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# Feedback linearization of nonlinear differential-algebraic control systems

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**Abstract**

In this article, we study feedback linearization problems for nonlinear differential-algebraic control systems (DACSs). We consider two kinds of feedback equivalences, namely, external feedback equivalence, which is defined (locally) on the whole generalized state space, and internal feedback equivalence, which is defined on the locally maximal controlled invariant submanifold (i.e., on the set where solutions exist). We define a notion called explicitation with driving variables, which is a class of ordinary differential equation control systems (ODECSs) attaching to a given DACS. Then we give necessary and sufficient conditions for both internal and external feedback linearization problems of the DACS. We show that the feedback linearizability of the DACS is closely related to the involutivity of the linearizability distributions of the explicitation systems. Finally, we illustrate the results of the by an academic example and a constrained mechanical system.

**KEYWORDS**

controlled invariant submanifolds, constrained mechanical system, differential-algebraic control systems, explicitation, external and internal feedback equivalence, feedback linearization

## 1 | INTRODUCTION

Consider a nonlinear differential-algebraic control system (DACS) of the form

$$\Xi^u : E(x)\dot{x} = F(x) + G(x)u, \tag{1}$$

where $x \in X$ is called the generalized state and $(x, \dot{x}) \in TX$, where $TX$ is the tangent bundle of an open subset $X$ in $\mathbb{R}^n$ (or, more general, of an $n$-dimensional smooth manifold $X$), and $u \in \mathbb{R}^m$ is the vector of inputs, and where $E : TX \to \mathbb{R}^l$, $F : X \to \mathbb{R}^l$ and $G : X \to \mathbb{R}^{l\times m}$ are smooth maps. The word "smooth" will always mean $C^\infty$-smooth throughout the article. We denote a DACS of the form (1) by $\Xi^u_{l,n,m} = (E, F, G)$ or, simply, $\Xi^u$. A linear DACS is of the form

$$\Delta^u : E\dot{x} = Hx + Lu, \tag{2}$$

where $E, H \in \mathbb{R}^{l\times n}$ and $L \in \mathbb{R}^{l\times m}$. Denote a linear DACS by $\Delta^u_{l,n,m} = (E, H, L)$ or, simply, $\Delta^u$. Linear DACSs have been studied for decades, there is a rich literature devoted to them (see, e.g., the surveys[1,2] and textbook[3]).

In the context of this article, we will need results about canonical forms,[4-6] controllability,[7-9] and geometric subspaces.[10,11] The motivation of studying linear and nonlinear DACSs is their frequent presence in mathematical models of practical systems as constrained mechanics,[12] chemical processes,[13] electrical circuits,[14] and so forth.

Early efforts of studying solutions of nonlinear DAEs are the works of Rheinboldt[15] and Reich,[16] which regard a nonlinear DAE as an implicit description of a vector field on a manifold. In References 16 and 17, the concept of regularity in the linear DAE case was generalized for nonlinear DAEs to characterize the existence and uniqueness of DAE solutions. All the papers on nonlinear DAE solutions as[12,14,17-20] lead to a geometrical reduction method (see Definition 3 below). The use of such a reduction method in the control context can be consulted in References 21-24 in order to get a state space representation of a given DACS. The map $E$ of a DACS (1) can be nonsquare (i.e., $l \neq n$) and noninvertible. As a consequence, some free variables and constrained variables can be implicitly present in the generalized state $x$ (and also some constrained control variables can exist in the input $u$). We have proposed two normal forms to distinguish the different roles of variables for nonlinear DACSs in Reference 20. It was noted that although the free variables of $x$ may perform like an input, we will distinguish them from the real active control variables $u$. In this article, we will study feedback linearizable problems by considering the differences of the two kinds of inputs of DACSs (see Remark 3 below).

In the case of $E(x) = I_n$, the DACS (1) becomes an ordinary differential equation control system (ODECS)

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{3}$$

where $f = F$ and $g_i$, $1 \leq i \leq m$, being the columns of $G$, become vector fields on $X$. The feedback linearization problem of nonlinear ODECSs (i.e., when there exist a local change of coordinates in the state space and a feedback transformation such that the transformed system has a linear form in the new coordinates) has drawn the attention of researchers for decades (e.g., see survey papers[25,26] and books[27,28]). The solution of the feedback linearization problem of ODECSs was first given in Brockett's paper[29] and developed by Jakubczyk and Respondek,[30] Su,[31] Hunt et Su.[32] Compared to the ODECSs, fewer results on the linearization problems of DACSs can be found. Xiaoping[33] transformed a nonlinear DACS into a linear one by state space transformations, Kawaji[34] gave sufficient conditions for the feedback linearization of a special class of DACSs, Wang and Chen[35] considered a semiexplicit differential-algebraic equation (DAE) and linearized the differential part of the DAE. The linearization of semiexplicit DAEs under equivalence of different levels is studies in Reference 36. The feedback linearization technique was also applied for stabilization[37] and tracking[21] problems of semiexplicit nonlinear DACS. The authors of Reference 38 gave a comprehensive review for feedback linearization problems of DACSs and proposed a feedback linearized normal form using the notions of $M$ derivative and $M$ bracket, such a normal form can be used for studying adaptive control problems for semiexplicit nonlinear DACS.

In this article, our purpose is to find when a given DACS of the form (1) is locally feedback equivalent to a linear completely controllable one (see the definition of the complete controllability of linear DACSs in Reference 7 or see Definition 9 below). In particular, we will consider two kinds of equivalence relations, namely, the external feedback equivalence given in Definition 5 and the internal feedback equivalence given in Definition 6. Note that the words "external" and "internal", appearing throughout this article, basically mean that we consider the DACS on an open neighborhood of the generalized state space $X$ and on the *locally maximal controlled invariant submanifold* $M^*$ (see Definition 2), respectively. We have discussed in detail the differences and relations of the two equivalence relations for linear DAEs,[6] and for semiexplicit DAEs.[36] We will use a notion called the *explicitation with driving variables* (see Definition 7, firstly proposed in Reference 39 for linear DACSs) to connect nonlinear DACSs with nonlinear ODECSs. Via the explicitation with driving variables, we can interpret the linearizability of a DACS under internal or external feedback equivalence as that of an explicitation system under system feedback equivalence (see Definition 8).

The article is organized as follows: In Section 2, we define the external and the internal feedback equivalences and discuss their relations with solutions. In Section 3, we use the notion of explicitation with driving variables to connect DACSs with ODECSs. Necessary and sufficient conditions for both the external and the internal feedback linearization problems of DACSs are given in Section 4. We illustrate the results of Section 4 by the two examples in Section 5. The conclusions and perspectives of this article are given in Section 6 and a technical proof is given in Appendix.

## 2 | EXTERNAL AND INTERNAL FEEDBACK EQUIVALENCE

We use the following notations in this article: We denote by $T_xM \in \mathbb{R}^n$ the tangent space at $x \in M$ of a differentiable submanifold $M$ of $\mathbb{R}^n$. We use $C^k$ to denote the class of $k$-times differentiable functions and $GL(n, \mathbb{R})$ to denote the group of nonsingular matrices of $\mathbb{R}^{n\times n}$. For a smooth map $f : X \to \mathbb{R}$, we denote its differential by $\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mathrm{d}x_i = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}]$. For a map $A : X \to \mathbb{R}^{m\times n}$, $\ker A(x)$, $\mathrm{Im}\, A(x)$ and $\mathrm{rank}\, A(x)$ are the kernel, the image and the rank of $A$ at $x$, respectively. For a full row rank map $R : X \to \mathbb{R}^{r\times n}$, we denote by $R^\dagger : X \to \mathbb{R}^{n\times r}$ the right inverse of $R$, that is, $RR^\dagger = I_r$. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$. We assume the reader is familiar with basic notions of differential geometry such as smooth embedded submanifolds, involutive distributions and refer the reader for example, to the book[40] for the formal definitions of such notions.

**Definition 1** (solutions and admissible set). For a DACS $\Xi^u_{l,n,m} = (E, F, G)$, a curve $(x, u) : I \to X \times \mathbb{R}^m$ defined on an open interval $I \subseteq \mathbb{R}$ with $x(\cdot) \in C^1$ and $u(\cdot) \in C^0$, is called a solution of $\Xi^u$ if for all $t \in I$, $E(x(t))\dot{x}(t) = F(x(t)) + G(x(t))u(t)$. We call a point $x_a \in X$ *admissible* if there exists at least one solution $(x(\cdot), u(\cdot))$ such that $x(t_a) = x_a$ for a certain $t_a \in I$. The set of all admissible points will be called the admissible set (or the consistency set) of $\Xi^u$ and denoted by $S_a$.

A smooth connected embedded submanifold $M$ is called *controlled invariant* if for any point $x_0 \in M$, there exists a solution $(x, u) : I \to M \times \mathbb{R}^m$ such that $x(t_0) = x_0$ for a certain $t_0 \in I$ and $x(t) \in M$, $\forall\, t \in I$. Fix an admissible point $x_a \in X$, a smooth connected embedded submanifold $M$ containing $x_a$ is called *locally controlled invariant* if there exists a neighborhood $U$ of $x_a$ such that $M \cap U$ is controlled invariant.

**Definition 2** (locally maximal controlled invariant submanifold). A locally controlled invariant submanifold $M^*$, around an admissible point $x_a$, is called *maximal* if there exists a neighborhood $U$ of $x_a$ such that for any other locally controlled invariant submanifold $M$, we have $M \cap U \subseteq M^* \cap U$.

The locally maximal controlled invariant submanifold $M^*$ of a DACS can be constructed via the following *geometric reduction method*:

**Definition 3** (geometric reduction method[19,20]). For a DACS $\Xi^u_{l,n,m} = (E, F, G)$, fix a point $x_p \in X$. Let $U_0$ be a connected subset of $X$ containing $x_p$. Step 0: Set $M_0 = X$ and $M_0^c = U_0$. Step $k$ ($k > 0$): Suppose that a sequence of smooth connected embedded submanifolds $M^c_{k-1} \subsetneq \cdots \subsetneq M^c_0$ of $U_{k-1}$ for a certain $k-1$, have been constructed. Define recursively

$$M_k := \left\{ x \in M^c_{k-1} \mid F(x) \in E(x)T_xM^c_{k-1} + \mathrm{Im}\ G(x) \right\}.$$

As long as $x_p \in M_k$, let $M^c_k = M_k \cap U_k$ be a smooth embedded connected submanifold for some neighborhood $U_k \subseteq U_{k-1}$ of $x_p$.

**Proposition 1** (20). *In the above geometric reduction method, there always exists a smallest $k^*$ such that either $k^*$ is the smallest integer for which $x_p \notin M_{k^*+1}$ or $k^*$ is the smallest integer such that $x_p \in M^c_{k^*+1}$ and $M^c_{k^*+1} \cap U_{k^*+1} = M^c_{k^*} \cap U_{k^*+1}$. In the latter case, denote $M^* = M^c_{k^*+1}$ and assume that there exists an open neighborhood $U^* \subseteq U_{k^*+1}$ of $x_p$ such that* $\dim E(x)T_xM^* = const.$ *and* $E(x)T_xM^* + \mathrm{Im}\, G(x) = const.$ *for all $x \in M^* \cap U^*$, then*

*(i) $x_p$ is an admissible point, that is, $x_p = x_a$ and $M^*$ is the locally maximal controlled invariant submanifold around $x_p$;*
*(ii) $M^*$ coincides locally with the admissible set $S_a$, that is, $M^* \cap U^* = S_a \cap U^*$.*

By item (ii) of Proposition 1, the admissible set $S_a$ locally coincides with $M^*$ on the neighborhood $U^*$ of $x_p$. So any point $x_0 \in U^* \setminus M^*$ is not admissible and there exist no solutions passing through $x_0$. Thus to study solutions of a DACS, it is convenient to consider only the restriction of the DACS to its locally maximal controlled invariant submanifold $M^*$. We have shown how to restrict a DACS to the submanifold $M^*$ in Remark 3.4(iv) and Theorem 4.4(i) of Reference 20 with the help of normal forms, now we define formally the notion of local restriction as follows.

Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$ and fix an admissible point $x_a \in X$. Let $M^*$ be the $n^*$-dimensional maximal controlled invariant submanifold of $\Xi^u$ around $x_a$. Assume that there exists a neighborhood $U$ of $x_a$ such that for all $x \in M^* \cap U$,

**(CR)** $\dim E(x)T_xM^* = const. = r^*$ and $E(x)T_xM^* + \mathrm{Im}\, G(x) = const. = r^* + (m - m^*)$.

Let $\psi : U \to \mathbb{R}^n$ be a local diffeomorphism and $z = \psi(x) = (z_1, z_2)$ be local coordinates on $U$ such that $M^* \cap U = \{z_2 = 0\}$, thus $z_1$ are local coordinates on $M^* \cap U$. Then in the new $z$-coordinates, the DACS $\Xi^u$ becomes a system $\tilde{\Xi}^u_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$, given by

$$\begin{bmatrix} \tilde{E}_1(z_1, z_2) & \tilde{E}_2(z_1, z_2) \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \tilde{F}(z_1, z_2) + \tilde{G}(z_1, z_2)u,$$

where $\tilde{E}_1 : U \to \mathbb{R}^{l\times n^*}$, $\tilde{E}_2 : U \to \mathbb{R}^{l\times(n-n^*)}$, $\tilde{E}\circ\psi = \begin{bmatrix} \tilde{E}_1\circ\psi & \tilde{E}_2\circ\psi \end{bmatrix} = E \cdot \left(\frac{\partial\psi}{\partial x}\right)^{-1}$, $\tilde{F}\circ\psi = F$ and $\tilde{G}\circ\psi = G$. Set $z_2 = 0$ to have the following system (which is defined on $M^*$)

$$\begin{bmatrix} \tilde{E}_1(z_1, 0) & \tilde{E}_2(z_1, 0) \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ 0 \end{bmatrix} = \tilde{F}(z_1, 0) + \tilde{G}(z_1, 0)u. \tag{4}$$

By **(CR)**, there exist a neighborhood $U_1 \subseteq U$ of $x_a$ and $Q : M^* \cap U_1 \to GL(l, \mathbb{R})$ such that $\tilde{E}^1_1(z_1)$ and $\tilde{G}_2(z_1)$ below are of full row rank,

$$Q(z_1) \begin{bmatrix} \tilde{E}_1(z_1, 0) & \tilde{F}(z_1, 0) & \tilde{G}(z_1, 0) \end{bmatrix} = \begin{bmatrix} \tilde{E}^1_1(z_1) & \tilde{F}_1(z_1) & \tilde{G}_1(z_1) \\ 0 & \tilde{F}_2(z_1) & \tilde{G}_2(z_1) \\ 0 & \tilde{F}_3(z_1) & 0 \end{bmatrix},$$

where $\tilde{E}^1_1$, $\tilde{G}_2$ are smooth functions defined on $M^* \cap U_1$ with values in $\mathbb{R}^{r^*\times n^*}$ and $\mathbb{R}^{(m-m^*)\times m}$, respectively, and $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_3$ and $\tilde{G}_1$ are matrix-valued functions of appropriate sizes. Since $\tilde{G}_2(z_1)$ is of full row rank, we can always assume $\begin{bmatrix} \tilde{G}_1(z_1) \\ \tilde{G}_2(z_1) \end{bmatrix} = \begin{bmatrix} \tilde{G}^1_1(z_1) & \tilde{G}^2_1(z_1) \\ \tilde{G}^1_2(z_1) & \tilde{G}^2_2(z_1) \end{bmatrix}$ with $\tilde{G}^2_2 : M^* \cap U_1 \to GL(m - m^*, \mathbb{R})$ (if not, we permute the components of $u$ such that $\tilde{G}^2_2(z_1)$ is invertible), where $\tilde{G}^1_1$, $\tilde{G}^2_1$ and $\tilde{G}^1_2$ are of appropriate sizes. Thus, via $Q$ and the following feedback transformation (note that $a^u, b^u$ are defined on $M^*$ and $b^u(z_1)$ is invertible),

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = a^u(z_1) + b^u(z_1)u = \begin{bmatrix} 0 \\ \tilde{F}_2(z_1) \end{bmatrix} + \begin{bmatrix} I_{m^*} & 0 \\ \tilde{G}^1_2(z_1) & \tilde{G}^2_2(z_1) \end{bmatrix} u,$$

the DACS (4) is transformed into

$$\begin{bmatrix} \bar{E}^1_1(z_1) \\ 0 \\ 0 \end{bmatrix} \dot{z}_1 = \begin{bmatrix} \overline{F}_1(z_1) \\ 0 \\ \overline{F}_3(z_1) \end{bmatrix} + \begin{bmatrix} \overline{G}^1_1(z_1) & \overline{G}^2_1(z_1) \\ 0 & I_{m-m^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{5}$$

where $\bar{E}^1_1 = \tilde{E}^1_1, \overline{F}_3 = \tilde{F}_3, \overline{F}_1 = \tilde{F}_1 - \tilde{G}^2_1(\tilde{G}^2_2)^{-1}\tilde{F}_2, \overline{G}^1_1 = \tilde{G}^1_1 - \tilde{G}^2_1(\tilde{G}^2_2)^{-1}\tilde{G}^1_2$ and $\overline{G}^2_1 = \tilde{G}^2_1(\tilde{G}^2_2)^{-1}$.

**Definition 4** (restriction). Consider a DACS $\Xi^u = (E, F, G)$ with an $n^*$-dimensional maximal controlled invariant submanifold around an admissible point $x_a$. Assume that condition **(CR)** holds for all $x \in M^*$ around $x_a$. Then the local $M^*$-restriction of $\Xi^u$, denoted by $\Xi^u|_{M^*}$, is given by

$$\Xi^u|_{M^*} = \Xi^{u^*} : E^*(z^*)\dot{z}^* = F^*(z^*) + G^*(z^*)u^*. \tag{6}$$

where $z^* = z_1$, $u^* = u_1$, $E^* = \bar{E}^1_1 : M^* \to \mathbb{R}^{r^*\times n^*}$, $F^* = \overline{F}_1 : M^* \to \mathbb{R}^{r^*}$ and $G^* = \overline{G}^1_1 : M^* \to \mathbb{R}^{r^*\times m^*}$ come from (5), and where the map $E^*$ is of full row rank $r^*$.

*Remark* 1. The restriction $\Xi^u|_{M^*}$ is a DACS of the form (1) with associated dimensions $r^*, n^*, m^*$, that is, $\Xi^u|_{M^*} = \Xi^{u^*}_{r^*,n^*,m^*}$. It is important to know that $\Xi^u$ and $\Xi^u|_{M^*}$ has isomorphic solutions (see Theorem 4.4(i) of[20]). More specifically, a curve

$(x(\cdot), u(\cdot))$ is a solution of $\Xi^u$ passing through a point $x_0 \in X$ if and only if $(z^*(\cdot), u^*(\cdot))$ is a solution of $\Xi^u|_{M^*}$ passing through $z_0^* \in M^*$, where $(z^*(\cdot), 0) = \psi(x(\cdot))$, $(z_0^*, 0) = \psi(x_0)$ and $(u^*(\cdot), 0) = a^u(z^*(\cdot)) + b^u(z^*(\cdot))u(\cdot)$.

Now we define the external and the internal feedback equivalences for nonlinear DACSs and compare them by discussing their relations with solutions.

**Definition 5** (external feedback equivalence). Two DACSs $\Xi^u_{l,n,m} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on $X$ and $\tilde{X}$, respectively, are called externally feedback equivalent, shortly ex-fb-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$ and smooth functions $Q : X \to GL(l, \mathbb{R})$, $\alpha^u : X \to \mathbb{R}^m$, $\beta^u : X \to GL(m, \mathbb{R})$ such that

$$\tilde{E}(\psi(x)) = Q(x)E(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}, \quad \tilde{F}(\psi(x)) = Q(x)\left(F(x) + G(x)\alpha^u(x)\right), \quad \tilde{G}(\psi(x)) = Q(x)G(x)\beta^u(x). \tag{7}$$

The ex-fb-equivalence of two DACSs $\Xi^u$ and $\tilde{\Xi}^{\tilde{u}}$ is denoted by $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods $U$ of a point $x_p$ and $\tilde{U}$ of a point $\tilde{x}_p = \psi(x_p)$, and $Q(x)$, $\alpha^u(x)$, $\beta^u(x)$ are defined on $U$, we will talk about local ex-fb-equivalence.

**Definition 6** (internal feedback equivalence). Consider two DACSs $\Xi^u = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on $X$ and $\tilde{X}$, respectively. Fix two admissible points $x_a \in X$ and $\tilde{x}_a \in \tilde{X}$. Assume that

(A1) $M^*$ and $\tilde{M}^*$ are locally maximal controlled invariant submanifolds of $\Xi^u$ around $x_a$ and of $\tilde{\Xi}^{\tilde{u}}$ around $\tilde{x}_a$, respectively.
(A2) $M^*$ and $\tilde{M}^*$ satisfy the constant rank condition **(CR)** around $x_a$ and $\tilde{x}_a$, respectively.

Then, $\Xi^u$ and $\tilde{\Xi}^{\tilde{u}}$ are called locally internally feedback equivalent, shortly locally in-fb-equivalent, if their restrictions $\Xi^u|_{M^*}$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$ are ex-fb-equivalent. We will denote the locally in-fb-equivalence of two DACSs by $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$.

*Remark* 2. The dimensions of two locally in-fb-equivalent DACSs $\Xi^u$ and $\tilde{\Xi}^{\tilde{u}}$ are not necessarily the same. However, since $\Xi^u|_{M^*} = \Xi^{u^*}_{l^*,n^*,m^*}$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*} = \tilde{\Xi}^{\tilde{u}^*}_{\tilde{l}^*,\tilde{n}^*,\tilde{m}^*}$ are required to be external feedback equivalent, their dimensions have to be the same, that is, $r^* = \tilde{r}^*$, $n^* = \tilde{n}^*$ and $m^* = \tilde{m}^*$.

Both the ex-fb-equivalence and the in-fb-equivalence preserve solutions of DACSs. Indeed, consider two ex-fb-equivalent DACSs $\Xi^u$ and $\tilde{\Xi}^{\tilde{u}}$, the diffeomorphism $\tilde{x} = \psi(x)$ and the feedback transformation $u = \alpha^u(x) + \beta^u(x)\tilde{u}$ (defined on $X$) establish a one to one correspondence between solutions $(x, u)$ of $\Xi^u$ and solutions $(\tilde{x}, \tilde{u})$ of $\tilde{\Xi}^{\tilde{u}}$, that is, $\tilde{x} = \psi(x)$ and $u = \alpha^u(x) + \beta^u(x)\tilde{u}$. For two locally in-fb-equivalent DACSs $\Xi^u$ and $\tilde{\Xi}^{\tilde{u}}$, by $\Xi^u|_{M^*} \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$, there exist a diffeomorphism $\tilde{z}^* = \psi^*(z^*)$ between $M^*$ and $\tilde{M}^*$, and a feedback transformation $u^* = \alpha^{u^*}(z^*) + \beta^{u^*}(z^*)\tilde{u}^*$ defined on $M^*$ mapping solutions $(z^*, u^*)$ of $\Xi^u|_{M^*}$ into solutions $(\tilde{z}^*, \tilde{u}^*)$ of $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$. Recall from Remark 1 that the DACSs $\Xi^u$ and $\tilde{\Xi}^{\tilde{u}}$ have isomorphic solutions with their restrictions $\Xi^u|_{M^*}$ and $\tilde{\Xi}^{\tilde{u}}$, respectively. So solutions $(x, u)$ of $\Xi^u$ are also in a one-to-one correspondence with solutions $(\tilde{x}, \tilde{u})$ of $\tilde{\Xi}^{\tilde{u}}$ if $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$.

Conversely, if solutions of two DACSs $\Xi^u$ and $\tilde{\Xi}^{\tilde{u}}$ are in a one-to-one correspondence via a diffeomorphism and a feedback transformation, then the two DACSs are in-fb-equivalent, however, they are *not* necessarily ex-fb-equivalence. The reason is that solutions of DACSs exist on maximal controlled invariant submanifolds only, by assuming two DACSs have corresponding solutions, we only have the information that the two restrictions $\Xi^u|_{M^*}$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$ can be transformed into each other via a $Q$-transformation and a feedback transformation defined on $M^*$, together with a diffeomorphism between $M^*$ and $\tilde{M}^*$, we do not know, however, if those transformations can be extended outside the submanifolds $M^*$ and $\tilde{M}^*$.

**Example 1.** Consider two DACSs $\Xi^u_{3,3,1} = (E, F, G)$ defined on $X = \mathbb{R}^3$ and $\tilde{\Xi}^{\tilde{u}}_{3,3,1} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on $\tilde{X} = \mathbb{R}^3$, where

$$E(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F(x) = \begin{bmatrix} (x_1)^2 \\ e^{x_1}x_2 \\ x_3 \end{bmatrix}, \quad G(x) = \begin{bmatrix} e^{x_2} \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{E}(\tilde{x}) = \begin{bmatrix} 1 & \tilde{x}_2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tilde{F}(\tilde{x}) = \begin{bmatrix} \tilde{x}_2 \\ e^{\tilde{x}_1}\tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}, \quad \tilde{G}(\tilde{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

It is seen that $M^* = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = x_3 = 0\}$ and $\tilde{M}^* = \{(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathbb{R}^3 \mid \tilde{x}_2 = \tilde{x}_3 = 0\}$. The restrictions $\Xi^u|_{M^*}$ : $\dot{x}_1 = (x_1)^2 + u$ and $\tilde{\Xi}^{\tilde{u}}|_{\tilde{M}^*}$ : $\dot{\tilde{x}}_1 = \tilde{u}$ are ex-fb-equivalent via $Q(x_1) = 1$, $\tilde{x}_1 = \psi(x_1) = x_1$ and $\tilde{u} = (x_1)^2 + u$. Thus we have $\Xi^u \overset{in-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$. It is clear that solutions $((x_1, 0, 0), u)$ of $\Xi^u$ and solutions $((\tilde{x}_1, 0, 0), \tilde{u})$ of $\tilde{\Xi}^{\tilde{u}}$ have a one-to-one correspondence.

However, the two DACSs are *not* ex-fb-equivalent since rank $E(x) \neq$ rank $\tilde{E}(\tilde{x})$ (the matrix-valued functions $E(x)$ and $\tilde{E}(\tilde{x})$ of two ex-fb-equivalent DACSs should have the same rank).

Both the external and the internal feedback equivalences play an important role for DACSs. The internal feedback equivalence is convenient when we are only interested in solutions passing through an admissible point and evolving on $M^*$. The ex-fb-equivalence is useful when the initial point $x_0 \notin M^*$, that is, $x_0$ is not admissible, then there are no solutions passing through $x_0$ but there may still exist a jump from the inadmissible point $x_0$ to an admissible one on $M^*$, see our recent publication,[41] where we use external equivalence to study jump solutions of nonlinear DAEs. Note that if the initial state $x_0$ is not admissible, the jump of $x_0$ at $t = t_0$ will cause a distributional term, that is, the Dirac impulse $\delta$ in the derivatives $\dot{x}$. For linear DAEs/DACSs, such impulsive terms can be explained by the distributional solution (generalized function) theory. However, for a nonlinear DACS being feedback equivalent to a linear one with distributional solutions, the interpretations of the impulsive solutions in the nonlinear coordinates are still unclear and out of the scope of this article. The distributional solution theory may not be a suitable setting for nonlinear systems because the image of a nonlinear map on the Dirac impulse $\delta$ is in general *not* well-defined.

## 3 | EXPLICITATION OF NONLINEAR DIFFERENTIAL-ALGEBRAIC CONTROL SYSTEMS

We have proposed the notion of explicitation (with driving variables) for linear DACS in Reference 39 (or see Chapter 3 of Reference 42), we now extend this notion to nonlinear DACSs.

**Definition 7** (explicitation with driving variables). Given a DACS $\Xi^u_{l,n,m} = (E, F, G)$, fix a point $x_p \in X$. Assume that rank $E(x) = const. = r$ around $x_p$. Then locally there exists $Q : X \to GL(l, \mathbb{R})$ such that $E_1$ of $Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$ is of full row rank $r$, denote

$$Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}, \quad Q(x)G(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix}.$$

Define locally the maps $f : X \to \mathbb{R}^n$, $g^u : X \to \mathbb{R}^{n\times m}$, $g^v : X \to \mathbb{R}^{n\times s}$, $h : X \to \mathbb{R}^p$, $l^u : X \to \mathbb{R}^{p\times m}$, where $s = n - r$ and $p = l - r$, such that

$$f(x) = E_1^\dagger(x)F_1(x), \quad g^u(x) = E_1^\dagger(x)G_1(x), \quad \mathrm{Im}\, g^v(x) = \ker E_1(x), \quad h(x) = F_2(x), \quad l^u(x) = G_2(x),$$

where $E_1^\dagger$ is a right inverse of $E_1$. By a $(Q, v)$-explicitation, we will call any ODECS

$$\Sigma^{uv} : \begin{cases} \dot{x} &= f(x) + g^u(x)u + g^v(x)v, \\ y &= h(x) + l^u(x)u, \end{cases} \tag{8}$$

where $v \in \mathbb{R}^{s\times n}$ is called the vector of driving variables. System (8) is denoted by $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ or, simply, $\Sigma^{uv}$.

Clearly, in the above definition, the choices of the invertible map $Q$, the right inverse $E_1^\dagger$ and the map $g^v$ satisfying $\mathrm{Im}\, g^v = \ker E_1 = \ker E$, are not unique. The following proposition shows that a $(Q, v)$-explicitation of a given DACS $\Xi^u$ is an ODECS defined up to a feedback transformation, an output multiplication and a generalized output injection, that is, a class of control systems. Throughout the class of all $(Q, v)$-explicitations of $\Xi^u$ will be called the explicitation class. For a particular ODECS $\Sigma^{uv}$ belonging to the explicitation class $\mathbf{Expl}(\Xi^u)$ of $\Xi^u$, we will write $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u)$.

**Proposition 2.** *Assume that an ODECS $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ is a $(Q, v)$-explicitation of a DACS $\Xi^u = (E, F, G)$ corresponding to the choice of invertible matrix $Q(x)$, right inverse $E_1^\dagger(x)$ and matrix $g^v(x)$. We have that an ODECS $\tilde{\Sigma}^{u,\tilde{v}}_{n,m,p} = (\tilde{f}, \tilde{g}^u, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^u)$ is a $(\tilde{Q}, \tilde{v})$ -explicitation of $\Xi^u$ corresponding to the choice of invertible matrix $\tilde{Q}(x)$, right inverse $\tilde{E}_1^\dagger(x)$ and matrix $\tilde{g}^{\tilde{v}}(x)$ if and only if $\Sigma^{uv}$ and $\tilde{\Sigma}^{u,\tilde{v}}$ are equivalent via a v-feedback transformation of the form $v = \alpha^v(x) + \lambda(x)u + \beta^v(x)\tilde{v}$, a generalized output injection $\gamma(x)y = \gamma(x)(h(x) + l^u(x)u)$ and an output multiplication $\tilde{y} = \eta(x)y$, which map*

$$f \mapsto \tilde{f} = f + \gamma h + g^v \alpha^v, \quad g^u \mapsto \tilde{g}^u = g^u + \gamma l^u + g^v \lambda, \quad g^v \mapsto \tilde{g}^{\tilde{v}} = g^v \beta^v, \quad h \mapsto \tilde{h} = \eta h, \quad l^u \mapsto \tilde{l}^u = \eta l^u.$$

*where $\alpha^v(x)$, $\beta^v(x)$, $\gamma(x)$, $\lambda(x)$, $\eta(x)$ are smooth matrix-valued functions, and $\beta^v(x)$ and $\eta(x)$ are invertible.*

We omit the proof of Proposition 2 since it follows the same line as that of Proposition 2.3 in Reference 39. Now we will define an equivalence relation for two ODECSs of the form (8).

**Definition 8** (system feedback equivalence). Two ODECSs $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$ and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}})$ defined on $X$ and $\tilde{X}$, respectively, are called system feedback equivalent, or shortly sys-fb-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$, smooth functions $\alpha^u(x)$, $\alpha^v(x)$, $\lambda(x)$ and $\gamma(x)$ with values in $\mathbb{R}^m$, $\mathbb{R}^s$, $\mathbb{R}^{s\times m}$ and $\mathbb{R}^{n\times p}$, respectively, and invertible smooth matrix-valued functions $\beta^u(x)$, $\beta^v(x)$ and $\eta(x)$ with values in $GL(m, \mathbb{R})$, $GL(s, \mathbb{R})$ and $GL(p, \mathbb{R})$, respectively, such that

$$\begin{bmatrix} \tilde{f}\circ\psi & \tilde{g}^{\tilde{u}}\circ\psi & \tilde{g}^{\tilde{v}}\circ\psi \\ \tilde{h}\circ\psi & \tilde{l}^{\tilde{u}}\circ\psi & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial x}\gamma \\ 0 & \eta \end{bmatrix} \begin{bmatrix} f & g^u & g^v \\ h & l^u & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \alpha^u & \beta^u & 0 \\ \alpha^v + \lambda\alpha^u & \lambda\beta^u & \beta^v \end{bmatrix}. \tag{9}$$

The sys-fb-equivalence of two control systems will be denoted by $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods $U$ of a point $x_p$ and $\tilde{U}$ of a point $\tilde{x}_p = \psi(x_p)$, and $\alpha^u$, $\alpha^v$, $\lambda$, $\gamma$, $\beta^u$, $\beta^v$, $\eta$ are defined on $U$, we will speak about local sys-fb-equivalence.

The two ODECSs $\Sigma^{uv}$ and $\tilde{\Sigma}^{u\tilde{v}}$ of Proposition 2 are, by definition, system feedback equivalent with $\psi$ being identity, $\alpha^u = 0$ and $\beta^u = I_m$. The following observation is crucial and will play an important role for studying the feedback linearization problems of DACSs in Section 4, which points out that the feedback transformations of explicitation systems of DACSs have a *triangular form* which are different from those of classical (ODE) control systems.

*Remark* 3. Observe that, in (9), there are two kinds of feedback transformations. Namely,

$$u = \alpha^u(x) + \beta^u(x)\tilde{u} \;\text{ and }\; v = \alpha^v(x) + \lambda(x)u + \beta^v(x)\tilde{v},$$

which can be written together as a feedback transformation of $(u, v)$ with a (lower) triangular form:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha^u(x) \\ \alpha^v(x) \end{bmatrix} + \begin{bmatrix} \beta^u(x) & 0 \\ \lambda(x) & \beta^v(x) \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}. \tag{10}$$

It implies that there are two kinds of inputs in the ODECSs of the form (8), one input (the driving variable $v$) is more "powerful" than the other input (the original control variable $u$), since when transforming $v$, we can use both $u$ and $x$, but when transforming $u$, we are *not* allowed to use $v$. Another difference between $u$ and $v$ is that the input $u$ is injected into the output $y$ via $l^u u$, but the driving variable $v$ is not directly injected into the output $y$. In a practical system, the variables $u$ are predefined control inputs, such as external forces, which can be changed actively in order to act on the system. The driving variables $v$ are, roughly speaking, the derivatives of the free variables in the generalized state $x$, such free variables may come from unknown constraint forces or some redundancies of mathematical modeling. It can be seen from Example 3 below that $u = (F_x, F_y)$ are the translation force generated by some actuators as electrical motors, the driving variable $v = \dot{F}_f$, where $F_f$ is a friction force which is an unknown constraint force.

The following theorem connects ex-fb-equivalence of two DACSs with sys-fb-equivalence of two ODECSs (explicitations). Note that the results of Theorem 1 is a general framework to use classic nonlinear control theory to study nonlinear DACSs, we will use it for the feedback linearization problems discussed in Section 4.

**Theorem 1.** *Consider two DACSs $\Xi^u_{l,n,m} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on $X$ and $\tilde{X}$, respectively. Assume that* $\operatorname{rank} E(x) = const. = r$ *in a neighborhood $U$ of a point $x_p \in X$ and* $\operatorname{rank} \tilde{E}(\tilde{x}) = r$ *in a neighborhood $\tilde{U}$ of a point $\tilde{x}_p \in \tilde{X}$. Then, given any ODECSs $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u) \in$ **Expl**$(\Xi^u)$ and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}}) \in$ **Expl**$(\tilde{\Xi}^{\tilde{u}})$, we have that locally $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ if and only if $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$.*

*Proof.* By the assumptions that rank $E(x)$ and rank $\tilde{E}(x)$ are constant and equal to $r$ around $x_p$ and $\tilde{x}_p$, respectively, there exist invertible matrix-valued functions $Q : U \to GL(l, \mathbb{R})$ and $\tilde{Q} : \tilde{U} \to GL(l, \mathbb{R})$, defined on neighborhoods $U$ of $x_p$ and $\tilde{U}$ of $\tilde{x}_p$, respectively, such that $E'(x) = Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$ and $\tilde{E}'(\tilde{x}) = \tilde{Q}(\tilde{x})\tilde{E}(\tilde{x}) = \begin{bmatrix} \tilde{E}_1(\tilde{x}) \\ 0 \end{bmatrix}$, where $E_1 : U \to R^{r\times n}$ and $\tilde{E}_1 : \tilde{U} \to R^{r\times n}$ are of full row rank. We have $\Xi^u \overset{ex-fb}{\sim} \Xi^{u'} = (E', F', G')$ and $\tilde{\Xi}^{\tilde{u}} \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}'} = (\tilde{E}', \tilde{F}', \tilde{G}')$ via $Q(x)$ and $\tilde{Q}(\tilde{x})$, respectively, where

$$F'(x) = QF(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}, \quad G'(x) = QG(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix}, \quad \tilde{F}'(\tilde{x}) = \tilde{Q}\tilde{F}(\tilde{x}) = \begin{bmatrix} \tilde{F}_1(\tilde{x}) \\ \tilde{F}_2(\tilde{x}) \end{bmatrix}, \quad \tilde{G}'(\tilde{x}) = \tilde{Q}\tilde{G}(\tilde{x}) = \begin{bmatrix} \tilde{G}_1(\tilde{x}) \\ \tilde{G}_2(\tilde{x}) \end{bmatrix}.$$

In this proof, without loss of generality, we will assume that $\Xi^u = \Xi^{u'}$ and $\tilde{\Xi}^{\tilde{u}} = \tilde{\Xi}^{\tilde{u}'}$, since $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ if and only if $\Xi^{u\prime} \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}\prime}$.

Moreover, choose maps $f, g^u, g^v, h, l^u$ and $\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}}$ such that

$$\begin{aligned} &f(x) = E_1^\dagger(x)F_1(x), \quad g^u(x) = E_1^\dagger(x)G_1(x), \quad \operatorname{Im} g^v(x) = \ker E_1(x), \quad h(x) = F_2(x), \quad l^u(x) = G_2(x), \\ &\tilde{f}(\tilde{x}) = \tilde{E}_1^\dagger(\tilde{x})\tilde{F}_1(\tilde{x}), \quad \tilde{g}^{\tilde{u}}(\tilde{x}) = \tilde{E}_1^\dagger(\tilde{x})\tilde{G}_1(\tilde{x}), \quad \operatorname{Im} \tilde{g}^{\tilde{v}}(\tilde{x}) = \ker \tilde{E}_1(\tilde{x}), \quad \tilde{h}(\tilde{x}) = \tilde{F}_2(\tilde{x}), \quad \tilde{l}^{\tilde{u}}(\tilde{x}) = \tilde{G}_2(\tilde{x}), \end{aligned} \tag{11}$$

where $E_1^\dagger(x)$ and $\tilde{E}_1^\dagger(\tilde{x})$ are right inverses of $E_1(x)$ and $\tilde{E}_1(\tilde{x})$, respectively. Then by Definition 7,

$$\Sigma^{uv} = (f, g^u, g^v, h, l^u) \in \mathbf{Expl}(\Xi^u), \qquad \tilde{\Sigma}^{\tilde{u}\tilde{v}} = (\tilde{f}, \tilde{g}^{\tilde{u}}, \tilde{g}^{\tilde{v}}, \tilde{h}, \tilde{l}^{\tilde{u}}) \in \mathbf{Expl}(\tilde{\Xi}^{\tilde{u}}).$$

It is seen from Proposition 2 that any control system in $\mathbf{Expl}(\Xi^u)$ is sys-fb-equivalent to $\Sigma^{uv}$ and that any control system in $\mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$ is sys-fb-equivalent to $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$. Without loss of generality, in the remaining part of the proof, we use $\Sigma^{uv}$ and $\tilde{\Sigma}^{\tilde{u}\tilde{v}}$ with system matrices given by (11) to represent two ODECSs in $\mathbf{Expl}(\Xi^u)$ and $\mathbf{Expl}(\tilde{\Xi}^{\tilde{u}})$, respectively. Throughout the proof below, we may drop the argument $x$ for the functions $E(x)$, $F(x)$, $G(x)$,..., for ease of notation.

*If.* Suppose that locally $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$. Then there exist a local diffeomorphism $\tilde{x} = \psi(x)$ and matrix-valued functions $\alpha^u$, $\alpha^v$, $\lambda$, $\gamma$, $\beta^u$, $\beta^v$, $\eta$ defined on a neighborhood $U$ of $x_p$ such that the system matrices satisfy relations (9) of Definition 8.

First, consider $\tilde{g}^{\tilde{v}} \circ \psi = \frac{\partial \psi}{\partial x} g^v \beta^v$. By $\operatorname{Im} g^v = \ker E_1$, $\operatorname{Im} \tilde{g}^{\tilde{v}} = \ker \tilde{E}_1$, we have $\ker \tilde{E}_1 \circ \psi = \frac{\partial \psi}{\partial x} \ker E_1$. Thus there exists $Q_1 : U \to GL(r, \mathbb{R})$ such that

$$\tilde{E}_1 \circ \psi = Q_1 E_1 \left(\frac{\partial \psi}{\partial x}\right)^{-1}. \tag{12}$$

Then, by (9), the following relation holds:

$$\begin{bmatrix} \tilde{f} \circ \psi & \tilde{g}^{\tilde{u}} \circ \psi \\ \tilde{h} \circ \psi & \tilde{l}^{\tilde{u}} \circ \psi \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial x}\gamma \\ 0 & \eta \end{bmatrix} \begin{bmatrix} f & g^u & g^v \\ h & l^u & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha^u & \beta^u \\ \alpha^v + \lambda\alpha^u & \lambda\beta^u \end{bmatrix}.$$

Substituting (11) into the above equation, we get

$$\begin{bmatrix} \tilde{E}_1^\dagger \circ \psi \cdot \tilde{F}_1 \circ \psi & \tilde{E}_1^\dagger \circ \psi \cdot \tilde{G}_1 \circ \psi \\ \tilde{F}_2 \circ \psi & \tilde{G}_2 \circ \psi \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial x}\gamma \\ 0 & \eta \end{bmatrix} \begin{bmatrix} E_1^\dagger F_1 & E_1^\dagger G_1 & g^v \\ F_2 & G_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha^u & \beta^u \\ \alpha^v + \lambda\alpha^u & \lambda\beta^u \end{bmatrix}.$$

Premultiply the above equation by

$$\begin{bmatrix} \tilde{E}_1 \circ \psi & 0 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} Q_1 E_1 \left(\frac{\partial \psi}{\partial x}\right)^{-1} & 0 \\ 0 & I_p \end{bmatrix}$$

to get

$$\begin{bmatrix} \tilde{F}_1\circ\psi & \tilde{G}_1\circ\psi \\ \tilde{F}_2\circ\psi & \tilde{G}_2\circ\psi \end{bmatrix} = \begin{bmatrix} Q_1 & Q_1E_1\gamma \\ 0 & \eta \end{bmatrix} \begin{bmatrix} F_1 & G_1 \\ F_2 & G_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha^u & \beta^u \end{bmatrix}. \tag{13}$$

Now from Equations (12), (13) and Definition 5, it can be seen that $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ via the transformations defined by $\tilde{x} = \psi(x)$, $Q = \begin{bmatrix} Q_1 & Q_1E_1\gamma \\ 0 & \eta \end{bmatrix}$, $\alpha^u$ and $\beta^u$.

*Only if.* Suppose that $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$ (in a neighborhood $U$ of $x_p$). Assume that $\Xi^u$ and $\tilde{\Xi}^u$ are ex-fb-equivalent via an invertible matrix-valued function $Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$, $\tilde{x} = \psi(x)$, $\alpha^u$, $\beta^u$, where $Q_1 : U \to \mathbb{R}^{r\times r}$ and $Q_2$, $Q_3$, $Q_4$ are matrix-valued functions of appropriate sizes. Then by

$$QE = \tilde{E}\circ\psi \frac{\partial \psi}{\partial x} \Rightarrow \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{E}_1\circ\psi \\ 0 \end{bmatrix} \frac{\partial \psi}{\partial x},$$

we can deduce that

$$\tilde{E}_1\circ\psi = Q_1E_1\left(\frac{\partial \psi}{\partial x}\right)^{-1}. \tag{14}$$

Moreover, we have $Q_3 = 0$ and $Q_1$ is invertible (since both $E_1$ and $\tilde{E}_1$ are of full row rank), which implies that $Q_4$ is invertible as well (since $Q$ is invertible). Subsequently, by

$$\tilde{F}\circ\psi = Q(F + G\alpha^u) \Rightarrow \begin{bmatrix} \tilde{F}_1\circ\psi \\ \tilde{F}_2\circ\psi \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \left( \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \alpha^u \right),$$

we have

$$\tilde{F}_1\circ\psi = Q_1(F_1 + G_1\alpha^u) + Q_2(F_2 + G_2\alpha^u) \tag{15}$$

and

$$\tilde{F}_2\circ\psi = Q_4(F_2 + G_2\alpha^u). \tag{16}$$

Moreover, by

$$\tilde{G}\circ\psi = QG\beta^u \Rightarrow \begin{bmatrix} \tilde{G}_1\circ\psi \\ \tilde{G}_2\circ\psi \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \beta^u,$$

we have

$$\tilde{G}_1\circ\psi = Q_1G_1\beta^u + Q_2G_2\beta^u \tag{17}$$

and

$$\tilde{G}_2\circ\psi = Q_4G_2\beta^u. \tag{18}$$

Recall the system matrices given in (11). First, from $\operatorname{Im} g^v = \ker E_1$, $\operatorname{Im} \tilde{g}^{\tilde{v}}\circ\psi = \ker \tilde{E}_1\circ\psi$, and Equation (14), it is seen that there exists $\beta^v : U \to GL(s, \mathbb{R})$ such that

$$\tilde{g}^{\tilde{v}} \circ \psi = \frac{\partial \psi}{\partial x} g^{v} \beta^{v}. \tag{19}$$

Secondly, by Equations (14) and (15), we have

$$\begin{aligned}\tilde{f} \circ \psi &= \tilde{E}_1^{\dagger} \circ \psi \tilde{F}_1 \circ \psi = \frac{\partial \psi}{\partial x} E_1^{\dagger} Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} F_1 + G_1 \alpha^u \\ F_2 + G_2 \alpha^u \end{bmatrix} = \frac{\partial \psi}{\partial x} E_1^{\dagger} Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} F_1 + G_1 \alpha^u + E_1 g^v (\lambda \alpha^u + \alpha^v) \\ F_2 + G_2 \alpha^u \end{bmatrix} \\ &= \frac{\partial \psi}{\partial x} \left( f + g^u \alpha^u + g^v \left( \lambda \alpha^u + \alpha^v \right) + \gamma \left( h + l^u \alpha^u \right) \right),\end{aligned} \tag{20}$$

where $\gamma = E_1^{\dagger} Q_1^{-1} Q_2$, and $\alpha^v$ and $\lambda$ are matrix-valued functions of appropriate sizes. Thirdly, by Equation (17), we have

$$\tilde{g}^{\tilde{u}} \circ \psi = \tilde{E}_1^{\dagger} \circ \psi \tilde{G}_1 \circ \psi = \frac{\partial \psi}{\partial x} E_1^{\dagger} Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} G_1 \beta^u \\ G_2 \beta^u \end{bmatrix} = \frac{\partial \psi}{\partial x} E_1^{\dagger} Q_1^{-1} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} G_1 \beta^u + E_1 g^v \lambda \\ G_2 \beta^u \end{bmatrix} = \frac{\partial \psi}{\partial x} \left( g^u \beta^u + g^v \lambda + \gamma l^u \beta^u \right). \tag{21}$$

Note that we use the equations $E_1 g^v (\lambda \alpha^u + \alpha^v) = 0$ and $E_1 g^v \lambda = 0$ to deduce (20) and (21). At last, by Equations (16) and (18) we have

$$\tilde{h} \circ \psi = \tilde{F}_2 \circ \psi = Q_4 (F_2 + G_2 \alpha^u) = Q_4 \left( h + l^u \alpha^u \right) \tag{22}$$

and

$$\tilde{l}^{\tilde{u}} \circ \psi = \tilde{G}_2 \circ \psi = Q_4 G_2 \beta^u = Q_4 l^u \beta^u. \tag{23}$$

Finally, it can be seen from (20), (21), (22), and (23), that $\Sigma^{uv} \overset{sys-fb}{\sim} \tilde{\Sigma}^{\tilde{u}\tilde{v}}$ via $\tilde{x} = \psi(x)$, $\alpha^v$, $\beta^v$, $\alpha^u$, $\beta^u$, $\lambda$, $\gamma = E_1^{\dagger} Q_1^{-1} Q_2$ and $\eta = Q_4$. ■

## 4 | EXTERNAL AND INTERNAL FEEDBACK LINEARIZATION

In this section, we discuss the problem that when a nonlinear DACS of the form (1) is locally externally or internally feedback equivalent to a linear DACS of the form (2) with complete controllability. First, we review some definitions and criteria for the complete controllability of linear DACSs. We denote by $A^{-1}\mathscr{B}$, the preimage of a space $\mathscr{B}$ under a linear map $A$. The augmented Wong sequences (see e.g., References 2,7,39) of a linear DACS $\Delta^u_{l,n,m} = (E, H, L)$, given by (2), are

$$\mathscr{V}_0 := \mathbb{R}^n, \qquad \mathscr{V}_{i+1} := H^{-1}(E\mathscr{V}_i + \operatorname{Im} L), \quad i \geq 0; \tag{24}$$

$$\mathscr{W}_0 := 0, \qquad \mathscr{W}_{i+1} := E^{-1}(H\mathscr{W}_i + \operatorname{Im} L), \quad i \geq 0. \tag{25}$$

Additionally, recall the following sequence of subspaces (see e.g., Reference 2):

$$\hat{\mathscr{W}}_1 := \ker E, \qquad \hat{\mathscr{W}}_{i+1} := E^{-1}(H\hat{\mathscr{W}}_i + \operatorname{Im} L), \; i \geq 1. \tag{26}$$

For simplicity of notation, we denote $K_\beta = \operatorname{diag}\{K_{\beta_1}, \dots, K_{\beta_k}\} \in \mathbb{R}^{(|\beta|-k)\times|\beta|}$, $L_\beta = \operatorname{diag}\{L_{\beta_1}, \dots, L_{\beta_k}\} \in \mathbb{R}^{(|\beta|-k)\times|\beta|}$, $\mathcal{E}_\beta = \operatorname{diag}\{e_{\beta_1}, \dots, e_{\beta_k}\} \in \mathbb{R}^{|\beta|\times k}$, $N_\beta = \operatorname{diag}\{N_{\beta_1}, \dots, N_{\beta_k}\} \in \mathbb{R}^{|\beta|\times|\beta|}$, where $\beta$ is a multi-index $\beta = (\beta_1, \dots, \beta_k)$ and $|\beta| = \sum_{i=1}^k \beta_i$, and where

$$K_{\beta_i} = \begin{bmatrix} 0 & I_{\beta_i-1} \end{bmatrix} \in \mathbb{R}^{(\beta_i-1)\times\beta_i}, \quad e_{\beta_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\beta_i}, \quad L_{\beta_i} = \begin{bmatrix} I_{\beta_i-1} & 0 \end{bmatrix} \in \mathbb{R}^{(\beta_i-1)\times\beta_i}, \quad N_{\beta_i} = \begin{bmatrix} 0 & 0 \\ I_{\beta_i-1} & 0 \end{bmatrix} \in \mathbb{R}^{\beta_i\times\beta_i}.$$

Definition 5 applied to linear systems says that two linear DACSs $\Delta^{u}_{l,n,m} = (E, H, L)$ and $\tilde{\Delta}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{H}, \tilde{L})$ are ex-fb-equivalent if there exist constant invertible matrices $Q, P, S$ and a matrix $R$ such that $\tilde{E} = QEP^{-1}$, $\tilde{H} = Q(H + LR)P^{-1}$, $\tilde{L} = QLS$.

**Definition 9** (complete controllability in Reference 7). A linear DACS $\Delta^{u}_{l,n,m} = (E, H, L)$ is completely controllable if for any $x_0, x_1 \in \mathbb{R}^n$, there exist a solution $(x, u)$ of $\Delta^u$ and $t \in \mathbb{R}^+$ such that $x(0) = x_0$ and $x(t) = x_1$.

**Lemma 1** (7). *For a linear DACS $\Delta^{u}_{l,n,m} = (E, H, L)$, the following statements are equivalent:*

*(i)* $\Delta^u$ *is completely controllable.*
*(ii)* $\operatorname{Im} E + \operatorname{Im} H + \operatorname{Im} L = \operatorname{Im} E + \operatorname{Im} L$ *and* $\operatorname{Im}_{\mathbb{C}} E + \operatorname{Im}_{\mathbb{C}} H + \operatorname{Im}_{\mathbb{C}} L = \operatorname{Im}_{\mathbb{C}}(\lambda E - H) + \operatorname{Im}_{\mathbb{C}} L$, $\forall \lambda \in \mathbb{C}$.
*(iii)* $\mathscr{V}^* \cap \mathscr{W}^* = \mathbb{R}^n$, *where* $\mathscr{V}^*$ *and* $\mathscr{W}^*$ *are the limits of the augmented Wong sequences (24) and (25), respectively;*
*(iv)* $\Delta^u$ *is ex-fb-equivalent (under linear transformations) to*

$$\begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\overline{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} N_{\rho}^T & 0 \\ 0 & K_{\overline{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_{\rho} & 0 \\ 0 & 0 \\ 0 & I_{m-m^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

*where $\rho = (\rho_1, \dots, \rho_{m^*})$ and $\overline{\rho} = (\overline{\rho}_1, \dots, \overline{\rho}_{s^*})$ are multi-indices, and $s^* = n - \operatorname{rank} E$.*

We define (locally) internal and (locally) external feedback linearizability of nonlinear DACSs as follows.

**Definition 10.** Consider a DACS $\Xi^{u}_{l,n,m} = (E, F, G)$ and fix an admissible point $x_a \in X$. Then $\Xi^u$ is called locally internally (resp. externally) feedback linearizable around $x_a$ if $\Xi^u$ is locally in-fb-equivalent (resp. ex-fb-equivalent) to a linear DACS with complete controllability around $x_a$.

We consider an ODECS $\Sigma^{uv}_{n,m,s,p} = (f, g^u, g^v, h, l^u)$, given by (8). If $\Sigma^{uv}$ has no outputs, we denote it by $\Sigma^{uv}_{n,m,s} = (f, g^u, g^v)$. Then for $\Sigma^{uv}_{n,m,s} = (f, g^u, g^v)$, define the following two sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$, called the *linearizability distributions* of $\Sigma^{uv}$,

$$\begin{cases} \mathcal{D}_0 := \{0\}, \\ \mathcal{D}_1 := \operatorname{span}\left\{g_1^u, \dots, g_m^u, g_1^v, \dots, g_s^v\right\}, \\ \mathcal{D}_{i+1} := \mathcal{D}_i + [f, \mathcal{D}_i], \quad i = 1, 2, \dots, \end{cases} \qquad \begin{cases} \hat{\mathcal{D}}_1 := \operatorname{span}\left\{g_1^v, \dots, g_s^v\right\}, \\ \hat{\mathcal{D}}_{i+1} := \mathcal{D}_i + [f, \hat{\mathcal{D}}_i], \quad i = 1, 2, \dots. \end{cases} \tag{27}$$

*Remark* 4. Consider a linear DACS $\Delta^u = (E, H, L)$, denote $\mathscr{W}_i(\Delta^u)$ and $\hat{\mathscr{W}}_i(\Delta^u)$ as the subspaces $\mathscr{W}_i$, given by (25), and $\hat{\mathscr{W}}_i$, given by (26), of $\Delta^u$, respectively. For a linear ODECS $\Lambda^{uv} = (A, B^u, B^v, C, D^u)$ (of the form (8) but with constant system matrices), define the following two sequences of subspaces

$$\mathcal{W}_0 := \{0\}, \quad \mathcal{W}_{i+1} := \begin{bmatrix} A & B^w \end{bmatrix} \left( \begin{bmatrix} \mathcal{W}_i \\ \mathbb{R}^{m+s} \end{bmatrix} \cap \ker \begin{bmatrix} C & D^w \end{bmatrix} \right), \ i \geq 0,$$

and

$$\hat{\mathcal{W}}_1 := \operatorname{Im} B^v, \quad \hat{\mathcal{W}}_{i+1} := \begin{bmatrix} A & B^w \end{bmatrix} \left( \begin{bmatrix} \hat{\mathcal{W}}_i \\ \mathbb{R}^{m+s} \end{bmatrix} \cap \ker \begin{bmatrix} C & D^w \end{bmatrix} \right), \ i \geq 1,$$

where $w = (u, v)$, $B^w = [B^u, B^v]$ and $D^w = [D^u, 0]$. We have proved in Proposition 2.10 of Reference 39 that if $\Lambda^{uv} \in$ **Expl**$(\Delta^u)$, then

$$\mathscr{W}_i(\Delta^u) = \mathcal{W}_i(\Lambda^{uv}), \quad \forall i \geq 0, \qquad \hat{\mathscr{W}}_i(\Delta^u) = \hat{\mathcal{W}}_i(\Lambda^{uv}), \quad \forall i \geq 1.$$

Apparently, $\mathcal{W}_i$ and $\hat{\mathcal{W}}_i$ are linear counterparts of $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$, respectively, but they are for linear systems with outputs.

**Theorem 2** (internal feedback linearization). *Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$, fix an admissible point $x_a \in X$. Let $M^*$ be the $n^*$-dimensional locally maximal controlled invariant submanifold of $\Xi^u$ around $x_a$. Assume that the constant rank assumption* (**CR**) *is satisfied for $x \in M^*$ around $x_a$. Then $\Xi^u|_{M^*}$ is a DACS $\Xi^{u^*}_{r^*,n^*,m^*} = (E^*, F^*, G^*)$ of the form (6) and its explicitation* **Expl**$(\Xi^u|_{M^*})$ *is a class of ODECSs without outputs. The DACS $\Xi^u$ is locally internally feedback linearizable if and only if for one (and thus any) ODECS $\Sigma^{u^*v^*} = (f^*, g^{u^*}, g^{v^*}) \in$* **Expl**$(\Xi^u|_{M^*})$*, the linearizability distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ of $\Sigma^{u^*v^*}$ satisfy the following conditions on $M^*$ around $x_a$:*

*(FL1) $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ are of constant rank for $1 \le i \le n^*$.*
*(FL2) $\mathcal{D}_{n^*} = \hat{\mathcal{D}}_{n^*} = TM^*$.*
*(FL3) $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ are involutive for $1 \le i \le n^* - 1$.*

*Proof.* Since $\Xi^u$ satisfies condition **(CR)** around $x_a$, its $M^*$-restriction $\Xi^u|_{M^*}$ by Definition 4 is a DACS $\Xi^u|_{M^*} = \Xi^{u^*}_{r^*,n^*,m^*} = (E^*, F^*, G^*)$ of the form (6) with $E^*$ being of full row rank $r^*$. It follows by the full row rankness of $E^*$ that the maps $h = F_2$ and $l^{u^*} = G_2$ are absent in the explicitation systems of $\Xi^{u^*}$, which means that the output $y = h(x) + l^{u^*}(x)u^*$ is absent as well (see Definition 7). Thus an ODECS $\Sigma^{u^*v^*}_{n^*,m^*,s^*} = (f^*, g^{u^*}, g^{v^*}) \in$ **Expl**$(\Xi^u|_{M^*})$ is a control system without outputs, which is in the form

$$\Sigma^{w^*} : \dot{z}^* = f^*(z^*) + g^{u^*}(z^*)u^* + g^{v^*}(z^*)v^*,$$

where $w^* = (u^*, v^*)$, $f^* = (E^*)^\dagger F^*$, $g^{u^*} = (E^*)^\dagger G^*$, $\operatorname{Im} g^{v^*} = \ker E^*$ and $s^* = n^* - r^*$.

*Only if.* Suppose that $\Xi^u$ is locally internally feedback linearizable, which means that its $M^*$-restriction $\Xi^u|_{M^*}$, given by (6), is locally ex-fb-equivalent to a completely controllable linear DACS

$$\Delta^{\tilde{u}^*} : E^*\dot{\tilde{z}}^* = H^*\tilde{z}^* + L^*\tilde{u}^*,$$

where $E^*, H^*, L^*$ are constant matrices of appropriate sizes. Then a linear ODECS $\Lambda^{\tilde{w}^*} = (A^*, B^{\tilde{u}^*}, B^{\tilde{v}^*}) \in$ **Expl**$(\Delta^{\tilde{u}^*})$, where $\tilde{w}^* = (\tilde{u}^*, \tilde{v}^*)$, is of the form

$$\Lambda^{\tilde{w}^*} : \dot{\tilde{z}}^* = A^*\tilde{z}^* + B^{\tilde{u}^*}\tilde{u}^* + B^{\tilde{v}^*}\tilde{v}^*.$$

where $A^* = (E^*)^\dagger H^*$, $B^{\tilde{u}^*} = (E^*)^\dagger L^*$ and $\operatorname{Im} B^{\tilde{v}^*} = \ker E^*$. By Lemma 1, the complete controllability of $\Delta^{\tilde{u}^*}$ implies $\hat{\mathscr{W}}_{n^*}(\Delta^{\tilde{u}^*}) = \mathscr{W}_{n^*}(\Delta^{\tilde{u}^*}) = \mathbb{R}^{n^*}$. By Proposition 2.10 of Reference 39 (see also Remark 4(ii)), we get

$$\hat{\mathcal{W}}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathcal{W}_{n^*}(\Lambda^{\tilde{w}^*}) = \hat{\mathscr{W}}_{n^*}(\Delta^{\tilde{u}^*}) = \mathscr{W}_{n^*}(\Delta^{\tilde{u}^*}) = \mathbb{R}^{n^*}.$$

Since $\Lambda^{\tilde{w}^*}$ is a linear control system without outputs, we have $\hat{\mathcal{D}}_{n^*}(\Lambda^{\tilde{w}^*}) = \hat{\mathcal{W}}_{n^*}(\Lambda^{\tilde{w}^*})$, $\mathcal{D}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathcal{W}_{n^*}(\Lambda^{\tilde{w}^*})$. Hence, $\hat{\mathcal{D}}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathcal{D}_{n^*}(\Lambda^{\tilde{w}^*}) = \mathbb{R}^{n^*}$. Thus $\Lambda^{\tilde{w}^*}$ satisfies (FL2). Moreover, since $\Lambda^{\tilde{w}^*}$ is a linear control system, it satisfies (FL1) and (FL3) in an obvious way. Notice that the nonlinear system $\Sigma^{w^*}$ is locally sys-fb-equivalent to $\Lambda^{\tilde{w}^*}$ by Theorem 1 because $\Sigma^{w^*} \in$ **Expl**$(\Xi^u|_{M^*})$, $\Lambda^{\tilde{w}^*} \in$ **Expl**$(\Delta^{\tilde{u}^*})$ and $\Xi^u|_{M^*} \overset{ex\text{-}fb}{\sim} \Delta^{\tilde{u}^*}$. Since $\Sigma^{w^*}$ and $\Lambda^{\tilde{w}^*}$ are control systems without outputs, sys-fb-equivalence reduces to feedback equivalence. Thus $\Sigma^{w^*}$ and $\Lambda^{\tilde{w}^*}$ are locally feedback equivalent (via $\tilde{z}^* = \psi(z^*)$ and two kinds of feedback transformations defined by $\alpha^{u^*}, \alpha^{v^*}, \beta^{u^*}, \beta^{v^*}, \lambda$, see Remark 3). It is easy to verify by a direct calculation that if $\hat{\mathcal{D}}_i$ and $\mathcal{D}_i$ are involutive, then the two distribution sequences are invariant for the two feedback equivalent control systems $\Sigma^{w^*}$ and $\Lambda^{\tilde{w}^*}$, that is, $\frac{\partial \psi}{\partial z^*}\hat{\mathcal{D}}_i(\Sigma^{w^*}) = \hat{\mathcal{D}}_i(\Delta^{\tilde{w}^*})\circ\psi$ and $\frac{\partial \psi}{\partial z^*}\mathcal{D}_i(\Sigma^{w^*}) = \mathcal{D}_i(\Delta^{\tilde{w}^*})\circ\psi$. So the system $\Sigma^{w^*}$ being feedback equivalent to $\Lambda^{\tilde{w}^*}$ satisfies conditions (FL1)–(FL3) as well. It is seen from Proposition 2 that any other ODECS $\hat{\Sigma}^{\hat{w}^*} \in$ **Expl**$(\Xi^u|_{M^*})$ is sys-fb-equivalent to $\Sigma^{w^*}$, which means $\Sigma^{w^*}$ is feedback equivalent (via two kinds of feedback transformations) to $\hat{\Sigma}^{\hat{w}^*}$ as any explicitation system in **Expl**$(\Xi^u|_{M^*})$ has no outputs. So any other explicitation system $\hat{\Sigma}^{\hat{w}^*}$ satisfies (FL1)-(FL3) of Theorem 2 as well.

*If*. Suppose that an ODECS $\Sigma^{u^*v^*} \in$ **Expl**$(\Xi^u|_{M^*})$ satisfies (FL1)–(FL3) around $x_a$. Then the following lemma holds.

**Lemma 2.** *The ODECS $\Sigma^{w^*} = \Sigma^{u^*v^*}_{n^*,m^*,s^*} = (f^*, g^{u^*}, g^{v^*})$ is locally feedback equivalent, via two kinds of feedback transformations (see Remark 3), to the Brunovský canonical form Reference 43 around $x_a$, which is given by*

$$\Sigma_{Br}^{\tilde{w}^*} = \Sigma_{Br}^{\tilde{u}^*\tilde{v}^*} : \begin{cases} \dot{\xi}_1 = N_\rho^T \xi_1 + \mathcal{E}_\rho \tilde{u}^*, \\ \dot{\xi}_2 = N_{\overline{\rho}}^T \xi_2 + \mathcal{E}_{\overline{\rho}} \tilde{v}^*, \end{cases} \tag{28}$$

*where* $\tilde{w}^* = (\tilde{u}^*, \tilde{v}^*)$, *and* $\rho = (\rho_1, \dots, \rho_a)$ *and* $\overline{\rho} = (\overline{\rho}_1, \dots, \overline{\rho}_b)$ *are multi-indices.*

The proof of Lemma 2 is technical and is put into Appendix. Now we will prove that the $M^*$-restriction $\Xi^u|_{M^*}$, given by (6), is locally ex-fb-equivalent to a linear DACS

$$\Delta^{\tilde{u}^*} : \begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\overline{\rho}} \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\overline{\rho}} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho \\ 0 \end{bmatrix} \tilde{u}^*. \tag{29}$$

Notice that by Lemma 1, the linear DACS $\Delta^{\tilde{u}^*}$ is completely controllable. We have $\Sigma_{Br}^{\tilde{w}^*} \in \mathbf{Expl}(\Delta^{\tilde{u}^*})$ because the $\xi_1$-subsystems of $\Sigma_{Br}^{\tilde{w}^*}$ and $\Delta^{\tilde{u}^*}$ coincide, $N_{\overline{\rho}}^T = L_{\overline{\rho}}^\dagger K_{\overline{\rho}}$ and $\ker L_{\overline{\rho}} = \operatorname{Im} \mathcal{E}_{\overline{\rho}}$. Recall that $\Sigma^{w^*}$ is locally sys-fb-equivalent to $\Sigma_{Br}^{\tilde{w}^*}$ (by Lemma 2) and $\Sigma^{w^*} \in \mathbf{Expl}(\Xi^u|_{M^*})$, it is seen that $\Xi^u|_{M^*}$ is locally ex-fb-equivalent to $\Delta^{\tilde{u}^*}$ around $x_a$ by Theorem 1. Hence $\Xi^u$ is locally in-fb-equivalent to the complete controllable linear DACS $\Delta^{\tilde{u}^*}$, that is, $\Xi^u$ is locally internally feedback linearizable. ∎

**Theorem 3** (external feedback linearization). *Consider a DACS* $\Xi_{l,n,m}^u = (E, F, G)$, *fix an admissible point* $x_a \in X$. *Then* $\Xi^u$ *is locally externally feedback linearizable around* $x_a$ *if and only if there exists a neighborhood* $U \subseteq X$ *of* $x_a$ *in which the following conditions are satisfied.*

*(EFL1)* $\operatorname{rank} E(x)$ *and* $\operatorname{rank} [E(x), G(x)]$ *are constant.*
*(EFL2)* $F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x)$ *or, equivalently, the locally maximal invariant submanifold* $M^* = M_0^c = U$.
*(EFL3) For one (and thus any) control system* $\Sigma^{uv} \in \mathbf{Expl}(\Xi^u|_{M^*})$, *which is a system with no outputs on* $M^* = U$, *the linearizability distributions* $\mathcal{D}_i$ *and* $\hat{\mathcal{D}}_i$ *satisfy (FL1)–(FL3) of Theorem 2.*

*Proof. Only if.* Suppose that $\Xi^u$ is locally externally feedback linearizable. By definition, the DACS $\Xi^u$ is locally ex-fb-equivalent to a linear completely controllable DACS (via $Q(x)$, $z = \psi(x)$ and $u = \alpha^u(x) + \beta^u(x)\tilde{u}$)

$$\Delta^{\tilde{u}} : \tilde{E}\dot{z} = \tilde{H}z + \tilde{L}\tilde{u}. \tag{30}$$

Thus by Definition 5, we have

$$Q(x)E(x) = \tilde{E} \cdot \frac{\partial \psi(x)}{\partial x}, \quad Q(x)(F(x) + G(x)\alpha^u(x)) = \tilde{H} \cdot \psi(x), \quad Q(x)G(x)\beta^u(x) = \tilde{L}. \tag{31}$$

It is clear that $\Delta^{\tilde{u}}$ satisfies (EFL1). So the system $\Xi^u$ satisfies (EFL1) as well because the ranks of $E(x)$ and $[E(x), G(x)]$ are invariant under ex-fb-equivalence. The complete controllability of $\Delta^{\tilde{u}}$ implies $\tilde{H}z \in \operatorname{Im} \tilde{E} + \operatorname{Im} \tilde{L}$ (see Lemma 1(ii)). By substituting (31), we get

$$\begin{aligned} Q(F + G\alpha^u)(x) \in \operatorname{Im} QE\left(\frac{\partial \psi}{\partial x}\right)^{-1}(x) + \operatorname{Im} QG\beta^u(x) &\Rightarrow F(x) + G(x)\alpha^u(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x) \\ &\Rightarrow F(x) \in \operatorname{Im} E(x) + \operatorname{Im} G(x). \end{aligned}$$

Thus $\Xi^u$ satisfies (EFL2). Notice that by (EFL2), we have that the locally maximal controlled invariant submanifold $M^*$ around $x_a$ coincides with the neighborhood $U$. Observe that the restriction $\Delta^{\tilde{u}}|_{M^*} = \Delta^{\tilde{u}}|_U$, whose canonical form is given by

$$\begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\overline{\rho}} \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\overline{\rho}} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho \\ 0 \end{bmatrix} u^*,$$

is also a linear completely controllable DACS as $\Delta^{\tilde{u}}$. This means that $\Xi^u$ is locally internally feedback linearizable. Thus by Theorem 2, the DACS $\Xi^u$ satisfies (EFL3) on $M^* = U$.

*If.* Suppose that in a neighborhood $U$ of $x_a$, the DACS $\Xi^u$ satisfies (EFL1)–(EFL3). Denote $\operatorname{rank} E(x) = r$, $\operatorname{rank}\,[E(x), G(x)] = r + \tilde{m}^*$ and $m^* = m - \tilde{m}^*$. Then, by (EFL1), there exist an invertible $Q(x)$ defined on $U$ and a partition of $u = (u_1, u_2)$ such that

$$Q(x)E(x)\dot{x} = Q(x)F(x) + Q(x)G(x)u \Rightarrow \begin{bmatrix} E_1(x) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{bmatrix} + \begin{bmatrix} G_1^1(x) & G_1^2(x) \\ G_2^1(x) & G_2^2(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $E_1(x)$ is of full row rank $r$ and $G_2^2(x)$ is a $\tilde{m}^* \times \tilde{m}^*$ invertible matrix-valued function defined on $U$. Moreover, by (EFL2), we have $F_3(x) = 0$ for $x \in U$. Now we use the feedback transformation

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2(x) \end{bmatrix} + \begin{bmatrix} I_{m^*} & 0 \\ G_2^1(x) & G_2^2(x) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and the system becomes

$$\begin{bmatrix} E_1(x) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} \tilde{F}_1(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1^1(x) & \tilde{G}_1^2(x) \\ 0 & I_{\tilde{m}^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix},$$

where $\tilde{F}_1 = F_1 - G_1^2(G_2^2)^{-1}F_2$, $\tilde{G}_1^1 = G_1^1 - G_1^2(G_2^2)^{-1}G_2^1$ and $\tilde{G}_1^2 = G_1^2(G_2^2)^{-1}$.

Premultiply the above equation by $\begin{bmatrix} I_r & -\tilde{G}_1^2(x) & 0 \\ 0 & I_{\tilde{m}^*} & 0 \\ 0 & 0 & I_{l-r-\tilde{m}^*} \end{bmatrix}$ to get

$$\begin{bmatrix} E^*(x) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} F^*(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} G^*(x) & 0 \\ 0 & I_{\tilde{m}^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^* \\ \tilde{u}^* \end{bmatrix}, \tag{32}$$

where $E^* = E_1$, $F^* = \tilde{F}_1$, $G^* = \tilde{G}_1^1$, $u^* = \tilde{u}_1$ and $\tilde{u}^* = \tilde{u}_2$. Then by Definition 4, we have that $\Xi^u|_{M^*} = \Xi^u|_U$ is the following system:

$$\Xi^u|_{M^*} \, : \, E^*(x)\dot{x} = F^*(x) + G^*(x)u^*.$$

By Theorem 2 and condition (EFL3), $\Xi^u|_{M^*}$ is locally ex-fb-equivalent (on $M^* = U$) to a linear DACS $\Delta^{\tilde{u}^*}$ of the form (29). It follows from (32) that $\Xi^u$ is locally on $U$ ex-fb-equivalent to

$$\begin{bmatrix} I_{|\rho|} & 0 \\ 0 & L_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} N_\rho^T & 0 \\ 0 & K_{\bar{\rho}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \mathcal{E}_\rho & 0 \\ 0 & 0 \\ 0 & I_{\tilde{m}^*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^* \\ \tilde{u}^* \end{bmatrix},$$

which is completely controllable by Lemma 1. Therefore, $\Xi^u$ is locally externally feedback linearizable by Definition 10. ∎

*Remark* 5. (i) By conditions (EFL1) and (EFL2), the locally maximal controlled invariant submanifold $M^*$ around $x_a$ is a neighborhood $U$ of $x_a$. So condition (EFL3) is actually, satisfied if and only if conditions (FL1)–(FL3) are satisfied on $M^* = U$, that is, locally around $x_a$.

(ii) Note that when applying the geometric reduction method of Definition 3 to a linear DACS $\Delta^u = (E, H, L)$, we get a sequence of subspaces $\mathscr{V}_i = M_i$, which is actually the augmented Wong sequence $\mathscr{V}_i$ defined by (24). Thus the locally maximal controlled invariant submanifold $M^*$ is a nonlinear generalization of the limit $\mathscr{V}^*$ of $\mathscr{V}_i$. So condition (EFL2) together with condition $\hat{\mathcal{D}}_{n^*} = \mathcal{D}_{n^*} = TM^*$ of (FL2) are the nonlinear counterparts of condition $\mathscr{V}^* \cap \mathscr{W}^* = \mathbb{R}^n$ of Lemma 1, which assures that the linearized DACS is completely controllable. The sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ can thus be seen as nonlinear generalizations of the augmented Wong sequence $\mathscr{W}_i$ of (25) and the sequence $\hat{\mathscr{W}}_i$ of (26), respectively.

(iii) If $E(x) = I_n$, a DACS $\Xi^u = (E, F, G)$ becomes an ODECS of the form (3). Suppose that $G(x) = \begin{bmatrix} g_1(x) & \dots & g_m(x) \end{bmatrix}$ is of constant rank. We have that conditions (EFL1)–(EFL2) of Theorem 3 are clearly satisfied and that condition (EFL3) reduces to the feedback linearizability conditions in the classical sense. Indeed, we have $\Xi^u \in \mathbf{Expl}(\Xi^u|_{M^*}) = \mathbf{Expl}(\Xi^u)$ because $\Xi^u$ with $E(x) = I_n$ is already an ODECS. Thus the vector of driving variables $v$ is absent and the two linearizability distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ satisfy $\hat{\mathcal{D}}_{i+1} = \mathcal{D}_i$ for $i \geq 1$. Hence conditions (FL1)–(FL3) become (FL1)' $\mathcal{D}_i$ are of constant rank for $1 \leq i \leq n$; (FL2)' $\dim \mathcal{D}_n = n$; (FL3)' $\mathcal{D}_i$ are involutive for $1 \leq i \leq n-1$, which are the feedback linearizability conditions for classical nonlinear (ODE) control systems, see for example, References 27,28,30,44.

## 5 | EXAMPLES

**Example 2.** Consider the following academic example borrowed from Reference 45. For a DACS $\Xi^u$, defined on $X = \mathbb{R}^3$, given by

$$\begin{bmatrix} x_2 & x_1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (x_2)^2 - (x_1)^3 + x_3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{33}$$

where $u = (u_1, u_2)$, we fix an admissible point $x_a = (x_{1a}, x_{2a}, x_{3a}) = (1, 0, 0) \in X$. Clearly, there exists a neighborhood $U$ ($x_1 \neq 0$ for all $x \in U$) of $x_a$ such that conditions (EFL1) and (EFL2) of Theorem 3 are satisfied. Subsequently, via $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$, the DACS $\Xi^u$ is ex-fb-equivalent to

$$\begin{bmatrix} x_2 & x_1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ (x_2)^2 - (x_1)^3 + x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}.$$

Observe that the locally maximal invariant submanifold $M^* = U$ and

$$\Xi^u|_{M^*} = \Xi^u|_U : \begin{bmatrix} x_2 & x_1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ (x_2)^2 - (x_1)^3 + x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u^*,$$

where $u^* = \tilde{u}_1$. Now an ODECS $\Sigma^{u^*v} \in \mathbf{Expl}(\Xi^u|_{M^*})$ can be taken as

$$\Sigma^{u^*v} : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (x_2)^2 - (x_1)^3 + x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2/x_1 \\ 0 \end{bmatrix} u^* + \begin{bmatrix} x_1 \\ -x_2 \\ -x_1 \end{bmatrix} v,$$

where $v$ is a driving variable. We calculate the distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ for the system $\Sigma^{u^*v}$ to get

$$\hat{\mathcal{D}}_1 = \operatorname{span}\{g^v\}, \quad \mathcal{D}_1 = \operatorname{span}\{g^{u^*}, g^v\}, \quad \mathcal{D}_2 = \hat{\mathcal{D}}_2 = \operatorname{span}\{g^{u^*}, g^v, ad_f g^v\},$$

where

$$g^{v}=\begin{bmatrix}x_1\\-x_2\\-x_1\end{bmatrix},\quad g^{u^*}=\begin{bmatrix}0\\2/x_1\\0\end{bmatrix},\quad ad_f g^{v}=\begin{bmatrix}0\\0\\3(x_1)^3+2(x_2)^2+x_1\end{bmatrix}.$$

Clearly, the distributions above are of constant rank and $\mathcal{D}_2=\hat{\mathcal{D}}_2=T_xU$ for all $x\in U$. Additionally, $[g^{u^*},g^{v}]=0\in\mathcal{D}_1$ and $\hat{\mathcal{D}}_1$ is of rank one, so the distributions $\hat{\mathcal{D}}_1,\mathcal{D}_1,\hat{\mathcal{D}}_2$ are all involutive. Thus, condition (EFL3) of Theorem 3 is satisfied. Therefore, system $\Xi^{u}$ is externally feedback linearizable.

In fact, we can choose $\varphi^{u^*}(x)$ and $\varphi^{v}(x)$ such that

$$\operatorname{span}\left\{d\varphi^{v}\right\}=\mathcal{D}_1^{\perp},\quad \operatorname{span}\left\{d\varphi^{v},d\varphi^{u^*}\right\}=\hat{\mathcal{D}}_1^{\perp}.$$

Furthermore, use the following coordinates change and feedback transformation (note that the feedback transformation below has a triangular form as we discussed in Remark 3)

$$\xi=\varphi^{u^*}(x)=x_1x_2,\quad z_1=\varphi^{v}(x)=x_1+x_3,\quad z_2=L_f\varphi^{v}(x)=-(x_1)^3+(x_2)^2+x_3,$$

$$\begin{bmatrix}\tilde{u}^*\\\tilde{v}\end{bmatrix}=\begin{bmatrix}2&0\\\frac{4x_2}{x_1}&-3(x_1)^3-x_1-2(x_2)^2\end{bmatrix}\begin{bmatrix}u^*\\v\end{bmatrix}+\begin{bmatrix}0\\(x_2)^2-(x_1)^3+x_3\end{bmatrix},$$

the system $\Sigma^{uv}$ becomes

$$\Lambda^{\tilde{u}^*\tilde{v}}:\dot{\xi}=\tilde{u}^*,\quad \dot{z}_1=z_2,\quad \dot{z}_2=\tilde{v}.$$

Now by Theorem 1, $\Xi^{u}|_{M^*}$ is ex-fb-equivalent to the following linear DACS

$$\Delta^{\tilde{u}^*}:\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}\dot{\xi}\\\dot{z}_1\\\dot{z}_2\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}\xi\\z_1\\z_2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}\tilde{u}^*,$$

since $\Sigma^{u^*v}\in\mathbf{Expl}(\Xi^{u}|_{M^*})$, $\Lambda^{\tilde{u}^*\tilde{v}}\in\mathbf{Expl}(\Delta^{\tilde{u}^*})$, and $\Sigma^{u^*v}\overset{sys-fb}{\sim}\Lambda^{\tilde{u}^*\tilde{v}}$. Therefore, the original DACS $\Xi^{u}$ is ex-fb-equivalent to the following completely controllable linear DACS:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{\xi}\\\dot{z}_1\\\dot{z}_2\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}\xi\\z_1\\z_2\end{bmatrix}+\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}\tilde{u}^*\\\tilde{u}_2\end{bmatrix}$$

via

$$Q=\begin{bmatrix}1&1&0\\0&0&1\\0&1&0\end{bmatrix},\quad \begin{bmatrix}\xi\\z_1\\z_2\end{bmatrix}=\begin{bmatrix}x_1x_2\\x_1+x_3\\-(x_1)^3+(x_2)^2+x_3\end{bmatrix},\quad \begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}1/2&0\\-1&1\end{bmatrix}\begin{bmatrix}\tilde{u}^*\\\tilde{u}_2\end{bmatrix}.$$

**Example 3.** Consider the model of a 3-link manipulator[46] with active joints 1 and 2, and a passive joint 3 (see Figure 1 below), the same model was used in Reference 20 to illustrate an applicable algorithm for the geometric reduction method, we will use it for the internal feedback linearization of DACSs in this article.

The dynamic equations of the manipulator are given by:

$$\begin{cases}m\ddot{x}-ml\sin\theta\ddot{\theta}-ml\dot{\theta}^2\cos\theta&=F_x,\\ m\ddot{y}+ml\cos\theta\ddot{\theta}-ml\dot{\theta}^2\sin\theta&=F_y,\\ -ml\sin\theta\ddot{x}+ml\cos\theta\ddot{y}+ml^2\ddot{\theta}&=\tau_\theta+F_f,\end{cases}\tag{34}$$

where the mass $m$ and the half length of the free-link $l$ are constants, $x$ and $y$ are the position variables of the free joint, and $\theta$ is the angle between the base frame and the link frame, $F_x$ and $F_y$ are the translation force at the free joint in the direction of $x$ and $y$, respectively, and $\tau_\theta$ is the torque applied to the free joint (we take $\tau_\theta = 0$ implying that joint 3 is free). We additionally consider the friction force $F_f$ caused by the rotation of the free link. We regard $(F_x, F_y)$ as the active control inputs to the system. The friction force $F_f$ is a generalized state variable rather than an active control input since we cannot change it arbitrarily.

We consider system (34) subjected to the following constraint:

$$x - y = 0. \tag{35}$$

We combine (34) together with (35) as a DACS $\Xi^u_{7,7,2} = (E, F, G)$ of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & -ml\sin\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sin\theta_1 & 0 & \cos\theta_1 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} x_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ \frac{F_f}{ml} \\ x_1 - y_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

For the DACS $\Xi^u$, the generalized states $\xi = (x_1, x_2, y_1, y_2, \theta_1, \theta_2, F_f) \in X = \mathbb{R}^6 \times S$ and the vector of control inputs is $(F_x, F_y)$. Consider $\Xi^u$ around a point $\xi_p = (x_{1p}, x_{2p}, y_{1p}, y_{2p}, \theta_{1p}, \theta_{2p}, F_{fp}) = 0$. The system $\Xi^u$ is *not* locally externally feedback linearizable since condition (EF2) of Theorem 3 is not satisfied around $\xi_p$. Now we apply the geometric reduction method of Definition 3 to get

$$M_0^c = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^6, \quad M_1^c = \left\{\xi \in M_0^c \mid x_1 - y_1 = 0\right\}, \quad M_2^c = \left\{\xi \in M_1^c \mid x_2 - y_2 = 0\right\}, \quad M_3^c = M_2^c.$$

Thus by Proposition 1, $M^* = M_3^c = M_2^c$ is the locally maximal controlled invariant submanifold around $x_p \in M^*$ (so $x_p$ is admissible). Choose new coordinates $\xi_2 = (\tilde{x}_1, \tilde{x}_2) = (x_1 - y_1, x_2 - y_2)$ and keep the remaining coordinates $\xi_1 = (y_1, y_2, \theta_1, \theta_2, F_f)$ unchanged, the system represented in the new coordinates is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 & 0 & m \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta_1 - \sin\theta_1 & 0 & l & 0 & 0 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \\ \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 + y_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ \frac{F_f}{ml} \\ \tilde{x}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

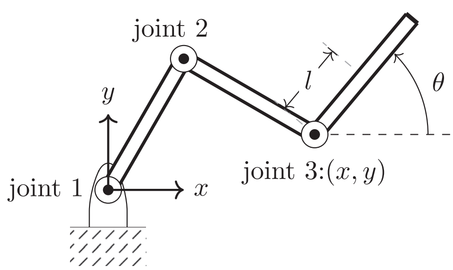


**FIGURE 1** A 3-link manipulator with a free joint

Set $\xi_2 = (\tilde{x}_1, \tilde{x}_2) = 0$ to get a DACS of the form (4)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \cos\theta_1 - \sin\theta_1 & 0 & l & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ \frac{F_f}{ml} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}.$$

By using $Q(\xi_1)$ and the feedback transformations defined on $M^*$ as

$$Q(\xi_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \sin\theta_1 & 0 & -\cos\theta_1 & 0 & m \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_f/l \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix},$$

we bring the system into

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ ml\theta_2^2\sin\theta_1 + \frac{F_f}{l}\sec\theta_1 \\ \theta_2 \\ ml\theta_2^2\cos\theta_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \tan\theta_1 & -\sec\theta_1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The local $M^*$-restriction $\Xi^u|_{M^*} = (E^*, F^*, G^*)$ by Definition 4 (compare Example 5.1 of Reference 20) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & m & 0 & -ml\sin\theta_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ \frac{F_f}{l}\sec\theta_1 + ml\theta_2^2\sin\theta_1 \\ \theta_2 \\ ml\theta_2^2\cos\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \tan\theta_1 \\ 0 \\ 1 \end{bmatrix} u_1. \tag{36}$$

An explicitation system $\Sigma^{u_1 v} \in \mathbf{Expl}(\Xi^u|_{M^*})$ can be chosen as

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ \frac{F_f\tan\theta_1 + ml^2\theta_2^2}{ml(\cos\theta_1 + \sin\theta_1)} \\ \theta_2 \\ \frac{F_f\sec\theta_1 + ml^2\theta_2^2(\sin\theta_1 - \cos\theta_1)}{ml^2(\cos\theta_1 + \sin\theta_1)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\sec\theta_1}{m(\cos\theta_1 + \sin\theta_1)} \\ 0 \\ \frac{\tan\theta_1 - 1}{ml(\cos\theta_1 + \sin\theta_1)} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v.$$

Define a new control

$$u^* := \frac{F_f \tan\theta_1 + ml^2\theta_2^2}{ml(\cos\theta_1 + \sin\theta_1)} + \frac{\sec\theta_1}{m(\cos\theta_1 + \sin\theta_1)} u_1.$$

Then the system $\Sigma^{u_1 v}$ under the new control is $\Sigma^{u^* v} = (f, g^{u^*}, g^v)$:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{F}_f \end{bmatrix} = \begin{bmatrix} y_2 \\ 0 \\ \theta_2 \\ \frac{F_f}{ml^2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{l}(\sin\theta_1 - \cos\theta_1) \\ 0 \end{bmatrix} u^* + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v.$$

Now calculate the distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ for the system $\Sigma^{u^* v}$ to get $\hat{\mathcal{D}}_1 = \text{span}\{g^v\}$, $\mathcal{D}_1 = \text{span}\left\{g^{u^*}, g^v\right\}$, $\hat{\mathcal{D}}_2 = \text{span}\left\{g^{u^*}, g^v, ad_f g^v\right\}$, $\mathcal{D}_2 = \text{span}\left\{g^v, g^{u^*}, ad_f g^v, ad_f g^{u^*}\right\}$, $\mathcal{D}_3 = \hat{\mathcal{D}}_2 = TM^*$. where

$$g^v = \frac{\partial}{\partial F_f}, \quad ad_f g^v = -\frac{1}{ml^2}\frac{\partial}{\partial\theta_2}, \quad g^{u^*} = \frac{\partial}{\partial y_2} + \frac{1}{l}(\sin\theta_1 - \cos\theta_1)\frac{\partial}{\partial\theta_2},$$
$$ad_f g^{u^*} = -\frac{\partial}{\partial y_1} - \frac{1}{l}(\sin\theta_1 - \cos\theta_1)\frac{\partial}{\partial\theta_1} + \frac{1}{l}(\sin\theta_1 + \cos\theta_1)\theta_2\frac{\partial}{\partial\theta_2}.$$

Clearly, the distributions above are of constant rank and are all involutive around $\xi_p$. Thus, conditions (FL1)-(FL3) of Theorem 2 are satisfied. Therefore, system $\Xi^u$ is locally internally feedback linearizable around $\xi_p$. Indeed, choose $\varphi^{u^*}(x)$ and $\varphi^v(x)$ such that

$$\text{span}\left\{d\varphi^v\right\} = \mathcal{D}_2^\perp, \quad \text{span}\left\{d\varphi^v, d\varphi^{u^*}\right\} = \hat{\mathcal{D}}_2^\perp.$$

Then define the following coordinates change and feedback transformation (which has a triangular form as desired):

$$\tilde{y}_1 = \varphi^v(\xi_1) = y_1 - l\int a(\theta_1)\mathrm{d}\theta_1, \quad \tilde{y}_2 = L_f\varphi^v(\xi_1) = y_2 - la(\theta_1)\theta_2, \quad \tilde{F}_f = L_f^2\varphi^v(\xi_1) = -a(\theta_1)F_f - a'(\theta_1)l\theta_2^2,$$
$$\tilde{\theta}_1 = \varphi^{u^*}(\xi_1) = \theta_1, \quad \tilde{\theta}_2 = L_f\varphi^{u^*}(\xi_1)\theta_2,$$
$$\begin{bmatrix} \tilde{u}^* \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} \frac{1}{l}(\sin\theta_1 - \cos\theta_1) & 0 \\ -2a'(\theta_1)(\sin\theta_1 - \cos\theta_1)\theta_2 & -a(\theta_1) \end{bmatrix}\begin{bmatrix} u^* \\ v \end{bmatrix} + \begin{bmatrix} \frac{F_f}{ml^2} \\ -3a'(\theta_1)\theta_2 F_f - a''(\theta_1)\theta_2^3 l \end{bmatrix},$$

where $a(\theta_1) = \frac{1}{\sin\theta_1 - \cos\theta_1}$, $a'(\theta_1) = \frac{\mathrm{d}a(\theta_1)}{\mathrm{d}\theta_1}$, $a''(\theta_1) = \frac{\mathrm{d}^2 a(\theta_1)}{\mathrm{d}\theta_1^2}$. We transform $\Sigma^{u^* v}$ into a linear control system in the Brunovský form

$$\Lambda^{\tilde{u}^*\tilde{v}} : \begin{cases} \dot{\tilde{y}}_1 = \tilde{y}_2, \\ \dot{\tilde{y}}_2 = \tilde{F}_f, \\ \dot{\tilde{F}}_f = \tilde{v}, \\ \dot{\tilde{\theta}}_1 = \tilde{\theta}_2, \\ \dot{\tilde{\theta}}_2 = \tilde{u}^*. \end{cases}$$

Thus by Theorem 1, the restriction $\Xi^u|_{M^*}$, given by (36), is locally ex-fb-equivalent to the following completely controllable linear DACS $\Delta^{\tilde{u}^*}$,

$$\Delta^{\tilde{u}^*} : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \\ \dot{\tilde{F}}_f \\ \dot{\tilde{\theta}}_1 \\ \dot{\tilde{\theta}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{F}_f \\ \tilde{\theta}_1 \\ \tilde{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\tilde{u}^*.$$

because $\Sigma^{u^*v} \overset{sys-fb}{\sim} \Sigma^{u_1 v} \in \mathbf{Expl}(\Xi^u|_{M^*})$, $\Lambda^{\tilde{u}^*\tilde{v}} \in \mathbf{Expl}(\Delta^{\tilde{u}})$ and $\Sigma^{u^*v} \overset{sys-fb}{\sim} \Lambda^{\tilde{u}^*\tilde{v}}$. Hence the DACS $\Xi^u$ is locally in-fb-equivalent to the linear DACS $\Delta^{\tilde{u}^*}$, that is, $\Xi^u$ is locally internally feedback linearizable.

## 6 | CONCLUSIONS AND PERSPECTIVES

In this article, we give necessary and sufficient conditions for the problem that when a nonlinear DACS is locally internally or locally externally feedback equivalent to a completely controllable linear DACS. The conditions are based on an ODECS constructed by the explicitation with driving variables. Two examples are given to illustrate how to externally or internally feedback linearize a nonlinear DACS.

A natural problem for future works is that of when a nonlinear DACS is ex-fb-equivalent to a linear one which is not necessarily completely controllable. Actually, this problem is more involved than the problem of external feedback linearization with complete controllability. Indeed, since in Theorem 3, the maximal controlled invariant submanifold $M^*$ on $U$ is $M^* = U$, it follows that the algebraic constraints are directly governed by some variables of $u$. Thus the in-fb-equivalence is very close to the ex-fb-equivalence. However, if $M^* \neq U$, then the algebraic constraints may affect the generalized state. Moreover, since the explicitation is defined up to a generalized output injection, it may happen that one system of the explicitation is feedback linearizable but another is not. The general feedback linearizability problem remains open and, in view of the above points, is challenging.

Some further problems of nonlinear DACSs can be investigated based on the results of this article. For an external feedback linearizable DACS, besides discussing classical control problems as stabilization and tracking of solutions from admissible points, we can study how to design control laws to steer inadmissible initial values to the maximal controlled invariant submanifold. Moreover, the explicitation method used in this article can also be applied to dynamic feedback linearization problems, which are well-studied for nonlinear ODECS (see e.g., References 27 and 28 but not for DACSs. The two kinds of inputs (see Remark 3) $u$ and $v$ of the $(Q, v)$-explicitation of DACSs may be a key difference between the definition of dynamic feedback equivalence for ODECSs and that of DACSs. Furthermore, connections between DACSs and infinite-dimensional differential geometry (or, differential flatness) are also interesting further topics.

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### CONFLICT OF INTEREST

The author declared that there is no conflict of interest.

### DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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## APPENDIX

*Proof of Lemma* 2. For ease of notation, we drop the index "$*$" for $z^*$, $u^*$, $v^*$ and $f^*$ of the system $\Sigma^{u^*v^*}_{n^*,m^*,s^*}$, that is, $\Sigma^{u^*v^*}$ becomes

$$\Sigma^{uv}\,:\,\dot z = f(z) + g^u(z)u + g^v(z)v.$$

The admissible point $x_a$ in the $z$-coordinates will be denoted by $z_a$. We will only show the proof for the case that

$$m^* = s^* = 1, \qquad \text{rank}\,[g^v(z_a)\;\; g^u(z_a)] = 2.$$

The proof for the general case (i.e., for any $m^* \geq 1$ and $s^* \geq 1$, and for rank $[g^v(z_a)\;\; g^u(z_a)] = m^* + s^*$) can be done in a similar fashion as that on page 233–238 of Reference 28 for the feedback linearization of nonlinear multi-inputs multi-outputs control systems. We now describe a procedure to construct a change of coordinates $\xi = \psi(z)$ and a feedback transformation:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha^u(z) \\ \alpha^v(z) \end{bmatrix} + \begin{bmatrix} \beta^u(z) & 0 \\ \lambda(z) & \beta^v(z) \end{bmatrix} \begin{bmatrix} \tilde u \\ \tilde v \end{bmatrix} \tag{A1}$$

to transform $\Sigma^{uv}$ into its Brunovský canonical form, where $\beta^u, \beta^v, \alpha^u, \lambda, \alpha^v$ are scalar functions, and $\beta^u(z)$ and $\beta^v(z)$ are nonzero around $z_a$, notice that the designed feedback transformation (A1) has a triangular form as in (10). Note that constructing (A1) is equivalent to finding the inverse feedback transformation

$$\begin{bmatrix} \tilde u \\ \tilde v \end{bmatrix} = \begin{bmatrix} a^u(z) \\ a^v(z) \end{bmatrix} + \begin{bmatrix} b^u(z) & 0 \\ \tilde\lambda(z) & b^v(z) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{A2}$$

where

$$a^u = -(\beta^u)^{-1}\alpha^u, \quad a^v = (\beta^v)^{-1}\lambda(\beta^u)^{-1}\alpha^u - (\beta^v)^{-1}\alpha^v \quad b^u = (\beta^u)^{-1}, \quad b^v = (\beta^u)^{-1}, \quad \tilde\lambda = -(\beta^v)^{-1}\lambda(\beta^u)^{-1}.$$

Below we will search for functions $a^u$, $a^v$, $\tilde\lambda$, and nonzero functions $b^u$, $b^v$ to construct (A2).

Consider the two sequences of distributions $\mathcal{D}_i$ and $\hat{\mathcal{D}}_i$ for $\Sigma^{uv}$, given by (27), and define

$$\rho := \max\left\{i \in \mathbb{N}^+ \mid \hat{\mathcal{D}}_i \neq \mathcal{D}_i\right\}, \qquad \overline{\rho} := \max\left\{i \in \mathbb{N}^+ \mid \mathcal{D}_{i-1} \neq \hat{\mathcal{D}}_i\right\}.$$

By $m^* = s^* = 1$, it is seen that, for each $i \geq 1$,

$$\dim\ \mathcal{D}_i - \dim \hat{\mathcal{D}}_i = \begin{cases} 0, & \text{if } \mathcal{D}_i = \hat{\mathcal{D}}_i \\ 1, & \text{if } \mathcal{D}_i \neq \hat{\mathcal{D}}_i \end{cases}, \qquad \dim\ \hat{\mathcal{D}}_i - \dim\ \mathcal{D}_{i-1} = \begin{cases} 0, & \text{if } \hat{\mathcal{D}}_i = \mathcal{D}_{i-1} \\ 1, & \text{if } \hat{\mathcal{D}}_i \neq \mathcal{D}_{i-1} \end{cases}. \tag{A3}$$

It follows that $\rho + \overline{\rho} = n^*$. Then only two cases are possible: either $\rho \geq \overline{\rho}$ or $\rho < \overline{\rho}$.

Case 1: If $\rho \geq \overline{\rho}$, then we have

$$\mathcal{D}_0 \subsetneq \hat{\mathcal{D}}_1 \subsetneq \ldots \subsetneq \mathcal{D}_{\overline{\rho}-1} \subsetneq \hat{\mathcal{D}}_{\overline{\rho}} \subsetneq \mathcal{D}_{\overline{\rho}} = \hat{\mathcal{D}}_{\overline{\rho}+1} \subsetneq \mathcal{D}_{\overline{\rho}+1} = \ldots \subsetneq \mathcal{D}_{\rho-1} = \hat{\mathcal{D}}_{\rho} \subsetneq \mathcal{D}_{\rho} = \hat{\mathcal{D}}_{\rho+j} = \mathcal{D}_{\rho+j},\ j > 0.$$

It follows that $\mathcal{D}_{\rho} = \mathcal{D}_{n^*} = \hat{\mathcal{D}}_{n^*}$ Then by (FL2) of Theorem 2, we have $\mathcal{D}_{\rho} = TM^*$ and thus $\dim \mathcal{D}_{\rho} = n^*$. By $\hat{\mathcal{D}}_{\rho} \subsetneq \mathcal{D}_{\rho}$ and (A3), we have $\dim \hat{\mathcal{D}}_{\rho} = n^* - 1$. Now by the involutivity of $\hat{\mathcal{D}}_{\rho}$ (condition (FL3)), we can choose a scalar function $h^u(z)$ such that

$$\text{span}\,\{dh^u\} = \hat{\mathcal{D}}_{\rho}^{\perp},$$

where $\hat{\mathcal{D}}_{\rho}^{\perp}$ denotes the annihilator of the distribution $\hat{\mathcal{D}}_{\rho}$. It follows that for all $z$ around $z_a$,

$$\left\langle dh^u(z), ad_f^i g^u(z) \right\rangle = 0, \quad 0 \le i \le \rho - 2, \quad \left\langle dh^u(z), ad_f^{\rho-1} g^u(z) \right\rangle \neq 0; \qquad \left\langle dh^u(z), ad_f^i g^v(z) \right\rangle = 0, \quad 0 \le i \le \rho - 1. \tag{A4}$$

Recall the following result:[27,28]

$$\left\langle dh(z), ad_f^i g(z) \right\rangle = 0, \ 0 \le i \le l-2 \Rightarrow \left\langle dh(z), ad_f^{l-1} g(z) \right\rangle = (-1)^i \left\langle dL_f^i h(z), ad_f^{l-1-i} g(z) \right\rangle, \ 0 \le i \le l-1, \tag{A5}$$

where $h(z)$ is a scalar function, $f(z)$ and $g(z)$ are vector fields.

It can be deduced from (A4) and (A5) that for all $z$ around $z_a$,

$$\begin{aligned}&\left\langle dL_f^i h^u(z), ad_f^j g^u(z) \right\rangle = 0, \ 0 \le i \le \rho-2, \ 0 \le j \le \rho - i - 2, \quad \left\langle dL_f^i h^u(z), ad_f^{\rho-i-1} g^u(z) \right\rangle \neq 0, \ 0 \le i \le \rho - 2;\\ &\left\langle dL_f^i h^u(z), ad_f^j g^v(z) \right\rangle = 0, \quad 0 \le i \le \rho - 1, \ 0 \le j \le \rho - i - 1.\end{aligned} \tag{A6}$$

By using (A6), we have the following table for the expressions of $\left\langle dL_f^i h^u, ad_f^j g^u \right\rangle$, $0 \le i \le \rho - \overline{\rho}$, $\overline{\rho} - 1 \le j \le \rho - 1$:

$$\begin{array}{ccccc} & ad_f^{\overline{\rho}-1} g^u & ad_f^{\overline{\rho}} g^u & \cdots & ad_f^{\rho-1} g^u \\ dh^u & 0 & 0 & \cdots & \left\langle dh^u, ad_f^{\rho-1} g^u \right\rangle \\ \cdots & \cdots & \cdots & * & \\ dL_f^{\rho-\overline{\rho}-1} h^u & 0 & \left\langle dL_f^{\rho-\overline{\rho}-1} h^u, ad_f^{\overline{\rho}} g^u \right\rangle & & \\ dL_f^{\rho-\overline{\rho}} h^u & \left\langle dL_f^{\rho-\overline{\rho}} h^u, ad_f^{\overline{\rho}-1} g^u \right\rangle & & ? & \end{array}.$$

Notice that all the antidiagonal elements of the above table are nonzero by (A6). It follows that the co-distribution

$$\Omega_1 = \text{span} \left\{ dL_f^i h^u, \quad 0 \le i \le \rho - \overline{\rho} \right\}$$

is of dimension $\rho - \overline{\rho} + 1$ around $z_a$. We have $\Omega_1 \subseteq \mathcal{D}_{\overline{\rho}-1}^{\perp}$ because

$$\begin{aligned}\left\langle dL_f^i h^u(z), ad_f^j g^u(z) \right\rangle &\overset{(A6)}{=} 0, \quad 0 \le i \le \rho - \overline{\rho}, \quad 0 \le j \le \overline{\rho} - 2,\\ \left\langle dL_f^i h^u(z), ad_f^j g^v(z) \right\rangle &\overset{(A6)}{=} 0, \quad 0 \le i \le \rho - \overline{\rho}, \quad 0 \le j \le \overline{\rho} - 2.\end{aligned}$$

It is seen that $\dim \mathcal{D}_{\overline{\rho}-1}^{\perp} - \dim \Omega_1 = (n^* - (2\overline{\rho} - 2)) - (\rho - \overline{\rho} + 1) = 1$ and $\Omega_1 \subsetneq \mathcal{D}_{\overline{\rho}-1}^{\perp}$. Then by the involutivity of $\mathcal{D}_{\overline{\rho}-1}$ (condition (FL3)), we can choose a scalar function $h^v(z)$ such that

$$\text{span}\left\{ dh^v \right\} + \Omega_1 = \mathcal{D}_{\overline{\rho}-1}^{\perp},$$

which implies that for all $z$ around $z_a$,

$$\left\langle dh^v(z), ad_f^i g^u(z) \right\rangle = 0, \ 0 \le i \le \overline{\rho} - 2; \quad \left\langle dh^v(z), ad_f^i g^v(z) \right\rangle = 0, \ 0 \le i \le \overline{\rho} - 2, \quad \left\langle dh^v(z), ad_f^{\overline{\rho}-1} g^v(z) \right\rangle \neq 0. \tag{A7}$$

It can be deduced by (A7) and (A5) that for all $z$ around $z_a$,

$$\begin{aligned}&\left\langle dL_f^i h^v(z), ad_f^j g^u(z) \right\rangle = 0, \quad 0 \le i \le \overline{\rho} - 2, \ 0 \le j \le \overline{\rho} - i - 2;\\ &\left\langle dL_f^i h^v(z), ad_f^j g^v(z) \right\rangle = 0, \ 0 \le i \le \overline{\rho} - 2, \quad 0 \le j \le \overline{\rho} - i - 2, \quad \left\langle dL_f^i h^v(z), ad_f^{\overline{\rho}-i-1} g^v(z) \right\rangle \neq 0, \ 0 \le i \le \overline{\rho} - 2.\end{aligned} \tag{A8}$$

By using (A6) and (A8), we can construct the following table:

| | $g^v$ | $g^u$ | $\cdots$ | $\cdots$ | $ad_f^{\overline{\rho}-1}g^v$ | $ad_f^{\overline{\rho}-1}g^u$ | $ad_f^{\overline{\rho}}g^u$ | $\cdots$ | $ad_f^{\rho-1}g^u$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{d}h^u$ | 0 | 0 | $\cdots$ | $\cdots$ | 0 | 0 | 0 | $\cdots$ | $\left\langle \mathrm{d}h^u, ad_f^{\rho-1}g^u \right\rangle$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $*$ | |
| $\mathrm{d}L_f^{\rho-\overline{\rho}-1}h^u$ | 0 | 0 | $\cdots$ | $\cdots$ | 0 | 0 | $\left\langle \mathrm{d}L_f^{\rho-\overline{\rho}-1}h^u, ad_f^{\overline{\rho}}g^u \right\rangle$ | | |
| $\mathrm{d}L_f^{\rho-\overline{\rho}}h^u$ | 0 | 0 | $\cdots$ | $\cdots$ | 0 | $\left\langle \mathrm{d}L_f^{\rho-\overline{\rho}}h^u, ad_f^{\overline{\rho}-1}g^u \right\rangle$ | | | ? |
| $\mathrm{d}h^v$ | 0 | 0 | $\cdots$ | $\cdots$ | $\left\langle \mathrm{d}h^v, ad_f^{\overline{\rho}-1}g^v \right\rangle$ | ? | | | |
| $\cdots$ | 0 | 0 | $\cdots$ | $*$ | | | | | |
| $\cdots$ | 0 | 0 | $*$ | ? | | | | | |
| $\mathrm{d}L_f^{\rho-1}h^u$ | 0 | $L_{g^u}L_f^{\rho-1}h^u$ | | | | | | | |
| $\mathrm{d}L_f^{\overline{\rho}-1}h^v$ | $L_{g^v}L_f^{\overline{\rho}-1}h^v$ | ? | ? | | ? | | | | |

. (A9)

Notice that all the antidiagonal elements of table (A9) are nonzero. It follows that the $(\rho+\overline{\rho})\times(\rho+\overline{\rho}) = n^*\times n^*$ matrix

$$\frac{\partial\psi}{\partial z}(z)\left[g^v \quad g^u \quad \cdots \quad \cdots \quad ad_f^{\overline{\rho}-1}g^v \quad ad_f^{\overline{\rho}-1}g^u \quad ad_f^{\overline{\rho}}g^u \quad \cdots \quad ad_f^{\rho-1}g^u\right](z)$$

is invertible around $z_a$, where

$$\psi = (h^u, \dots, L_f^{\rho-1}h^u, h^v, \dots, L_f^{\overline{\rho}-1}h^v). \tag{A10}$$

Thus the Jacobian matrix $\frac{\partial\psi(z)}{\partial z}$ is invertible around $z_a$ and $\psi$ is a local diffeomorphism. Then set

$$a^u(z) = L_f^{\rho}h^u(z), \quad b^u(z) = L_{g^u}L_f^{\rho-1}h^u(z), \quad a^v(z) = L_f^{\overline{\rho}}h^v(z), \quad b^v(z) = L_{g^v}L_f^{\overline{\rho}-1}h^v(z), \quad \tilde{\lambda}(z) = L_{g^u}L_f^{\overline{\rho}-1}h^v(z). \tag{A11}$$

Note that $b^u(z)$ and $b^v(z)$ are nonzero at $z_p$. It is seen that $\Sigma^{u^*v^*}$ is mapped, via the coordinates transformations $\xi = (\xi_1, \xi_2) = \psi(z)$ and the feedback transformation (A2), into the Brunovský form $\Sigma_{Br}^{w} = \Sigma_{Br}^{w^*}$ of (28) with indices $\rho$ and $\overline{\rho}$.

Case 2: If $\rho < \overline{\rho}$, then we have $\mathcal{D}_0 \subsetneq \hat{\mathcal{D}}_1 \subsetneq \dots \subsetneq \hat{\mathcal{D}}_\rho \subsetneq \mathcal{D}_\rho \subsetneq \hat{\mathcal{D}}_{\rho+1} = \mathcal{D}_{\rho+1} \subsetneq \dots = \mathcal{D}_{\overline{\rho}-1} \subsetneq \hat{\mathcal{D}}_{\overline{\rho}} = \mathcal{D}_{\overline{\rho}} = \hat{\mathcal{D}}_{\overline{\rho}+j} = \mathcal{D}_{\overline{\rho}+j}, j > 0$. It follows that $\hat{\mathcal{D}}_{\overline{\rho}} = \mathcal{D}_{\overline{\rho}} = \hat{\mathcal{D}}_{n^*} = \mathcal{D}_{n^*}$. Then by (FL2) of Theorem 2, we have $\hat{\mathcal{D}}_{\overline{\rho}} = TM^*$ and thus $\dim \hat{\mathcal{D}}_{\overline{\rho}} = n^*$. By $\mathcal{D}_{\overline{\rho}-1} \subsetneq \hat{\mathcal{D}}_{\overline{\rho}}$ and (A3), we have $\dim \mathcal{D}_{\overline{\rho}-1} = n^*-1$. Now by the involutivity of $\mathcal{D}_{\overline{\rho}}$ (condition (FL1)), we can choose a scalar function $h^v(z)$ such that

$$\mathrm{span}\left\{dh^v\right\} = \mathcal{D}_{\overline{\rho}-1}^{\perp}.$$

Then following a similar proof as in Case 1, we can show that the distribution

$$\Omega_2 = \mathrm{span}\left\{dL_f^ih^v, \quad 0 \le i \le \overline{\rho}-\rho-1\right\}$$

is of dimension $\rho-\overline{\rho}$ around $z_a$ and $\Omega_2 \subsetneq \hat{\mathcal{D}}_\rho^{\perp}$. Notice that $\dim \hat{\mathcal{D}}_\rho^{\perp} = n^* - (2\rho-1) = \overline{\rho}-\rho+1$, we have $\dim \hat{\mathcal{D}}_\rho^{\perp} - \dim \Omega_2 = 1$. Thus by the involutivity of $\hat{\mathcal{D}}_\rho$ (condition (FL2)), we can choose a scalar function $h^u(z)$ such that

$$\mathrm{span}\left\{\mathrm{d}h^u\right\} + \Omega_2 = \hat{\mathcal{D}}_\rho^{\perp}.$$

Then, similarly as in Case 1, we construct the following table:

| | $g^v$ | $g^u$ | $\cdots$ | $\cdots$ | $ad_f^{\rho-1}g^v$ | $ad_f^{\rho-1}g^u$ | $ad_f^{\rho}g^v$ | $\cdots$ | $ad_f^{\overline{\rho}-1}g^v$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{d}h^v$ | 0 | 0 | $\cdots$ | $\cdots$ | 0 | 0 | 0 | $\cdots$ | $\left\langle \mathrm{d}h^v, ad_f^{\overline{\rho}-1}g^v \right\rangle$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $*$ | ? |
| $\mathrm{d}L_f^{\overline{\rho}-\rho-1}h^v$ | 0 | 0 | $\cdots$ | $\cdots$ | 0 | 0 | $\left\langle \mathrm{d}L_f^{\overline{\rho}-\rho-1}h^v, ad_f^{\rho}g^v \right\rangle$ | | |
| $\mathrm{d}h^u$ | 0 | 0 | $\cdots$ | $\cdots$ | 0 | $\left\langle \mathrm{d}h^u, ad_f^{\rho-1}g^u \right\rangle$ | | | |
| $\mathrm{d}L_f^{\overline{\rho}-\rho}h^v$ | 0 | 0 | $\cdots$ | $\cdots$ | $\left\langle \mathrm{d}L_f^{\overline{\rho}-\rho}h^v, ad_f^{\rho-1}g^v \right\rangle$ | ? | | | ? |
| $\cdots$ | 0 | 0 | $\cdots$ | $*$ | | | | | |
| $\cdots$ | 0 | 0 | $*$ | | | | | | |
| $\mathrm{d}L_f^{\rho-1}h^u$ | 0 | $L_{g^u}L_f^{\rho-1}h^u$ | | | | | | | |
| $\mathrm{d}L_f^{\overline{\rho}-1}h^v$ | $L_{g^v}L_f^{\overline{\rho}-1}h^v$ | ? | ? | | | ? | | | ? |

and show that all the antidiagonal elements of the table are nonzero around $z_a$. Finally, we define a diffeomorphism $\psi$ and functions $a^u$, $b^u$, $a^v$, $b^v$, $\tilde{\lambda}$ in the same form as (A10) and (A11) of Case 1. It is seen that $\Sigma^{uv}$ can also be transformed into the Brunovský form $\Sigma_{Br}^{w} = \Sigma_{Br}^{w^*}$ of (28) with indices $\rho$ and $\overline{\rho}$ via the change of coordinates $\xi = \psi(z)$ and the feedback transformation (A2). ■