : [0, a] \to [0, \tilde{a}]$ is diffeomorphism, we get $J(0) = x_0^-$, $J(\tilde{a}) = x_0^+$ and $E(\tilde{J}(\tilde{\tau}))\frac{d\tilde{J}(\tilde{\tau})}{d\tilde{\tau}} = E(J(\tau))\frac{d\tau}{d\tilde{\tau}}\frac{dJ(\tau)}{d\tau} = \frac{d\tau}{d\tilde{\tau}}E(J(\tau))\frac{dJ(\tau)}{d\tau} = 0$, $\forall \tau \in [0, \tilde{a}]$, which implies that $\tilde{J}(\tilde{\tau})$ is another IFJ trajectory of Ξ. The upper bound a of the domain of $J(\tau)$ is not fixed since it can always be scaled by φ into any $\tilde{\alpha} > 0$, including $\tilde{\alpha} = +\infty$.

Remark 4.2. We can regard the notion of IFJ trajectory as a nonlinear generalization of that of jump vector $e_0 = x_0^+ - x_0^-$ of linear DAEs. The impulse-free jump rule $E \cdot e_0\delta_0 = 0$ of linear DAEs is generalized into $Ee_0u(\tau) = 0$ for some $u : [0, a] \to \mathbb{R}$ with $\int_0^a u(\tau)d\tau = 1$. With other words we can consider the term $\frac{dJ(\tau)}{d\tau}$ in (10) as a linear control system (see also (15) below)

$$\frac{dJ(\tau)}{d\tau} = e_0u(\tau), \ \forall \tau \in [0, a], \quad e_0 \in \ker E, \quad J(0) = x_0^-, \quad J(a) = x_0^+. \tag{11}$$

In the following example, we will show an important feature i.e., the coordinate-freeness of our jump rule defined by (10) by comparing it with two existing jump rules: one is

$$x_0^+ - x_0^- \in \ker E(x_0^+) \tag{12}$$



(a) x-coordinates (b) z-coordinates

Fig. 1. The jumps calculated by (10), (12) and MATLAB *decic* function, respectively, shown in different coordinates. The red quiver plot illustrates the direction of $\ker E(x)$ and $\ker \tilde{E}(z)$, the blue solid lines with arrows represent the evolution of $\mathcal{C}^1$ solutions and the dash-dotted magenta lines depicts the sets of singular/impasses points.

introduced in [18] and another is given by the MATLAB function *decic* [21], which calculates consistent initial values for DAEs via a numerical searching method [20].

Example 4.3. Consider a DAE $\Xi_{2,2} = (E, F)$, given by

$$\Xi : \begin{bmatrix} 1 & 3x_2^2 - 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}. \tag{13}$$

Fix a point $x_p = (x_{1p}, x_{2p}) = (0, 1)$, it is clear that $M^* = M_1^c = \left\{ x \in \mathbb{R}^2 \mid x_1 = 0, x_2 > \frac{\sqrt{3}}{3} \right\}$ (note that M^* should be connected) and that $\dim E(x) T_x M^* = 1$, $\forall x \in M^* \cap U^*$, where $U^* = \left\{ x \in \mathbb{R}^2 \mid x_2 > \frac{\sqrt{3}}{3} \right\}$. By Proposition 2.3, $x_p = x_c$ is consistent, and M^* is the locally maximal invariant submanifold (around x_c) and coincides with the consistency space S_c on U^*. The inconsistent initial point which we consider is $x_0^- = (x_{10}^-, x_{20}^-) = (1, 1) \in U^* \backslash M^*$. For an IFJ solution $J : [0, a] \to X$ we make the following choice

$$\frac{dJ(\tau)}{d\tau} = \begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 1 - 3x_2^2 \\ 1 \end{bmatrix}, \quad J(0) = x_0^-. \tag{14}$$

The solution of (14) is $J(\tau) = (\tau + 2 - (\tau + 1)^3, \tau + 1)$ on the interval $[0, a]$ with $a \approx 0.3247$, which is indeed an IFJ trajectory of Ξ since $J(a) = x_0^+ \approx (0, 1.3247) \in M^* \cap U^*$ and $\begin{bmatrix} 1-3x_2^2 \\ 1 \end{bmatrix} \in \ker \begin{bmatrix} 1 & 3x_2^2-1 \\ 0 & 0 \end{bmatrix}$. Hence $x_0^- = J(0) \to x_0^+ = J(a)$ is an impulse-free jump in the sense of Definition 4.1. Secondly, we follow the jump rule $\tilde{x}_0^+ - x_0^- \in \ker E(\tilde{x}_0^+)$ of (12) to get three possible jumps $x_0^- \to \tilde{x}_0^+$ with either $\tilde{x}_0^+ = (0, 0)$, $\tilde{x}_0^+ = (0, \frac{1+\sqrt{7/3}}{2}) \approx (0, 1.2638)$ or $\tilde{x}_0^+ = (0, \frac{1-\sqrt{7/3}}{2}) \approx (0, -0.2638)$, but only the second is contained in U^*.

Thirdly, we calculate the consistent initial point for Ξ by MATLAB using *decic* function, the result is $\bar{x}_0^+ = (0, 1)$. We draw those three different jumps reaching at the consistent points

$$x_0^+ = (0, 1.3247), \quad \tilde{x}_0^+ = (0, 1.2638), \quad \bar{x}_0^+ = (0, 1),$$

in Fig. 1(a).

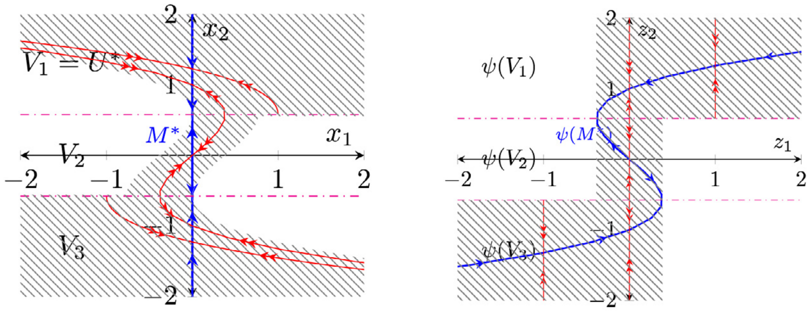
Now choose new coordinates $z = (z_1, z_2) = (x_1 + x_2^3 - x_2, x_2)$, then the DAE Ξ is ex-equivalent (on $V = U^*$), via the diffeomorphism $\psi(x) = z(x)$, to $\tilde{\Xi} = (\tilde{E}, \tilde{F})$ given by

$$\tilde{\Xi} : \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -z_2 \\ z_1 - z_2^3 + z_2 \end{bmatrix}.$$

Note that the DAE $\tilde{\Xi}$ is a degenerate form of the van der Pol oscillator equation, which was a well-studied case (see e.g., [13,17,23]) for analyzing discontinue solutions of semi-explicit DAEs. Under the new z-coordinates, the inconsistent initial point is $z_0^- = \psi(x_0^-) = (1, 1)$ and all three jump rules agree on the jump from z_0^- to $z_0^+ \approx (1, 1.3247)$. However, the transformed consistent points are given by, see also Fig. 1(b),

$$z_0^+ = \psi(x_0^+) = (1, 1.3247), \quad \tilde{z}_0^+ = \psi(\tilde{x}_0^+) = (0.7547, 1.2638), \quad \bar{z}_0^+ = \psi(\bar{x}_0^+) = (0, 1).$$

Clearly, $\tilde{z}_0^+$ and $\bar{z}_0^+$ do not coincide with the "correct" value z_0^+, which shows that the jump rule from [18] and MATLAB's *decic* jump rule are not invariant under coordinates transformations.



(a) x-coordinates (b) z-coordinates

Fig. 2. Semi-global impulse-free jump solutions of the DAE of Example 4.3 in different coordinates. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Remark 4.4.

(i) Recall from Remark 2.7 that the ex-equivalence preserves $\mathcal{C}^1$-solutions of DAEs. Now we show that for two ex-equivalent (via Q and ψ) DAEs Ξ and $\tilde{\Xi}$, there exists a one-to-one correspondence between any IFJ trajectory of Ξ and that of $\tilde{\Xi}$. More specifically, any IFJ trajectory $J(\tau)$ of Ξ is mapped via ψ into an IFJ trajectory $\tilde{J}(\tau) = \psi(J(\tau))$ of $\tilde{\Xi}$ (and vice versa) since by (6) and (10), we have

$$Q(J(\tau))E(J(\tau))\left(\frac{\partial \psi}{\partial x}(J(\tau))\right)^{-1}\frac{\partial \psi}{\partial x}(J(\tau))\cdot\frac{dJ(\tau)}{d\tau} = 0 \Rightarrow \tilde{E}(\psi(J(\tau)))\frac{d\psi(J(\tau))}{d\tau} = 0.$$

As a result, the impulse-jump $x_0^- \to x_0^+$ is mapped via ψ into $z_0^- = \psi(x_0^-) \to z_0^+ = \psi(x_0^+)$.

(ii) As the jumps defined by the rule (10) are invariant under coordinates transformations, we can choose suitable new coordinates such that the structure of the DAE is simplified in order to calculate IFJs. The DAE $\tilde{\Xi}$ in the new z-coordinates of Example 4.3 is easier for the analysis of IFJs since the distribution $\mathcal{E} = \ker E$ is rectified into $\operatorname{span}\left\{\frac{\partial}{\partial z_2}\right\}$ such that only z_2-variables are allowed to change. Observe in Fig. 1(b) that for any inconsistent initial point $z_0^- = (z_{10}^-, z_{20}^-) \in \psi(U^*)$ such that $z_{10}^- < -\frac{2\sqrt{3}}{9}$, there does not exist an impulse free jump on $\psi(U^*)$ since we cannot steer z_0^- into $\psi(M^*)$ on $\psi(U^*)$ without changing z_1-variables.

(iii) Note that although the analysis in Example 4.3 are local results as we consider local coordinates transformations defined on the neighborhood $U^* \subseteq \mathbb{R}^2$ only, we show below that those results can be extended to an open and dense subset of $\mathbb{R}^2$. Observe that the two DAE Ξ and $\tilde{\Xi}$ are locally ex-equivalent not only on $V_1 = U^*$, but also on the other two connected subsets $V_2 = \left\{x \in \mathbb{R}^2 \mid -\frac{\sqrt{3}}{3} < x_2 < \frac{\sqrt{3}}{3}\right\}$ and $V_3 = \left\{x \in \mathbb{R}^2 \mid x_2 < -\frac{\sqrt{3}}{3}\right\}$. Observe that $X = \mathbb{R}^2 = \bigcup_{i=1}^3 \mathrm{cl}(V_i)$, by an analysis for the inconsistent initial points on V_2 and V_3, we get a semi-global result of the existence of impulse-free jump solutions for almost all points of $\mathbb{R}^2$ (except for the singular set $\left\{x \in \mathbb{R}^2 \mid x_2 = \pm\frac{\sqrt{3}}{3}\right\}$). We draw the results of analysis in Fig. 2, where the shadow area depicts the set of inconsistent initial points which admits an impulse-free jump. Note that if we allow impulse-free jumps to cross the singular set, then we may find impulse-free jumps for the inconsistent points in the white area in Fig. 2, e.g., an inconsistent point $z_0^- = (1, 0)$ on Fig. 2(b) can then jump upwards to $z_0^+ \approx (1, 1.3247)$, nevertheless, we may loss the uniqueness of impulse-free jumps, e.g., for any point $(0, z_{20}^-)$ with $0 < z_{20}^- < 1$, it may jump upwards to $(0, 1)$ or downwards to $(0, 0)$ or $(0, -1)$ along z_2-axis.

In the following discussions, we will focus on impulse-free jumps in a neighborhood of a consistent point x_c to study their existence and uniqueness. Consider the jump rule (10) in Definition 4.1, the collection of all $\frac{dJ(\tau)}{d\tau}$ satisfying $E(J(\tau))\frac{dJ(\tau)}{d\tau} = 0$ is given by the differential inclusion $\frac{dJ(\tau)}{d\tau} \in \ker E(J(\tau))$. Assume that $\operatorname{rank} E(x) = const. = r$, then $\dim\ker E = const. = n - r$, we can choose locally $m = n - r$ independent vector fields $g_1, \ldots, g_m$ such that

$$\operatorname{span}\{g_1, \ldots, g_m\} = \ker E.$$

By introducing extra variables u_i, $i = 1, \ldots, m$, we parameterize the distribution $\ker E$ and thus all solutions of the differential inclusion $\frac{dJ(\tau)}{d\tau} \in \ker E(J(\tau))$ are given by all solutions of the drift-less control system (corresponding to all controls $u_i(\tau) \in \mathbb{R}$):

$$\Sigma: \quad \frac{dJ(\tau)}{d\tau} = \sum_{i=1}^{m} g_i(J(\tau))u_i(\tau), \quad x(0) = x_0^-. \tag{15}$$

So the existence of an IFJ solution of Ξ is equivalent to that of an input $u(\cdot)$ such that the solution $J(\cdot)$ of Σ staring from x_0^- can reach a consistent point $x_0^+ \in M^*$; such a problem is related to the reachability analysis of nonlinear control systems.

Remark 4.5. In practice, we may interpret the u-variables in the control system Σ as some unknown forces steering the inconsistent initial value x_0^- into the consistency set S_c of the DAE Ξ. The u-variables can be seen as an analog of the Dirac impulse δ in the distributional solutions of linear DAEs (compare Remark 4.2). Note that we may solve the linear ODE (11) with $u = \delta_0$ in the sense of distribution (generalized function) while it is hard to solve $\Sigma_\delta : \frac{dx}{d\tau} = \sum_{i=1}^m g_i(x)\delta$, which is a nonlinear ODE with distributions (generalized function) in coefficients (see some discussions on its solutions in Chapter 3 of [35]), or a control system with impulsive/measure inputs (see e.g. [36,37]).

In order to prove the existence and uniqueness of an impulse-free jump, let us first recall some notions as integral manifolds, involutivity, invariant distributions from differential geometry and the reachability analysis in nonlinear control theory (see e.g. Chapter 2 of [38] and Chapter 1 of [39]). A distribution $\mathcal{D}$ is said to be *invariant* under a vector field f if the Lie brackets $[f, g] \in \mathcal{D}$, $\forall g \in \mathcal{D}$. For a DAE $\Xi = (E, F)$, fix a consistent point $x_c \in X$, let $\mathcal{E} = \ker E = \operatorname{span}\{g_1, \dots, g_m\}$ and denote by $\langle g_1, \dots, g_m | \mathcal{E}\rangle$ the smallest invariant distribution under $g_1, \dots g_m$ which contains $\mathcal{E} = \ker E$. Then we introduce the following assumption:

(DS) there exists a neighborhood U of x_c such that the distribution $\mathcal{D} := \langle g_1, \dots, g_m | \mathcal{E}\rangle$ is nonsingular, i.e., $\dim \mathcal{D}(x) = const. = k \geq m$ for all $x \in U$.

Note that if **(DS)** is satisfied, then the distribution $\mathcal{D}$ is involutive (see Lemma 1.8.5 of [39]) and by Frobenius theorem, for any point $x_0^- \in U$, we can find a neighborhood $V \subseteq U$ of x_0^- and a coordinate transformation $z = \Phi(x) = (\phi_1(x), \dots, \phi_n(x))$ such that $\operatorname{span}\{d\phi_1, \dots, d\phi_{n-k}\} = \mathcal{D}^\perp$, where $\mathcal{D}^\perp$ denotes the co-distribution annihilating $\mathcal{D}$. The integral submanifold of the distribution $\mathcal{D}$ passing through x_0^- is given by

$$N_{x_0^-} = \left\{ x \in V \mid \phi_1(x) = \phi_1(x_0^-), \dots, \phi_{n-k}(x) = \phi_{n-k}(x_0^-) \right\}.$$

Note that $N_{x_0^-} \subseteq V$ coincides with the local *reachable space* $R^V(x_0^-)$ of Σ from x_0^- (see Propositions 3.12 and 3.15 of [38]). Now we are ready to present our results of the existence and uniqueness of local impulse-free jumps in a neighborhood V of a consistent point $x_c \in X$ for index-1 nonlinear DAEs.

Theorem 4.6. *Consider a DAE $\Xi = (E, F)$ and a consistent point $x_c \in X$. Assume that conditions* (**RE**), (**CR**), (**DS**) *are satisfied in a neighborhood U of x_c. Suppose that Ξ is index-1, implying (by Proposition 3.3) that $M^* = M_1^c \subsetneq U$ is a locally maximal invariant submanifold around x_c. Then for any point $x_0^- \in V \backslash M^*$ satisfying $N_{x_0^-} \cap M^* \neq \emptyset$ in a neighborhood $V \subseteq U$ of x_c, there exists an IFJ solution $J(\tau)$ of Ξ with*

$$J(0) = x_0^- \quad \text{and} \quad J(a) = x_0^+ \in M^* \cap N_{x_0^-},$$

where $N_{x_0^-} \subseteq V$ is the integral submanifold of the distribution $\mathcal{D} = \langle g_1, \dots, g_m | \ker E\rangle$. Moreover, the following statements are equivalent around x_c:

(i) The impulse-free jump $x_0^- \to x_0^+$ is unique.
(ii) The distribution $\mathcal{E} = \ker E$ is involutive.
(iii) Ξ is locally on V ex-equivalent to the following index-1 nonlinear Weierstrass form

$$(\mathbf{INWF}) : \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} F^*(\xi_1) \\ \xi_2 \end{bmatrix}, \tag{16}$$

where $M^* \cap V = \{\xi \in V \mid \xi_2 = 0\}$, $\xi = (\xi_1, \xi_2)$ and ξ_1 is a system of coordinates on $M^* \cap V$.

The proof is given in Section 6. Note that for DAE Ξ in Theorem 4.6, the reachability condition $N_{x_0^-} \cap M^* \neq \emptyset$ is necessary for the existence of the impulse-free jumps, we will discuss it in details in the following remark.

Remark 4.7.

(i) For a linear index-1 regular DAE $\Delta = (E, H)$, the submanifold M^* is the flat manifold passing through $x = 0$ with its tangent space being $\mathscr{V}^*$ (i.e., the limit of Wong sequence $\mathscr{V}_i$, see (8)) and $N_{x_0^-}$ is the flat manifold passing through x_0^- with its tangent space being $\ker E$. Note that we always have $\mathscr{V}^* \oplus \ker E = \mathbb{R}^n$ as Δ is index-1 and regular. Thus the intersection $N_{x_0^-} \cap M^*$ is non-empty and $\dim(\mathscr{V}^* \cap \ker E) = \dim(N_{x_0^-} \cap M^*) = 0$ proves that $N_{x_0^-} \cap M^*$ is a point. Moreover, the subspace $\ker E$ of Δ is clearly involutive and Δ is always ex-equivalent to the index-1 **(WF)** on $\mathbb{R}^n$.

(ii) The set $N_{x_0^-} \cap M^*$ could be empty in the nonlinear case due to the existence of singular points, e.g., any inconsistent initial point x_0^- in the white area of Fig. 2 cannot reach/jump impulse-freely to the blue line $M^* = S_c$ (unless it is allowed to cross the singular points) because on each V_i, $i = 1, 2, 3$, the local reachable set $N_{x_0^-} \subseteq V_i$ has no intersections with S_c. So the set of points from which there exists an impulse-free jump is a subset of V_i (e.g., the shadow area in Fig. 2), which we will call the local *admissible impulse-free jump set* in V_i.

(iii) For a DAE Ξ, being index-1 is not a necessary condition for the existence of impulse-free jumps. Take the following DAE for example: $\Xi : \begin{bmatrix} 0 & x & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$, which is of geometric index-2 since $M^* = M_2^c = \{x = y = z = 0\}$. The corresponding IFJ control system is $\begin{bmatrix} \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -x \end{bmatrix} u$, which for any $(x_0^-, y_0^-, z_0^-) \notin M^*$ with $x_0^- \neq 0$ is controllable to $(x_0^+, y_0^+, z_0^+) = (0, 0, 0)$, i.e. there exists $a > 0$ and an input $u(\cdot)$ such that the solution $J(\tau) = (x(\tau), y(\tau), z(\tau))$ satisfies $J(0) = (x_0^-, y_0^-, z_0^-)$ and $(x(a), y(a), z(a)) = (0, 0, 0)$. Clearly, $J(\cdot)$ is an IFJ trajectory of Ξ and $(x_0^-, y_0^-, z_0^-) \to (0, 0, 0)$ is an impulse-free jump.

For a linear regular DAE $\Delta_{n,n} = (E, H)$, its consistency projector [18,22] is defined by

$$\Pi_{E,H} := P^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} P,$$

where the dimension n_1 and the matrix P come from the **(WF)** of Δ, given by (7). We now generalize the above notion of consistency projector to nonlinear DAEs with the help of the **(INWF)**, given by (16).

Definition 4.8 (*Nonlinear Consistency Projector*)**.** Consider a DAE $\Xi_{l,n} = (E, F)$ and a consistent point $x_c \in X$. Assume that there exists a neighborhood V of x_c such that Ξ is locally (on V) ex-equivalent to the **(INWF)** of (16) via a Q-transformation and a local diffeomorphism ψ. The nonlinear (local) *consistency projector* $\Omega_{E,F} : V \backslash M^* \to V \cap M^*$ of Ξ is then defined by

$$\Omega_{E,F} := \psi^{-1} \circ \pi \circ \psi,$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection $(\xi_1, \xi_2) \mapsto (\xi_1, 0)$.

For a linear DAE Δ, any inconsistent initial value x_0^- of Δ jumps to $x_0^+ = \Pi x_0^-$ and the jump $x_0^- \to x_0^+$ is impulse-free, i.e., $e_0 = x_0^+ - x_0^-$, if and only if $E(I - \Pi) = 0$ (compare Theorem 3.8 of [18]), which actually is equivalent to that Δ is index-1. For a nonlinear DAE, in order that the existence and uniqueness of impulse-free jumps are satisfied, we need both that Ξ is index-1 and that $\ker E$ is involutive, as seen from the following corollary.

Corollary 4.9. *Consider a DAE $\Xi = (E, F)$ and a consistent point $x_c \in X$. Assume that the conditions* ***(RE)*** *and* ***(CR)*** *are satisfied in an open neighborhood U of x_c. Then there exists a neighborhood $V \subseteq U$ of x_c such that for any inconsistent initial point $x_0^- \in V/M^*$ satisfying $N_{x_0^-} \cap M^* \neq \emptyset$, there exists a unique impulse-free jump $x_0^- \to x_0^+$ if and only if Ξ is index-1 and $\mathcal{E} = \ker E$ is involutive. Let $\Omega_{E,F}$ be the consistency projector of Ξ defined on V, the unique impulse-free jump is given by*

$$x_0^- \to x_0^+ = \Omega_{E,F}(x_0^-) \in M^* \cap N_{x_0^-}.$$

Proof. "*Only if.*" Suppose that $x_0^- \to x_0^+$ is unique, that is, $N_{x_0^-} \cap M^*$ is a unique point x_0^+ on M^*. It follows that $\dim(M^* \cap N_{x_0^-}) = 0$, which implies that

$$T_{x_0^+} M^* \cap T_{x_0^+} N_{x_0^-} = T_{x_0^+} M^* \cap \ker E(x_0^+) = 0. \tag{17}$$

Thus we have that Ξ is index-1 by Proposition 3.3. Hence by Theorem 4.6, the impulse-free jump is unique implies that $\mathcal{E} = \ker E$ is involutive.

"*If.*" Suppose that the distribution $\mathcal{E} = \ker E$ is involutive, then the condition $\operatorname{rank} E(x) = const.$ of **(CR)** implies that $\mathcal{D} = \langle g_1, \ldots, g_m | \mathcal{E} \rangle = \mathcal{E} = \ker E$ is nonsingular (i.e., **(DS)** holds). Suppose additionally that Ξ is index-1, then by Theorem 4.6, Ξ is ex-equivalent to the **(INWF)**, given by (16), and there exists a unique impulse-free jump

$$x_0^- = \psi^{-1}(\xi_0^-) \to x_0^+ = \psi^{-1}(\xi_0^+) \in M^* \cap N_{x_0^-},$$

where $\xi_0^+ = \pi(\xi_0^-)$ since for the **(INWF)**, only ξ_2-variables are allowed to jump. It follows that $x_0^+ = \psi^{-1} \circ \pi \circ \psi(x_0^-) = \Omega(x_0^-)$. □

Remark 4.10. A similar definition of nonlinear consistency projector can be found in [40], where index-1 DAEs are studied and it is assumed that they are global equivalent (actually ex-equivalent using Definition 2.6) to a semi-explicit form. Such an assumption is equivalent to the involutivity assumption of $\ker E$ (see Theorem 3.13 of [28]) when the singular points are not considered. But it can be seen from Remarks 4.4(iii) and 4.7(iii) above, those singular points actually play important roles for the existence of impulse-free jumps. Note that under an additional Q-transformation, we can always transform the semi-explicit form in [40] into our **(INWF)**. Moreover, we have shown a way of constructing the (Q, ψ)-transformations to obtain the **(INWF)** and to define the nonlinear consistency projector in the proof of Theorem 4.6, those results are not discussed in [40].

Example 4.11. We reconsider the DAE Ξ, given by (13), of Example 4.3. It is clear that Ξ is of index-1 and that the distribution $\mathcal{E} = \ker E =$ is involutive. We have that Ξ with the initial point $x_0^- = (1, 1)$ is locally ex-equivalent to its

INWF represented by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -f(\xi_1, 0) \\ \xi_2 \end{bmatrix}, \ \xi_0^- = \psi(x_0^-) = (1, 1), \tag{18}$$

on $V = U^* = \left\{ x \in \mathbb{R}^2 \mid x_1 = 0, x_2 > \frac{\sqrt{3}}{3} \right\}$, via $\psi = \xi = (\xi_1, \xi_2) = (x_1 + x_2^3 - x_2, x_1)$ and $Q = \left[\begin{smallmatrix} 1 & f' \\ 0 & 1 \end{smallmatrix}\right]$, where $f(\xi) = f(\xi_1, 0) + f'(\xi)\xi_2$ and

$$f = \frac{1}{3}\left(a + \sqrt{a^2 - \frac{1}{27}}\right)^{-\frac{1}{3}} + \left(a + \sqrt{a^2 - \frac{1}{27}}\right)^{\frac{1}{3}}, \quad a(\xi_1, \xi_2) = \frac{\xi_1 - \xi_2}{2}.$$

Thus the nonlinear (local) consistency projector of Ξ is

$$\Omega = \psi^{-1} \circ \pi \circ \psi = \begin{bmatrix} 0 \\ f(x_1 + x_2^2 - x_2, 0) \end{bmatrix}.$$

Hence $x_0^+ = \Omega(x_0^-) \approx (0, 1.3247)$, which agrees with the result of Example 4.3.

5. A singular perturbed system approximation of nonlinear DAEs

Singular perturbation theory was frequently used (see e.g., [7,13,17,23]) to approximate DAEs of the semi-explicit form (3), the main idea is to regularize a DAE Ξ^{SE} of the form (3) by replacing the algebraic constraint $0 = f_2(x_1, x_2)$ with $\varepsilon\dot{x}_2 = f_2(x_1, x_2)$, where ε represents some ignored small modeling parameters (e.g., the small inductance of an inductor, see the electric circuits on page 367 of [17]). Then by rescaling time t to τ such that $\frac{d\tau}{dt} = \frac{1}{\varepsilon}$, we get a perturbed system in the time-scale τ as shown on the right-hand side of the following equations:

$$\Xi_\varepsilon^{SE} : \begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \varepsilon\dot{x}_2 = f_2(x_1, x_2). \end{cases} \quad \overset{\varepsilon = \frac{dt}{d\tau}}{\Leftrightarrow} \quad \begin{cases} \dfrac{dx_1}{d\tau} = \varepsilon f_1(x_1, x_2), \\ \dfrac{dx_2}{d\tau} = f_2(x_1, x_2). \end{cases}$$

Note that additionally to the requirement that f_1, f_2 are sufficiently smooth, there are, in general, two assumptions in the above approximation method of DAEs: (a) there exists a unique solution $(x_1(t), x_2(t))$ of Ξ^{SE} on the finite interval $[a, b]$ starting from a consistent initial point (x_{10}^+, x_{20}^+); (b) the Jacobian matrix $\frac{\partial f_2}{\partial x_2}(x_1(t), x_2(t))$ has all its eigenvalues $\lambda(t)$ satisfying $\mathrm{Re}\,\lambda(t) \leq 0$ for all $t \in [a, b]$. Assumption (a) means that the DAE Ξ^{SE} is internally regular, and assumption (b) implies that Ξ^{SE} is (locally) index-1 and the origin is asymptotically stable equilibrium point of the so-called boundary layer model $\frac{dy}{d\tau} = f_2(x_1(t), y + h(x_1(t)))$, where $x_2 = h(x_1)$ is the unique solution of $0 = f_2(x_1, x_2)$. Then under assumptions (a),(b), the well-known Tihkonov's theorem (see e.g., [23] and its infinite time interval extension in [41]) for sufficient small $\varepsilon > 0$ and for any (x_{10}^-, x_{20}^-) satisfying $x_{10}^- = x_{10}^+$ and $y_0^- = x_{20}^- - h(x_{10}^+)$ contained in a compact subset of the region of attraction of the boundary layer model, there exists a solution $(\bar{x}_1(t, \varepsilon), \bar{x}_2(t, \varepsilon))$ of Ξ_ε^{SE} starting from (x_{10}^-, x_{20}^-) such that

$$\lim_{\varepsilon \to 0} \|x_1(t) - \bar{x}_1(t, \varepsilon)\| = 0, \quad \text{and} \quad \lim_{\varepsilon \to 0} \|x_2(t) - \bar{x}_2(t, \varepsilon)\| = 0,$$

for all $a < c \leq t \leq b$. In this section, we will propose a singular perturbed system approximation for index-1 nonlinear DAEs Ξ with the help of the results in Proposition 3.3 and Theorem 4.6.

Definition 5.1 (*Singular Perturbed System*)**.** Consider a DAE $\Xi_{l,n} = (E, F)$ and fix a consistent point x_c. Assume that there exists a neighborhood V of x_c such that Ξ is locally (on V) ex-equivalent to the DAE (9) (or in particular, the **(INWF)** of (16)) via a Q-transformation and a local diffeomorphism ψ. Define the following singular perturbed system on V:

$$\Xi_\varepsilon : \dot{x} = E_W^{-1}(x, \varepsilon)F(x) \text{ with } E_W(x, \varepsilon) = E(x) + Q^{-1}(x)\begin{bmatrix} 0 & 0 \\ 0 & \varepsilon W^{-1} \end{bmatrix} \frac{\partial \psi(x)}{\partial x}, \tag{19}$$

where $W \in \mathbb{R}^{m \times m}$ a Hurwitz matrix. Then by rescaling time t to τ, where $\frac{d\tau}{dt} = \frac{1}{\varepsilon}$, we define

$$\Sigma_\varepsilon : \ \frac{dx}{d\tau} = f(x, \varepsilon), \tag{20}$$

where $f(x, \varepsilon) = \varepsilon E_W^{-1}(x, \varepsilon)F(x) = \left(\frac{\partial \psi(x)}{\partial x}\right)^{-1} \left[\begin{smallmatrix} \varepsilon F^* \circ \xi_1 \circ \psi(x) - E_2 \circ \psi \cdot W \cdot \xi_2 \circ \psi(x) \\ W \xi_2 \circ \psi(x) \end{smallmatrix}\right]$.

Remark 5.2. Any linear index-1 regular DAE $\Delta = (E, H)$ of the form (2) is always ex-equivalent, via two constant matrices Q and P, to the **(WF)** of (7) with $N = 0$, i.e., $QEP^{-1} = \left[\begin{smallmatrix} I_{n_1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $QHP^{-1} = \left[\begin{smallmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{smallmatrix}\right]$. Applying the construction of (19) to Δ and setting $W = -I_{n_2}$, we get the following singular perturbed system:

$$\Delta_\varepsilon : \dot{x} = E^{-1}(\varepsilon)Hx = P^{-1}\left[\begin{smallmatrix} I_{n_1} & 0 \\ 0 & -\frac{1}{\varepsilon}I_{n_2} \end{smallmatrix}\right] QHx = P^{-1}\left[\begin{smallmatrix} A_1 & 0 \\ 0 & -\frac{1}{\varepsilon}I_{n_2} \end{smallmatrix}\right] Px,$$

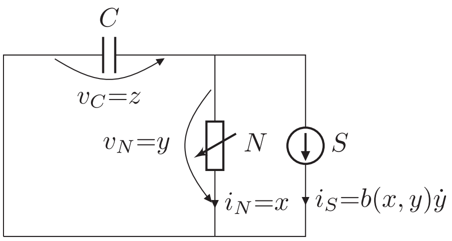


Fig. 3. An electric circuit with nonlinear resistor and controlled current source.

where $E(\varepsilon) = Q^{-1}\left[\begin{smallmatrix} I_{n_1} & 0 \\ 0 & -\varepsilon I_{n_2}\end{smallmatrix}\right]P$. Note that the above perturbed linear system Δ_ε is proposed in Section 4 of [42] as an ODE approximation of a linear DAE Δ.

The following theorem shows that the solution $\bar{x}(t)$ of the proposed perturbed system Ξ_ε, given by (19), coincides with the $\mathcal{C}^1$-solution $x(t)$ of Ξ when staring from a consistent point x_0^+ and that the solution $\bar{x}_W(\tau,\varepsilon)$ of Σ_ε in the rescaled time τ, given by (20), converges to an impulse-free jump trajectory $J_W(\tau)$ of Ξ when staring from an inconsistent point x_0^-.

Theorem 5.3. *Consider a DAE $\Xi = (E, F)$ and the singular perturbed systems (19) and (20) constructed in Definition 5.1. Assume that*

(SP) *for a certain compact subset $\mathfrak{V} \subseteq V$ and any inconsistent initial point $x_0^- \in \mathfrak{V}\backslash M^*$, the system Σ_ε, given by (20), has a unique solution $J_W : [0,+\infty) \to \mathfrak{V}$ for $\varepsilon = 0$ and a unique solution $\bar{x}_W(\cdot,\varepsilon) : [0,+\infty) \to \mathfrak{V}$ for all $0 < \varepsilon \leq \varepsilon^*$ with $\varepsilon^* \in \mathbb{R}^+$ sufficiently small.*

Then we have

$$\lim_{\varepsilon\to 0} \|\bar{x}_W(\tau,\varepsilon) - J_W(\tau)\| = 0, \quad \forall \tau \in [0,+\infty). \tag{21}$$

Suppose additionally that $J_W(+\infty) := \lim_{\tau\to\infty} J_W(\tau)$ is well defined, then $J_W(\tau)$ is an IFJ solution of Ξ and $x_0^+ := J_W(+\infty)$ is consistent and in general depends on the choice of the Hurwitz matrix W. However, if Ξ is locally (on V) ex-equivalent to the (**INWF**) *of (16), then the consistent point $x_0^+ = J_W(+\infty)$ is independent of the choice of W (actually $x_0^+ = \Omega_{E,F}(x_0^-)$ by Corollary 4.9).*

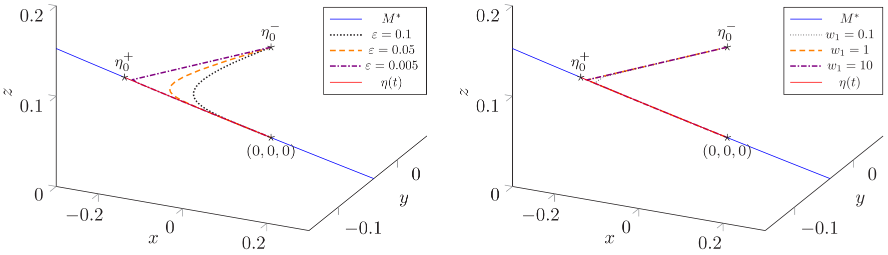
Furthermore, the solution $\bar{x}(t) : \mathcal{I} \to M^*$ of the perturbed system Ξ_ε, given by (19), starting from any consistent initial point on M^* coincides with the $\mathcal{C}^1$-solution $x(t)$ of Ξ, which does not depend on ε and W.

The proof is given is Section 6.

Remark 5.4. (i) Assumption **(SP)** is not only an existence and uniqueness condition for the solutions $J_W(\tau)$ and $\bar{x}_w(\tau)$ on the interval $[0,\infty)$, but also a stability assumption since we require the solutions lie entirely in the compact set $\mathfrak{V}$. Indeed, if the solutions are not stable, they will leave any compact set in finite time. Note that in classical singular perturbation theory with infinite time interval, one can find similar stability assumptions (see assumptions (VI) and (VII) in [41]) to guarantee the convergence of the difference of solutions. Actually, assumption **(SP)** automatically holds if there exists an asymptotically stable equilibrium $x_0^+ \in M^* \cap N_{x_0^-}$ for Σ_0, i.e., (20) with $\varepsilon = 0$, which can be proved by constructing a Lyapunov function $V(x)$ for Σ_0 and show that $\frac{\partial V(x)}{\partial x} f(x,\varepsilon) < 0$ for sufficient small ε (cf. the proof in [41]).

(ii) A simple choice of the Hurwitz matrix W is $W = \operatorname{diag}\{-w_1,\ldots,-w_m\}$ with $w_i \in \mathbb{R}^+$. The parameters w_i are weight coefficients indicating the rate of convergence of $J(\tau) \to x_0^+$ as $\tau \to \infty$. As seen from (31) below, the solution of the ξ_2^i-subsystem from ξ_{20}^{i-} is $\xi_2^i(\tau) = e^{-w_i\tau}\xi_{20}^{i-}$, so w_i is the rate of convergence for $\xi_2^i(\tau) \to 0$. Recall that an IFJ solution of Ξ can be seen as a solution of a control system $\frac{dJ(\tau)}{d\tau} = \sum_{i=1}^m g_i(J(\tau))u_i(\tau) = g(J(\tau))u(\tau)$ (see (15)), thus the choice of w_i can be regarded as some particular choices of the inputs u_i, e.g., we have that $g = \left[\begin{smallmatrix} E_2 \\ I_m \end{smallmatrix}\right]$ and $u(\tau) = W\xi_2(\tau)$ for (31), so $u(\tau)$ is a particular feedback which stabilizes the ξ_2-subsystem. As a consequence, the solutions $\bar{x}_W(\tau,\varepsilon)$ corresponding to all W-matrices may not approximate all the possible impulse-free jumps, meaning that the set of all $x_0^+ = \lim_{\varepsilon\to 0}\bar{x}_W(+\infty,\varepsilon)$ corresponding to all W-matrices is a subset of $M^* \cap N_{x_0^-}$, the latter is the set of all points which can be jumped into from x_0^-, see Theorem 4.6.

Example 5.5. Inspired by the simple circuit discussed in [13,14,17], consider the electrical circuit shown in Fig. 3, which consists of a capacitor C and a nonlinear resistor N. A controlled current source S is additionally connected in parallel with N in order to generate nonlinear terms in $E(x)$ of the DAE model. Note that controlled current sources have been used in [43] for electric circuits analog of mechanical systems under non-holonomic constraints.



(a) Fix $w_1 = 1$, $w_2 = 1$ and change ε.

(b) Fix $\varepsilon = 0.001$, $w_2 = 1$ and change w_1.

Fig. 4. The solutions $\bar{\eta}_W(t, \varepsilon)$ of Ξ_ε with different parameters and the solution $\eta(t)$ of Ξ.

The relation between the current $i_N = x$ and the voltage $v_n = y$ of the nonlinear resistor N is characterized by the following algebraic equation

$$0 = a(x, y),$$

and the current i_S of S is equal to $b(x, y)\dot{y}$, where $a : \mathbb{R}^2 \to \mathbb{R}$ and $b : \mathbb{R}^2 \to \mathbb{R}$ are smooth maps . Using Kirchoff's law, we model the circuit as a DAE $\Xi_{3,3} = (E, F)$:

$$\begin{bmatrix} 0 & -b(x,y) & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} x \\ y+z \\ a(x,y) \end{bmatrix}.$$

Let $\eta = (x, y, z)$ and $\eta_c = (0, 0, 0)$, we consider two different cases, for which the distribution $\mathcal{E} = \ker E$ is involutive in Case 1 but is not in Case 2.

Case 1: Consider $a(x, y) = x - y^2 - 2y$, $b(x, y) = y$, $C = 1$, conditions **(RE)**, **(CR)** are satisfied on $U = \{\eta \in \mathbb{R}^3 \mid y < 1\}$ (note that $\dim E(\eta)T_\eta M_1^c = 0$ for $y = 1$). The locally maximal invariant submanifold M^* (around η_c) is

$$M^* = M_1^c = \left\{\eta \in U \mid y + z = x - y^2 - 2y = 0\right\}.$$

Since $\mathcal{E} = \ker E = \operatorname{span}\{\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\}$ is involutive and Ξ is of index-1, the DAE Ξ is locally (on $V = U$) ex-equivalent to the following DAE represented in **(INWF)**:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{y}} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} -2\tilde{z} \\ \tilde{y} \\ \tilde{x} \end{bmatrix}. \tag{22}$$

via

$$Q = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \psi = \tilde{\eta} = (\tilde{z}, \tilde{y}, \tilde{x}) = (-\frac{1}{2}y^2 + z, y + z, x - y^2 - 2y).$$

Following (19) of Definition 5.1, we construct a singular perturbed system Ξ_ε (we choose $W = \begin{bmatrix} -w_1 & 0 \\ 0 & -w_2 \end{bmatrix}$):

$$Q^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\varepsilon}{w_1} & 0 \\ 0 & 0 & -\frac{\varepsilon}{w_2} \end{bmatrix} \frac{\partial \psi}{\partial \eta} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} x \\ y+z \\ x-y^2-2y \end{bmatrix} \Rightarrow \Xi_\varepsilon : \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f_1(\eta,\varepsilon,w_1,w_2) \\ f_2(\eta,\varepsilon,w_1,w_2) \\ f_3(\eta,\varepsilon,w_1,w_2) \end{bmatrix},$$

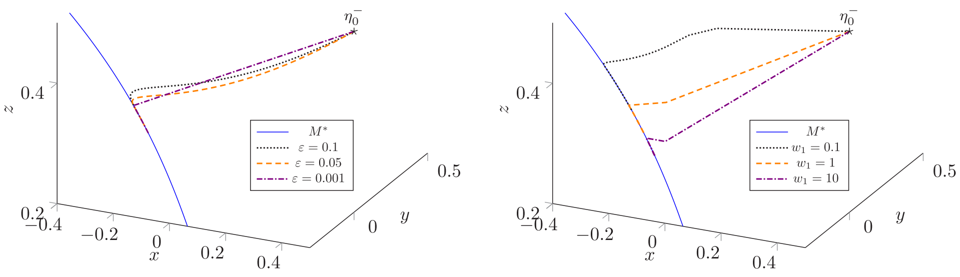
where $f_1 = -\frac{-w_2x+w_2y(2+y)-2\varepsilon(y^2-2z)-2w_1(y+z)}{\varepsilon}$, $f_2 = -\frac{w_1y+\varepsilon y^2-2\varepsilon z+w_1z}{\varepsilon+\varepsilon y}$, $f_3 = \frac{\varepsilon(y^2-2z)-w_1y(y+z)}{\varepsilon(1+y)}$. Consider an inconsistent initial point $\eta_0^- = (0, 0, 0.1) \in V\backslash M^*$, by Corollary 4.9, we get

$$\eta_0^+ = \Omega_{E,F}(\eta_0^-) = \psi^{-1} \circ \pi \circ \psi(\eta_0^-) = (-0.2, -0.1056, 0.1056),$$

which defines the unique impulse-free jump $\eta_0^- \to \eta_0^+$ of Ξ. Now we use MATLAB ode45 solver to simulate the solutions $\bar{\eta}_W(t, \varepsilon)$ of the perturbed system Ξ_ε for different ε, w_1 and w_2. First, we fix $w_1 = 1$ and $w_2 = 1$, and change ε from 0.1 to 0.05 and 0.005; as seen from Fig. 4(a), the solution $\bar{\eta}_W(t, \varepsilon)$ of Ξ_ε approaches the impulse-free jump $\eta_0^- \to \eta_0^+$ of Ξ closer as the perturbation parameter ε gets smaller, which agrees with the result (21) of Theorem 5.3. Then we fix $w_2 = 1$ and $\varepsilon = 0.001$, and change w_1 from 0.1 to 1 and 10; it is seen from Fig. 4(b) that $\bar{\eta}_W(t, \varepsilon)$ approaches the same jump $\eta_0^- \to \eta_0^+$ independently from the choice of w_1, which also agrees with the results of Theorem 5.3. Note that $\bar{\eta}_W(t, \varepsilon)$ coincides with the C^1-solution $\eta(t)$ of Ξ on M^*, which converges to $(0, 0, 0)$ indicating that the origin is an asymptotically stable point for C^1-solutions of Ξ.

Case 2: Consider $a(x, y) = x - y^3$, $b(x, y) = x$, $C = 1$, then conditions **(RE)**, **(CR)**, **(DS)** are satisfied on $U = \{\eta \in \mathbb{R}^3 \mid x > -1\}$. The locally maximal invariant submanifold M^* (around η_c) is

$$M^* = M_1^c = \left\{\eta \in U \mid y + z = x - y^3 = 0\right\}.$$



(a) Fixed $w_1 = 1$, $w_2 = 1$ and variant ε.

(b) Fixed $\varepsilon = 0.01$, $w_2 = 1$ and variant w_1.

Fig. 5. Trajectories $\eta_W(t,\varepsilon)$ of the perturbed system Ξ_ε with different values of parameters.

The distribution $\mathcal{E} = \ker E$ is *not* involutive but the DAE Ξ is index-1. By Proposition 3.3, Ξ is locally (on $V = U$) ex-equivalent to the following DAE of the form (9):

$$\begin{bmatrix} 1 & -\frac{x}{1+x} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{y}} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} \frac{x}{1+x} \\ \tilde{y} \\ \tilde{x} \end{bmatrix}, \tag{23}$$

where $x = \tilde{x} + \tilde{y}(\tilde{y}^2 - 3\tilde{y}\tilde{z} + 3\tilde{z}^2)$. Then we construct the singular perturbed system Ξ_ε by Definition 5.1 (we choose $W = \left[\begin{smallmatrix} -w_1 & 0 \\ 0 & -w_2 \end{smallmatrix}\right]$) to get

$$\Xi_\varepsilon : \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f_1(\eta,\varepsilon,w_1,w_2) \\ f_2(\eta,\varepsilon,w_1,w_2) \\ f_3(\eta,\varepsilon,w_1,w_2) \end{bmatrix},$$

where $f_1 = \frac{w_2(-x^2+(x+1)(y^3-1))-3y^2(\varepsilon x+w_1(y+z))}{\varepsilon(1+x)}$, $f_2 = -\frac{w_1 y+\varepsilon x+w_1 z}{\varepsilon(1+x)}$, $f_3 = \frac{x(\varepsilon-w_1(y+z))}{\varepsilon(1+x)}$. Consider an inconsistent initial point $\eta_0^- = (0.5, 0, 0.5) \in V\backslash M^*$, by Theorem 4.6, the impulse-free jump $\eta_0^- \to \eta_0^+$ is *not* unique and $\eta_0^+ \in M^* \cap N_{\eta_0^-}$, where $N_{\eta_0^-}$ is the integral submanifold of the distribution $\mathcal{D} = \langle g_1, \ldots, g_m | \mathcal{E}\rangle$. In this example, $\mathcal{E} = \ker E = \operatorname{span}\{g_1, \ldots, g_m\}$, where $g_1 = \frac{\partial}{\partial x}$, $g_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ and thus

$$\mathcal{D} = \operatorname{span}\{g_1, g_2, [g_1, g_2]\} = T_\eta U,$$

it follows that $N_{\eta_0^-} = U$ and that $\eta_0^+ \in M^* \cap N_{\eta_0^-}$ can be any point on M^*. Then we implement a similar simulation for the solution $\bar{\eta}_W(t,\varepsilon)$ of Ξ_ε as in Case 1 to get Fig. 5. Fig. 5(a) contains similar messages as Fig. 4(a): the solution $\bar{\eta}_W(t,\tau)$ approaches closer to an impulse-free jump of Ξ as $\varepsilon \to 0$. Nevertheless, as seen Fig. 5(b), the impulse-free jump $\eta_0^- \to \eta_0^+$ approximated by $\bar{\eta}_W(t,\varepsilon)$ is not unique and depends on w_1 (and thus on W), which verifies the results of Theorem 5.3 for DAEs with non-involutive $\ker E$. Observe that the ex-equivalent DAE (23) restricted to $\psi(M^*) = \left\{\tilde{\eta} \mid \tilde{x} = \tilde{y} = 0\right\}$ is $\dot{\tilde{z}} = 0$, so the $\mathcal{C}^1$-solution of Ξ is the initial consistent point $\eta(t) = \eta(0) = \eta_0^+$. Hence the solution $\bar{\eta}_W(t,\tau)$ of the perturbed system Ξ_ε on M^* will become a fixed point as $\varepsilon \to 0$.

6. Proofs of the results

Proof of Proposition 3.3. Note that our DAE Ξ is square by $l = n$ of **(RE)**. Following (4), we have (notice that $E_1(x)$ is of full row rank r)

$$\begin{aligned} M_1^c &= M_1 \cap U := \{x \in U \mid QF(x) \in \operatorname{Im} QE(x)\} = \left\{x \in U \,\middle|\, \left[\begin{smallmatrix} F_1(x) \\ F_2(x) \end{smallmatrix}\right] \in \operatorname{Im} \left[\begin{smallmatrix} E_1(x) \\ 0 \end{smallmatrix}\right]\right\} \\ &= \{x \in U \mid F_2(x) = 0\}. \end{aligned} \tag{24}$$

$(i) \Rightarrow (ii)$: It is a direct consequence of Definition 3.1 and Proposition 2.3.

$(ii) \Leftrightarrow (iii)$: Suppose that $M^* = M_1^c$ is locally maximal invariant. Since Ξ is locally internally regular (condition **(RE)**), we have that $\dim E(x)T_xM_1^c = \dim M_1^c$, $\forall x \in M_1^c$ by (5). Observe that $F_2 : U \to \mathbb{R}^{n-r}$ and $\operatorname{rank} DF_2(x) = const. \le n - r$, $\forall x \in M_1^c$. It follows that $\dim M_1^c = n - \operatorname{rank} DF_2 \ge n - (n-r) = r = \operatorname{rank} E(x)$. We conclude that $\operatorname{rank} E(x) = \dim E(x)T_xM_1^c$, $\forall x \in M_1^c$ by

$$\operatorname{rank} E(x) = r \le \dim M_1^c = \dim E(x)T_xM_1^c \le \operatorname{rank} E(x), \quad \forall x \in M_1^c. \tag{25}$$

Conversely, suppose that $\operatorname{rank} E(x) = \dim E(x)T_xM_1^c$, $\forall x \in M_1^c$, which implies that $\dim E_1(x)T_xM_1^c = \operatorname{rank} E_1(x)$, where E_1 comes from (24). It follows that $F_1(x) \in E_1(x)T_xM_1^c$, $\forall x \in M_1^c$. Observe that $F_2(x) = 0$, $\forall x \in M_1^c$, thus

$$M_2^c = M_2 \cap U = \left\{x \in M_1^c \,\middle|\, \left[\begin{smallmatrix} F_1(x) \\ F_2(x) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} E_1(x) \\ 0 \end{smallmatrix}\right] T_xM_1^c\right\} = M_1^c.$$

Then we conclude that $M^* = M_1^c$ is a locally maximal invariant submanifold by Proposition 2.3. Notice that the inequality $\dim M_1^c = \dim T_xM_1^c \geq \operatorname{rank} E(x)$ always holds for $x \in M_1^c$ (since $\operatorname{rank} \mathrm{D}F_2(x) \leq n-r$), hence $\ker E(x) \cap T_xM_1^c = 0$ if and only if $\operatorname{rank} E(x) = \dim E(x)T_xM_1^c$.

$(iii) \Rightarrow (iv)$: Suppose that item (iii) holds. The equivalence of (ii) and (iii) implies that $\dim M_1^c = \operatorname{rank} E(x) = r$, $\forall x \in M_1^c$. Thus $\operatorname{rank} \mathrm{D}F_2(x) = n - \dim M_1^c = n-r$, i.e., $\mathrm{D}F_2(x)$ is of full row rank for all $x \in M_1^c$. Now by $\ker \mathrm{D}F_2(x) = T_xM_1^c$ and $\ker E(x) \cap T_xM_1^c = 0$, $\forall x \in M_1^c$, it follows that $\operatorname{rank} \mathrm{D}F_2(x)Z(x) = \operatorname{rank} \mathrm{D}F_2(x) = n-r$ and $\operatorname{rank} \left[\begin{smallmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{smallmatrix}\right] = \operatorname{rank} E_1(x) + \operatorname{rank} \mathrm{D}F_2(x) = n$, $\forall x \in M_1^c$. Hence $\mathrm{D}F_2(x)Z(x)$ and $\left[\begin{smallmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{smallmatrix}\right]$ are invertible for all $x \in M_1^c$.

$(iv) \Rightarrow (v)$: Suppose that the matrix $A(x) = \mathrm{D}F_2(x)Z(x)$ or $B(x) = \left[\begin{smallmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{smallmatrix}\right]$ is invertible for all $x \in M_1^c$. It follows that $\mathrm{D}F_2(x)$ is of full row rank, i.e., $\operatorname{rank} \mathrm{D}F_2(x) = n-r = m$, $\forall x \in U$. Let $\xi_2 = F_2$, then there exist a neighborhood $U_1 \subseteq U$ of x_c and a smooth map $\xi_1 : U_1 \to \mathbb{R}^r$ such that $\psi(x) = (\xi_1(x), \xi_2(x))$ is a local diffeomorphism on U_1. Thus Ξ is ex-equivalent (via Q and ψ) to

$$Q(x)E(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1} \frac{\partial \psi(x)}{\partial x}\dot{x} = Q(x)F(x) \Leftrightarrow \begin{bmatrix} E_1^1(\xi) & E_1^2(\xi) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1(\xi) \\ \xi_2 \end{bmatrix}$$

where $E_1^1 : U_1 \to \mathbb{R}^{r\times r}$, $[E_1^1 \circ \psi \;\; E_1^2 \circ \psi] = E_1$ and $\tilde{F}_1 \circ \psi = F_1$. Observe that $A(x)$ or $B(x)$ is invertible implies $\ker E(x_c) \cap T_{x_c}M_1^c = 0$ since $\ker E(x_c) = \operatorname{Im} Z(x_c)$ and $\ker \mathrm{D}F_2(x_c) = T_{x_c}M_1^c$. Thus $\operatorname{rank} E_1^1(\psi(x_c)) = \dim E(x_c)T_{x_c}M_1^c = r$, i.e., $E_1^1(\psi(x_c))$ is invertible. Then by the smoothness of E_1^1, there exists a neighborhood $U_2 \subseteq U_1$ such that $E_1^1(\psi(x))$ is invertible $\forall x \in U_2$. Define $Q_1 := \left[\begin{smallmatrix} (E_1^1)^{-1} & 0 \\ 0 & I_m \end{smallmatrix}\right]$, then via the Q_1Q-transformation and the diffeomorphism $\xi = (\xi_1, \xi_2) = \psi(x)$, Ξ is locally (on $V = U_2$) ex-equivalent to (9) with $E_2 = (E_1^1)^{-1}E_1^2$ and $F^* = (E_1^1)^{-1}\tilde{F}_1$.

$(v) \Rightarrow (i)$: Note that the sequence of submanifolds M_k^c constructed by (4) is invariant under the ex-equivalence, i.e., for two ex-equivalent DAEs Ξ and $\tilde{\Xi}$, the submanifolds M_k^c of Ξ and $\tilde{M}_k^c$ of $\tilde{\Xi}$ satisfies $\tilde{M}_k^c = \psi(M_k^c)$. Thus the geometric index ν_g, which depends only on the sequence M_k^c, is also invariant under the ex-equivalence. By a direct calculation of the submanifolds M_1^c and M_2^c for (9), it is seen that (9) is index-1. Hence, the DAE Ξ, being ex-equivalent to (9), is also index-1. □

Proof of Theorem 4.6. As Ξ is of index-1, there exists a neighborhood $V \subseteq U$ of x_c such that Ξ is locally ex-equivalent (via a diffeomorphism ψ and a Q-transformation) to the DAE (9) on V. Note that for any point $x_0^- \in V\backslash M^*$, we have that $\xi_0^- = (\xi_{10}^-, \xi_{20}^-) = \psi(x_0^-)$ satisfies $\xi_{20}^- \neq 0$ since $M^* \cap V = \{\xi \in V \mid \xi_2 = 0\}$. Then consider the following control system defined on V with a vector of inputs $u \in \mathcal{C}^0$,

$$\begin{bmatrix} \frac{d\xi_1}{d\tau} \\ \frac{d\xi_2}{d\tau} \end{bmatrix} = \sum_{i=1}^{m} \tilde{g}_i(\xi)u_i = \begin{bmatrix} -E_2(\xi) \\ I_m \end{bmatrix}u, \quad \xi(0) = \xi_0^- = (\xi_{10}^-, \xi_{20}^-), \tag{26}$$

where $\operatorname{span}\left\{\tilde{g}_1 \circ \psi, \ldots, \tilde{g}_m \circ \psi\right\} = \ker(\tilde{E} \circ \psi) = \frac{\partial \psi}{\partial x} \ker E$. By condition **(DS)**, the k-dimensional distribution $\tilde{\mathcal{D}} = \left\langle \tilde{g}_1, \ldots, \tilde{g}_m \middle| \ker \tilde{E} \right\rangle$ is involutive, thus there exist $\tilde{\phi}_i : V \to \mathbb{R}$, $i = 1, \ldots, n-k$, such that $\operatorname{span}\left\{d\tilde{\phi}_1, \ldots, d\tilde{\phi}_{n-k}\right\} = \tilde{\mathcal{D}}^{\perp}$. Then let $\tilde{\xi}_1 = (\tilde{\phi}_1, \ldots, \tilde{\phi}_{n-k})$, it is directly seen from (26) that $\operatorname{span}\{d\xi_2\} \cap \tilde{\mathcal{D}}^{\perp} = 0$ and thus $d\tilde{\xi}_1$ and $d\xi_2$ are linearly independent. By taking a smaller V, if necessary, we can choose new local coordinates $\bar{\xi} = (\tilde{\xi}_1, \bar{\xi}_1, \xi_2)$ on V, where $\bar{\xi}_1 = (\tilde{\phi}_{n-k+1}, \ldots, \tilde{\phi}_{n-m})$ is chosen such that $\tilde{\Phi}(\xi) = (\tilde{\phi}_1(\xi), \ldots, \tilde{\phi}_{n-m}(\xi), \xi_2)$ is a local diffeomorphism. Then under the new local $\bar{\xi}$-coordinates, the control system (26) becomes

$$\begin{bmatrix} \frac{d\tilde{\xi}_1}{d\tau} \\ \frac{d\bar{\xi}_1}{d\tau} \\ \frac{d\xi_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{E}_1(\bar{\xi}) \\ I_m \end{bmatrix}u, \quad \bar{\xi}(0) = \tilde{\Phi}(\xi_0^-) = (\tilde{\xi}_{10}^-, \bar{\xi}_{10}^-, \xi_{20}^-) \in V\backslash M^*, \tag{27}$$

where $\bar{E}_1 : V \to \mathbb{R}^{(k-m)\times m}$. Note that by Propositions 3.12 and 3.15 of [38], system (27) restricted to $N_{\bar{\xi}_0^-} = \left\{\bar{\xi} \in V \mid \tilde{\xi}_1 = \tilde{\xi}_{10}^-\right\}$ is controllable. It follows that for any $\bar{\xi}_0^- = (\tilde{\xi}_{10}^-, \bar{\xi}_{10}^-, \xi_{20}^-) \in V\backslash M^*$ with $N_{\bar{\xi}_0^-} \cap M^* = \left\{\bar{\xi} \in V \mid \tilde{\xi}_1 = \tilde{\xi}_{10}^-, \xi_2 = 0\right\} \neq \emptyset$, there exist $u = u(\tau)$ and $a > 0$ such that the $\mathcal{C}^1$-solution $\bar{\xi}(\tau)$ of (27) under the input $u = u(\tau)$ satisfies $\bar{\xi}(0) = \bar{\xi}_0^-$ and $\bar{\xi}(a) = \bar{\xi}_0^+ = (\tilde{\xi}_{10}^+, \bar{\xi}_{10}^+, \xi_{20}^+) \in M^* \cap N_{x_0^-}$, i.e., $\tilde{\xi}_{10}^+ = \tilde{\xi}_{10}^-$, $\xi_{20}^+ = 0$ and $\bar{\xi}_{10}^+$ being arbitrary. Then by Definition 4.1, $\xi(\tau) = \tilde{\Phi}^{-1}(\bar{\xi}(\tau))$ is an IFJ trajectory of (9) since $\xi(0) = \xi_0^- = \tilde{\Phi}^{-1}(\bar{\xi}_0^-) \in V\backslash M^*$, $\xi(a) = \xi_0^+ = \tilde{\Phi}^{-1}(\bar{\xi}_0^+) \in M^* \cap V$ and $\left[\begin{smallmatrix} I & E_2(\xi(\tau)) \\ 0 & 0 \end{smallmatrix}\right] \frac{d\xi(\tau)}{d\tau} = 0$ for $\tau \in [0, a]$ (recall that M^* locally coincides with the consistency space S_c on V by Proposition 2.3). Since Ξ and (9) are ex-equivalent (via Q and ψ), we conclude that (see Remark 4.4(i)) for any inconsistent initial value $x_0^- = \psi^{-1}(\xi_0^-) \in V/M^*$ satisfying $M^* \cap N_{x_0^-} \neq \emptyset$, there exists an IFJ trajectory $J(\tau) = \psi^{-1}(\xi(\tau))$ satisfying that $J(0) = x_0^-$ and $J(a) = x_0^+ = \psi^{-1}(\xi_0^+) = \psi^{-1} \circ \tilde{\Phi}^{-1}(\bar{\xi}_0^+) \in N_{x_0^-} \cap M^*$.

$(i) \Rightarrow (ii)$: Suppose that for a fixed $x_0^- \in V\backslash M^*$, the impulse-free jump $x_0^- \to x_0^+$ of Ξ is unique. It follows that the impulse-free jump $\xi_0^- \to \xi_0^+$ of (9) is unique and so is the point $\bar{\xi}_0^+ = (\tilde{\xi}_{10}^+, \bar{\xi}_{10}^+, \xi_{20}^-) = \tilde{\Phi}(\xi_0^+)$. Thus $\bar{\xi}_1$-variables is not present in (27) since $\tilde{\xi}_{10}^{+} = \tilde{\xi}_{10}^{-}$ and $\xi_{20}^{-} = 0$ are fixed but $\bar{\xi}_{10}^{+}$ is arbitrary. Hence, we have $\dim \ker E = m = k = \dim \mathcal{D}$, which means that the distribution $\ker E(x)$ is involutive.

$(ii) \Rightarrow (iii)$: Suppose that the distribution $\ker E(x)$ is involutive. Choose $Q : U \to GL(n, \mathbb{R})$ such that $E_1 : U \to \mathbb{R}^{r\times n}$ of $QE = \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right]$ is of full row rank r and denote $QF = \left[\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right]$. Because Ξ is index-1, we have that $\left[\begin{smallmatrix} E_1(x_c) \\ \mathrm{D}F_2(x_c) \end{smallmatrix}\right]$ is invertible by Proposition 3.3. Since the distribution $\Xi = \ker E$ is involutive, by Frobenius theorem (see e.g., [25]), there exist a neighborhood $U_1 \subseteq U$ and a smooth map $\xi_1 : U_1 \to \mathbb{R}^r$ such that $\operatorname{span}\left\{\mathrm{d}\xi_1^1, \ldots, \mathrm{d}\xi_1^r\right\} = \mathcal{E}^{\perp}$, where $\mathrm{d}\xi_1^i$ are independent rows of $\mathrm{D}\xi_1$ and $\mathcal{E} = \ker E = \ker E_1$, i.e., $\mathrm{D}\xi_1(x) \ker E_1(x) = 0$, $\forall x \in U_1$. It follows that there exists $Q_1 : U_1 \to GL(r, \mathbb{R})$ such that $\mathrm{D}\xi_1(x) = Q_1(x)E_1(x)$. Set $\xi_2 = F_2$, then we have $\psi(x) = (\xi_1(x), \xi_2(x))$ is a local diffeomorphism on a neighborhood $U_2 \subseteq U_1$ of x_c since

$$\frac{\partial \psi(x_c)}{\partial x} = \begin{bmatrix} \mathrm{D}\xi_1(x_c) \\ \mathrm{D}F_2(x_c) \end{bmatrix} = \begin{bmatrix} Q_1(x_c) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_1(x_c) \\ \mathrm{D}F_2(x_c) \end{bmatrix}$$

is invertible. Define the new local coordinates $\xi = \psi = (\xi_1, \xi_2)$ on U_2, we get

$$\begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} \left(\frac{\partial \psi(x)}{\partial x}\right)^{-1} \frac{\partial \psi(x)}{\partial x} \dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} \Leftrightarrow \begin{bmatrix} E_1^1(\xi_1, \xi_2) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1(\xi_1, \xi_2) \\ \xi_2 \end{bmatrix}, \tag{28}$$

where $E_1^1 : U_2 \to \mathbb{R}^{r\times r}$, $[E_1^1 \circ \psi, E_1^2 \circ \psi] = E_1(\frac{\partial \psi}{\partial x})^{-1}$ with $E_1^2 \equiv 0$, $\tilde{F}_1 \circ \psi = F_1$. Notice that $E_1^2 \equiv 0$ because $\operatorname{Im} E_1^2(x) = E_1(x) \ker \mathrm{D}\xi_1(x) = 0$ and that $E_1^1(x)$ is invertible for $x \in U_2$ since $\operatorname{rank} E(x) = \mathit{const.} = r$, $\forall x \in U_2$. Let $\bar{F}_1 = (E_1^1)^{-1}\tilde{F}_1$, we can always find $\bar{F}_1' : U_2 \to \mathbb{R}^{r\times m}$ such that $\bar{F}_1(\xi_1, \xi_2) = \bar{F}_1(\xi_1, 0) + \bar{F}_1'(\xi_1, \xi_2)\xi_2$. Then via $\tilde{Q} = \left[\begin{smallmatrix} (E_1^1)^{-1} & -\bar{F}_1' \\ 0 & I \end{smallmatrix}\right]$, the DAE (28) is ex-equivalent to the **(INWF)** with $F^*(\xi_1) = \bar{F}_1(\xi_1, 0)$. Finally, it is seen that Ξ is locally (on $V = U_2$) ex-equivalent to the **(INWF)** via the $\tilde{Q}Q$-transformation and the diffeomorphism ψ.

$(iii) \Rightarrow (i)$: Suppose that Ξ is locally ex-equivalent to (16). Then via a similar analysis as the beginning of the present proof (we use now **(INWF)** rather than the form (9)), we can deduce that the $\bar{\xi}_1$-variables of (27) is absent, which implies that the impulse-free jump $\xi_0^- \to \xi_0^+$ (and thus $x_0^- \to x_0^+$) is unique. □

Proof of Theorem 5.3. Suppose that Ξ is locally (on V) ex-equivalent to (9) via Q and ψ. Consider the following perturbed system for (9),

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} I_r & E_2(\xi_1, \xi_2) \\ 0 & \varepsilon W^{-1} \end{bmatrix}^{-1} \begin{bmatrix} F^*(\xi_1, \xi_2) \\ \xi_2 \end{bmatrix} = \begin{bmatrix} F^*(\xi_1, \xi_2) - \frac{1}{\varepsilon} W E_2(\xi_1, \xi_2)\xi_2 \\ \frac{1}{\varepsilon} W \xi_2 \end{bmatrix}, \tag{29}$$

which is ex-equivalent (via $\left[\begin{smallmatrix} I & E_2 \\ 0 & \varepsilon W^{-1} \end{smallmatrix}\right] Q$ and ψ) to Ξ_ε of (19). By rescaling t to τ such that $\frac{d\tau}{dt} = \frac{1}{\varepsilon}$, we get

$$\tilde{\Sigma}_\varepsilon : \begin{bmatrix} \frac{d\xi_1}{d\tau} \\ \frac{d\xi_2}{d\tau} \end{bmatrix} = \begin{bmatrix} \varepsilon F^*(\xi_1, \xi_2) - E_2(\xi_1, \xi_2) W \xi_2 \\ W \xi_2 \end{bmatrix}. \tag{30}$$

Then consider the following system $\tilde{\Sigma}_0$ defined on $\psi(V)$,

$$\tilde{\Sigma}_0 : \begin{bmatrix} \frac{d\xi_1}{d\tau} \\ \frac{d\xi_2}{d\tau} \end{bmatrix} = \begin{bmatrix} -E_2(\xi_1, \xi_2) W \xi_2 \\ W \xi_2 \end{bmatrix}, \tag{31}$$

By assumption **(SP)**, there exists a compact subset $\tilde{\mathfrak{V}} \subseteq \psi(V)$, such that $\tilde{\Sigma}_\varepsilon$ has a unique solution $\tilde{\xi}_W(\cdot, \varepsilon) : [0, +\infty) \to \tilde{\mathfrak{V}}$ and $\tilde{\Sigma}_0$ has a unique solution $\tilde{J}_W : [0, +\infty) \to \tilde{\mathfrak{V}}$, given any inconsistent initial value $(\xi_{10}^-, \xi_{20}^-) \in \tilde{\mathfrak{V}} \backslash M^*$. Let $\tilde{J}_W(\tau) = (\xi_1(\tau), \xi_2(\tau)) = (\xi_1(\tau), e^{W\tau}\xi_{20}^-)$ be the solution of $\tilde{\Sigma}_0$ starting from (ξ_{10}^-, ξ_{20}^-). Define $\gamma_W(\tau, \varepsilon) := \tilde{\xi}_W(\tau, \varepsilon) - \tilde{J}_W(\tau)$, it follows that $\gamma_W'(\tau, \varepsilon) = \frac{d\gamma_W(\tau, \varepsilon)}{d\tau} = \left[\begin{smallmatrix} \varepsilon F^*(\xi_1(\tau), \xi_2(\tau)) \\ 0 \end{smallmatrix}\right]$. Then because $(\xi_1(\tau), \xi_2(\tau)) \in \tilde{\mathfrak{V}}$, $\forall \tau \in [0, \infty)$ and F^* is continues, we have $F^*(\xi_1(\tau), \xi_2(\tau))$ is bounded for all $\tau \in [0, \infty)$. So $\gamma_W'(\tau^*, \varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly for all $\tau^* \in [0, \tau]$. For each $\varepsilon \in (0, \varepsilon^*]$ there exists by the mean value theorem a $\tau_\varepsilon^* \in [0, \tau]$ such that $\gamma_W(\tau, \varepsilon) = \gamma_W'(\tau_\varepsilon^*, \varepsilon)\tau$ (because $\gamma_W(0, \varepsilon) = 0$) and hence we have

$$\lim_{\varepsilon\to 0} \|\tilde{\xi}_W(\tau, \varepsilon) - \tilde{J}_W(\tau)\| = \lim_{\varepsilon\to 0} \|\gamma_W(\tau, \varepsilon)\| = \lim_{\varepsilon\to 0} \|\gamma_W'(\tau_\varepsilon^*, \varepsilon)\tau\| = 0.$$

It is clear that $J_W(\tau) = \psi^{-1} \circ \tilde{J}_W(\tau)$ and $\bar{x}_W(\tau, \varepsilon) = \psi^{-1} \circ \tilde{\xi}_W(\tau, \varepsilon)$ are the solutions of Σ_0 and Σ_ε starting from x_0^-, respectively. Therefore, by Lipschitz condition of ψ^{-1} on the compact set $\tilde{\mathfrak{V}}$,

$$\lim_{\varepsilon\to 0} \|\bar{x}_W(\tau, \varepsilon) - J_W(\tau)\| = \lim_{\varepsilon\to 0} \|\psi^{-1} \circ \tilde{\xi}_W(\tau, \varepsilon) - \psi^{-1} \circ \tilde{J}_W(\tau)\| \le \lim_{\varepsilon\to 0} K \|\tilde{\xi}_W(\tau, \varepsilon) - \tilde{J}_W(\tau)\| = 0,$$

where K is a Lipschitz constant.

Furthermore, if $\tilde{J}_W(+\infty) = \psi \circ J_W(+\infty)$ is well defined, by W is Hurwitz, we then have

$$\tilde{J}_W(+\infty) = \lim_{\tau\to\infty} (\xi_1(\tau), \xi_2(\tau)) = (\xi_{10}^+, \xi_{20}^+) = (\xi_{10}^+, 0) \in M^* \cap \tilde{\mathfrak{V}}$$

(recall that $M^*\cap\tilde{\mathfrak{V}} = \{(\xi_2, \xi_2) \in \tilde{\mathfrak{V}} \mid \xi_2 = 0\}$). By definition, $\tilde{J}_W(\tau)$ is an IFJ trajectory of (9) since $\tilde{J}_W(0) = (\xi_{10}^-, \xi_{20}^-) \in \tilde{\mathfrak{V}}\backslash M^*$, $\tilde{J}_W(+\infty) = (\xi_{10}^+, \xi_{20}^+) \in \tilde{\mathfrak{V}} \cap M^*$ and $\left[\, I_r \;\; E_2(\tilde{J}_W(\tau)) \,\right] \frac{d\tilde{J}_W(\tau)}{d\tau} = 0$, $\forall \tau \in [0, +\infty)$. Since the ex-equivalence preserves jump trajectories (see Remark 4.4(ii)), we have that $J_W(\tau) = \psi^{-1}(\tilde{J}_W(\tau))$ is an IFJ trajectory of Ξ starting from $x_0^- = \psi^{-1}(\xi_{10}^-, \xi_{20}^-)$ and ending at $x_0^+ = \psi^{-1}(\xi_{10}^+, 0)$. Note that the consistent point $x_0^+ = \psi^{-1}(\xi_{10}^+, 0)$ depends on the choice of W since $(\xi_{10}^+, 0)$ is the converging point of the solution $J_W(\tau)$ (actually an equilibrium point of Σ_0), which depends on the choice W. If Ξ is locally (on V) ex-equivalent to the **(INWF)** of (16) via Q and ψ, then the matrix $E_2(\xi_1, \xi_2) \equiv 0$ of (31), which implies $\frac{d\xi_1}{d\tau} = 0$ and $\xi_1(\tau) = const. = \xi_{10}^-$. Therefore, we have $\lim_{\tau\to\infty} \xi_1(\tau) = \xi_{10}^+ = \xi_{10}^-$ is unique and does not depend on the choice of W, so $x_0^+ = \psi^{-1}(\xi_{10}^+, 0) = \psi^{-1} \circ \pi \circ \psi(x_0^-) = \Omega_{E,F}(x_0^-)$ is unique.

Furthermore, let $(\xi_1(\cdot, \varepsilon), \xi_2(\cdot, \varepsilon)) : \mathcal{I} \to V$ be the solution of (29) starting from any consistent point $(\xi_{10}^+, 0)$. We have $\xi_2(t, \varepsilon) = 0$, $\forall t \in \mathcal{I}$ (since $\xi_2(0) = 0$ is an equilibrium point of $\dot{\xi}_2 = \frac{1}{\varepsilon} W\xi_2$) and $\xi_1(t, \varepsilon)$ solves $\dot{\xi}_1 = F^*(\xi_1, 0)$. Hence both $\xi_1(t, \varepsilon)$ and $\xi_2(t, \varepsilon)$ do not depend on ε and W, and $(\xi_1(t), 0)$ is a $\mathcal{C}^1$-solution of (9). Since the ex-equivalence preserves also $\mathcal{C}^1$-solutions, it follows that $x(t) = \psi^{-1}(\xi_1(t), 0)$ is the solution of both the DAE Ξ and the perturbed system Ξ_ε staring from the consistent point $x_0^+ = \psi^{-1}(\xi_{10}^+, 0)$. $\square$

7. Conclusions and perspectives

In this paper, we study solutions of nonlinear DAEs with inconsistent initial values by regarding jumps as parametrized curves satisfying certain impulse-free conditions. We show that the impulse-free jump under a new proposed definition is invariant under the external equivalence of DAEs. We give some characterizations for the notion of geometric index-1. Then we show that the existence and uniqueness of impulse free jumps are closely related to the notion of geometric index-1 and the involutivity of the distribution defined by $\ker E$. We also generalize the consistency projector of linear DAEs to the nonlinear case by proposing a normal form called the index-1 nonlinear Weierstrass form **(INWF)**. At last, we propose a singular perturbation system approximation for nonlinear DAEs, the solutions of the perturbed system not only approximate the impulse-free jumps but also the C^1-solutions of the DAE. Our future research would be extending or applying our results of impulse-free jumps to problems like consistent initialization of switched nonlinear DAEs [18], solutions of control systems with impulsive inputs [37].

CRediT authorship contribution statement

Yahao Chen: Investigation, Writing – original draft, Conceptualization, Writing – review & editing. **Stephan Trenn:** Investigation, Writing – original draft, Conceptualization, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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