From Morse Triangular Form of ODE Control Systems to Feedback Canonical Form of DAE Control Systems

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Abstract

In this paper, we relate the feedback canonical form **FNCF** [24] of differential-algebraic control systems (DACSs) with the famous Morse canonical form **MCF** [28],[27] of ordinary differential equation control systems (ODECSs). First, a procedure called an explicitation (with driving variables) is proposed to connect the two above categories of control systems by attaching to a DACS a class of ODECSs with two kinds of inputs (the original control input u and a vector of driving variables v). Then, we show that any ODECS with two kinds of inputs can be transformed into its extended **MCF** via two intermediate forms: the extended Morse triangular form and the extended Morse normal form. Next, we illustrate that the **FNCF** of a DACS and the extended **MCF** of the explicitation system have a perfect one-to-one correspondence. At last, an algorithm is proposed to transform a given DACS into its **FBCF** via the explicitation procedure and a numerical example is given to show the efficiency of the proposed algorithm.

Keywords: differential-algebraic equations, ordinary differential equations, control systems, explicitation, Morse canonical form, feedback canonical form

2020 MSC: 15A21, 34A09, 34H05, 93C05, 93C15

1. Introduction

Consider a linear differential-algebraic control system (DACS) of the form

$$\Delta^u : E\dot{x} = Hx + Lu, \tag{1}$$

where $x \in \mathscr{X} \cong \mathbb{R}^n$ is called the "generalized" state, $u \in \mathbb{R}^m$ is the vector of control inputs, and where $E \in \mathbb{R}^{l\times n}$, $H \in \mathbb{R}^{l\times n}$ and $L \in \mathbb{R}^{l\times m}$. A linear DACS of the form (1) will be denoted by $\Delta^u_{l,n,m} = (E, H, L)$ or, simply, Δ^u. In the case of the control u being absent, the system becomes a linear differential-algebraic equation (DAE) $E\dot{x} = Hx$, which is called regular if $l = n$ and $sE - H \in \mathbb{R}^{n\times n}[s]\backslash 0$. A detailed exposition of the theory of linear DAEs and DACSs can be consulted in the textbooks [16],[13] and the survey paper [22]. Early results on linear DAEs can be traced back to two famous canonical forms of the matrix pencil $sE - H$ given by Weierstrass [34] and Kronecker [21]. The following literature discusses the normal forms and canonical forms of linear DAE systems. The authors of [20] proposed a canonical from for controllable and regular DACSs. Several forms for regular systems based on their controllability and impulse controllability were given in [19]. In [31], a canonical form of general DACSs was discussed. More recently, a normal form based on impulse-controllability and impulse-observability of DACSs was proposed in [32], and a quasi-Weierstrass and a quasi-Kronecker triangular/normal forms of DAEs were given in [6] and [9], respectively. In the present paper, we discuss the feedback canonical form **FBCF** obtained in [24] (we restate it as Theorem 4.4 of the present paper) for general linear DACSs, which plays an important role in, e.g. controllability analysis [7], regularization problems [12],[8], pole assignment [25],[10] and stabilization [4]. The **FBCF** of DACSs is actually an extension of the Kronecker canonical form of general linear DAEs. Some methods (most are numerical) of transforming a DAE into its Kronecker canonical form can be found in [17],[33],[3].

In [15], we proposed a notion, called explicitation, to connect DAEs with control systems. In the present paper, we will propose a new explicitation procedure called *explicitation with driving variables* (see Definition 2.2), and differences and relations of the two explicitation methods are discussed in Remark 2.5. Since the vector of driving variables v enters statically into the system (similarly as the control input u), we can regard it as another kind of input. More specifically, the *explicitation with driving variables* of a DACS is a class of ODECSs with two kinds of inputs of the form:

$$\Lambda^{uv}: \begin{cases} \dot{x} = Ax + B^u u + B^v v \\ y = Cx + D^u u, \end{cases} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B^u \in \mathbb{R}^{n\times m}$, $B^v \in \mathbb{R}^{n\times s}$, $C \in \mathbb{R}^{p\times n}$ and $D^u \in \mathbb{R}^{p\times m}$, where $u \in \mathbb{R}^m$ is the vector of control variables and $v \in \mathbb{R}^s$ is the vector of driving variables. An ODECS of the form (2) will be denoted by $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$ or, simply, Λ^{uv}. Note that although both u and v may be considered as inputs of system (2), we distinguish them because they play different roles for the system and, as a consequence, their feedback transformation rules are different (see Remark 2.8). Observe that we can express an ODECS Λ^{uv} of the form (2), as a classical ODECS $\Lambda^w = (A, B^w, C, D^w)$ of the form

$$\Lambda^u: \begin{cases} \dot{x} = Ax + B^w w \\ y = Cx + D^w w, \end{cases} \tag{3}$$

by denoting $w = [u^T, v^T]^T$, $B^w = [B^u \quad B^v]$ and $D^w = [D^u \quad 0]$. Throughout the paper, depending on the context, we will use either Λ^{uv} or Λ^w to denote an ODECS with two kinds of inputs.

We use Figure 1 to show the relations of the results of the paper. The purpose of this paper is to find an efficient *geometric* way to transform a DACS Δ^u into its feedback canonical form **FBCF** via the explicitation procedure. As we have pointed out, the **FBCF** is a generalization, on one hand, of the classical Kronecker form (because a DACS is a differential-algebraic equation) and one the other hand, of the Brunovsky canonical form [11] (because a DACS is a control system). The explicitation procedure allows us to attach to a DACS a control system Λ^{uv} with an output y (defining the algebraic constraint as $y = 0$) and to study the double nature of a DACS (differential-algebraic and control-theoretic) simultaneously by analyzing Λ^{uv}. More specifically, instead of using transformations directly on a DACS, we will first transform an ODECS Λ^{uv}, given by the explicitation of our DACS, into its canonical form (called the extended Morse canonical form **EMCF**, see Theorem 4.1). Then by the relation between DACSs and ODECSs given in Section 2, we can easily get the **FBCF** from the **EMCF**. Moreover, inspired by the quasi-Kronecker triangular form of [9], we will propose a Morse triangular form **MTF** (see Proposition 3.1) to transform an ODECS (with one type of controls) into its Morse normal form **MNF** (see Proposition 3.2). Note that a procedure of transforming an ODECS Λ^u into its **MCF** was given by Morse [28] for $D^u = 0$ and by Molinari [27] for the general case $D^u \neq 0$. We propose to do it via two intermediate normal forms **MTF** and **MNF**.

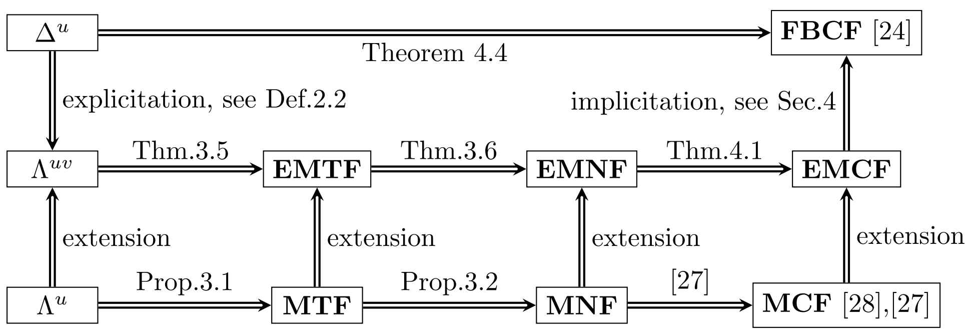

Figure 1: The relations of the results in the paper

We use the following abbreviations throughout the paper:

DAE	differential-algebraic equation	**MCF**	Morse canonical form
DACS	differential-algebraic control system	**EMTF**	extended Morse triangular form
ODECS	ordinary differential equation control system	**EMNF**	extended Morse normal form
MTF	Morse triangular form	**EMCF**	extended Morse canonical form
MNF	Morse normal form	**FBCF**	feedback canonical form

This paper is organized as follows. In Section 2, we introduce the explicitation with driving variables procedure and build geometric connections between DACSs and ODECSs. In Section 3, we show a method of constructing the **MTF** and the **MNF** for classical ODECSs of the form (3), then we extend them to the **EMTF** and the **EMNF** for ODECSs (with two kinds of inputs) of the form (2). In Section 4, we propose the **EMCF** for ODECSs of the form (2), which allows to construct the **FBCF** of DACSs as a corollary and we formulate the construction of the **FBCF** via the explication procedure as an algorithm. In Section 5, we give a numerical example to show the efficiency of the algorithm. Section 6 and 7 contain proofs and conclusions of the paper, respectively. The definitions of geometric invariant subspaces for ODECSs and DACSs are given in Appendix.

Throughout, we will use the following notations:

$\mathcal{C}^k$	the class of k-times differentiable functions
$\mathbb{N}$	the set of natural numbers with zero and $\mathbb{N}^+ = \mathbb{N}\backslash\{0\}$
$\mathbb{R}^{n\times m}$	the set of real valued matrices with n rows and m columns
$Gl\,(n,\mathbb{R})$	the group of nonsingular matrices of $\mathbb{R}^{n\times n}$
$\ker A$	the kernel of the map given by a matrix A
$\operatorname{Im} A$	the image of the map given by a matrix A
$\operatorname{rank} A$	the rank of a matrix A
I_n	the identity matrix of size $n \times n$ for $n \in \mathbb{N}^+$
$0_{n\times m}$	the zero matrix of size $n \times m$ for $n, m \in \mathbb{N}^+$
A^T	the transpose of a matrix A
A^{-1}	the inverse of a matrix A
$A\mathscr{B}$	$\{Ax \mid x \in \mathscr{B}\}$, the image of a space $\mathscr{B}$ under a map given by a matrix A
$A^{-1}\mathscr{B}$	$\{x \in \mathbb{R}^n \mid Ax \in \mathscr{B}\}$, the preimage of a space $\mathscr{B}$ under a map given by a matrix A
$A^{-T}\mathscr{B}$	$(A^T)^{-1}\mathscr{B}$
$\mathscr{A}^{\perp}$	$\{x \in \mathbb{R}^n \mid \forall a \in \mathscr{A} : x^T a = 0\}$, the orthogonal complement of a subspace $\mathscr{A} \subseteq \mathbb{R}^n$
$A^{\dagger}$	the right inverse of a full row rank matrix $A \in \mathbb{R}^{n\times m}$, i.e., $AA^{\dagger} = I_n$
$x^{(k)}$	k-th-order derivative of a function $x(t)$

2. Explicitation with driving variables for linear DACSs

A solution of Δ^u is a map $(x(t), u(t)) : \mathbb{R} \to \mathscr{X} \times \mathbb{R}^m$ with $x(t) \in \mathcal{C}^1$ and $u(t) \in \mathcal{C}^0$ satisfying $E\dot{x}(t) = Hx(t) + Lu(t)$. Notice that to some $\mathcal{C}^0$-controls $u(t)$, there may not correspond any $\mathcal{C}^1$-solution $x(t)$ because of algebraic relations between u_i's and x_j's present in Δ^u of the form (1).

Definition 2.1. Two DACSs $\Delta^u_{l,n,m} = (E, H, L)$ and $\tilde{\Delta}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{H}, \tilde{L})$ are called externally feedback equivalent, shortly ex-fb-equivalent, if there exist matrices $Q \in Gl(l,\mathbb{R})$, $P \in Gl(n,\mathbb{R})$, $F \in \mathbb{R}^{m\times n}$ and $G \in Gl(m,\mathbb{R})$ such that

$$\tilde{E} = QEP^{-1}, \quad \tilde{H} = Q(H + LF)P^{-1}, \quad \tilde{L} = QLG. \tag{4}$$

We denote the ex-fb-equivalence of two DACSs as $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$.

Now we introduce the *explicitation with driving variables* procedure for Δ^u as follows.

- Denote the rank of E by $q \in \mathbb{N}$, define $s = n - q$ and $p = l - q$. Then there exists a matrix $Q \in Gl(l,\mathbb{R})$ such that $QE = \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right]$, where $E_1 \in \mathbb{R}^{q\times n}$ and $\operatorname{rank} E_1 = q$. Via Q, DACS Δ^u is ex-fb-equivalent to

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} x + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} u, \tag{5}$$

  where $QH = \left[\begin{smallmatrix} H_1 \\ H_2 \end{smallmatrix}\right]$, $QL = \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$, and where $H_1 \in \mathbb{R}^{q\times n}$, $H_2 \in \mathbb{R}^{(l-q)\times n}$, $L_1 \in \mathbb{R}^{q\times m}$, $L_2 \in \mathbb{R}^{(l-q)\times m}$.

- Consider the differential part of (5):

$$E_1\dot{x} = H_1 x + L_1 u. \tag{5a}$$

The matrix E_1 is of full row rank q, so let $E_1^\dagger \in \mathbb{R}^{n\times q}$ denote its right inverse, i.e., $E_1 E_1^\dagger = I_q$. Set $A = E_1^\dagger H_1$ and $B^u = E_1^\dagger L_1$. In general, $w \in \mathbb{R}^n$ satisfies the linear equation $E_1 w = b$, where $E_1 : \mathbb{R}^n \to \mathbb{R}^q$ is of full row rank q, if and only if $w \in E_1^\dagger b + \ker E_1$. It follows that $x(t)$ satisfies (5a) if and only if

$$\dot{x} \in Ax + B^u u + \ker E_1. \tag{6}$$

- Choose a full column rank matrix $B^v \in \mathbb{R}^{n\times s}$ such that $\operatorname{Im} B^v = \ker E_1 = \ker E$ (note that the kernels of E_1 and E coincide since any invertible Q preserves the kernel). Then the vector $v \in \mathbb{R}^s$ of driving variables (see Remark 2.5 for a control-theory interpretation of v) parameterizes the subspace $\ker E_1 = \operatorname{Im} B^v$ via $B^v v$ and the solutions of the differential inclusion (6), and thus of (5a), correspond to the solutions of

$$\dot{x} = Ax + B^u u + B^v v. \tag{7}$$

- We claim, see Proposition 2.4 below, that all solutions of (5) (and thus of the original DAE Δ^u) are in one-to-one correspondence with all solutions (corresponding to all driving variables $v(t)$) of

$$\begin{cases} \dot{x} = Ax + B^u u + B^v v \\ 0 = Cx + D^u u, \end{cases} \tag{8}$$

where $C = H_2 \in \mathbb{R}^{p\times n}$ and $D^u = L_2 \in \mathbb{R}^{p\times m}$. Recall that a control system of the form (2) is denoted by $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$. It is immediately to see that equation (8) can be obtained from the ODECS Λ^{uv} by setting the output $y = 0$. In the above way, we attach an ODECS Λ^{uv} to a DACS Δ^u.

The above procedure of attaching a control system $\Lambda^{u,v}$ to a DACS Δ^u will be called *explicitation with driving variables* and is formalized as follows.

Definition 2.2. Given a DACS $\Delta^u_{l,n,m} = (E, H, L)$, by a (Q, v)-explicitation, we will call a control system $\Lambda^{uv} = (A, B^u, B^v, C, D^u)$, with

$$A = E_1^\dagger H_1, \quad B^u = E_1^\dagger L_1, \quad \operatorname{Im} B^v = \ker E_1 = \ker E, \quad C = H_2, \quad D^u = L_2,$$

where

$$QE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad QH = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad QL = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.$$

The class of all (Q, v)-explicitations will be called the *explicitation with driving variables class* or, shortly *explicitation class*, of Δ^u, denoted by $\mathbf{Expl}(\Delta^u)$. If a particular ODECS Λ^{uv} belongs to the explicitation class $\mathbf{Expl}(\Delta^u)$, we will write $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$.

The definition of the explicitation class $\mathbf{Expl}(\Delta^u)$ suggests that a given Δ^u has many (Q, v)-

explicitations. Indeed, the construction of $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$ is not unique at three stages: there is a freedom in choosing Q, $E_1^\dagger$, and B^v. We show in the following proposition that $\mathbf{Expl}(\Delta^u)$ is actually an ODECS defined up to a v-feedback transformation, an *output injection* and *an output transformation*, that is, a class of ODECSs.

Proposition 2.3. *Assume that an ODECS $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$ is a (Q,v)-explicitation of a DACS $\Delta^u_{l,n} = (E,H,L)$ corresponding to a choice of invertible matrix Q, right inverse $E_1^\dagger$, and matrix B^v. Then $\tilde{\Lambda}^{u\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^u, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^u)$ is a $(\tilde{Q},\tilde{v})$-explicitation of Δ^u corresponding to a choice of invertible matrix $\tilde{Q}$, right inverse $\tilde{E}_1^\dagger$, and matrix $\tilde{B}^{\tilde{v}}$ if and only if Λ^{uv} and $\tilde{\Lambda}^{u\tilde{v}}$ are equivalent via a v-feedback transformation of the form $v = F_v x + Ru + T_v^{-1}\tilde{v}$, an output injection $Ky = K(Cx + D^u u)$ and an output multiplication $\tilde{y} = T_y y$, which map*

$$\begin{aligned} A \mapsto \tilde{A} = A + KC + B^v F_v, \quad B^u \mapsto \tilde{B}^u = B^u + B^v R + KD^u, \quad B^v \mapsto \tilde{B}^{\tilde{v}} = B^v T_v^{-1}, \\ C \mapsto \tilde{C} = T_y C, \quad D^u \mapsto \tilde{D}^u = T_y D^u, \end{aligned} \tag{9}$$

where F_v, K, R, T_v, T_y are matrices of appropriate sizes, and T_v and T_y are invertible.

The following proposition shows that solutions of any DACS are in one-to-one correspondence with solutions of its (Q,v)-explicitations.

Proposition 2.4. *Consider $\Delta^u_{l,n,m} = (E,H,L)$ and let an ODECS $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$ be a (Q,v)-explicitation of Δ^u, i.e., $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$. Then a curve $(x(t), u(t))$ with $x(t) \in \mathcal{C}^1$ and $u(t) \in \mathcal{C}^0$ is a solution of Δ^u if and only if there exists $v(t) \in \mathcal{C}^0$ such that $(x(t), u(t), v(t))$ is a solution of Λ^{uv} respecting the output constraints $y = 0$, i.e., a solution of (8).*

The proofs of Proposition 2.3 and Proposition 2.4 will be given in Section 6.1.

Remark 2.5. Notice that the definition of (Q,v)-explicitation in the present paper is different in two aspects from the (Q,P)-explicitation of [15] (or see Chapter II of [14]). First, in this paper we consider the explicitation of DACSs while in [15] we dealt with DAEs (with no controls). The second difference is that in (Q,v)-explicitation, we keep the original generalized state variables x and add new driving variables v while in (Q,P)-explicitation of [15], we look for a partition $(z_1, z_2) = z = Px$ into state- and control- variables. More specifically, consider a DACS $\Delta^u_{l,n,m} = (E,H,L)$, then via two invertible matrices Q and P, the system Δ^u is ex-fb-equivalent with $F = 0$ and $G = I_m$ (or ex-equivalent, according to the terminology of [15], since here we do not use feedback transformation for Δ^u) to a pure semi-explicit PSE DACS

$$\Delta^u_{PSE} : \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}^1 \\ \dot{z}^2 \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} u,$$

with $z = \left[\begin{smallmatrix} z^1 \\ z^2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} P_1 x \\ P_2 x \end{smallmatrix}\right] = Px$, where P is any invertible map such that $\ker P_1 = \ker E$. We attach to Δ^u_{PSE}, the control system

$$\Lambda^{uz^2} : \begin{cases} \dot{z}^1 = H_1 z^1 + H_2 z^2 + L_1 u \\ y = H_3 z^1 + H_4 z^2 + L_2 u, \end{cases} \tag{10}$$

where $z^2 \in \mathscr{Z}_2 = \ker E$ is the vector of free variables (which perform like inputs), $z^1 \in \mathscr{Z}_1$ is the state such that $\mathscr{Z}_1 \oplus \mathscr{Z}_2 = \mathscr{X} \cong \mathbb{R}^n$, and y is the output. The system Λ^{uz^2} is called a (Q, P)-explicitation of Δ^u and we will write $\Lambda^{uz^2} \in \mathrm{Expl}(\Delta^u)$, where $\mathrm{Expl}(\Delta^u)$ is the explicitation class consisting of all (Q, P)-explicitations of Δ^u (clearly, for a given Δ^u, its (Q, P)-explicitation is not unique). Now by adding the equation $\dot{z}^2 = v$, we obtain the (dynamical) prolongation $\mathbf{\Lambda}^{uv}$ of Λ^{uz^2}

$$\mathbf{\Lambda}^{uv} : \begin{cases} \dot{z}^1 &= H_1 z^1 + H_2 z^2 + L_1 u \\ \dot{z}^2 &= v \\ \;y &= H_3 z^1 + H_4 z^2 + L_2 u, \end{cases} \tag{11}$$

which is actually an (I_l, v)-explicitation of Δ^u_{PSE}. We can summarize the relations between the notions of (Q, P)-explicitation and (Q, v)-explicitation by the following diagram.

$$\begin{tikzcd}
\Delta^u \arrow[rr, "\text{ex-equivalence via } (Q,P)"] \arrow[dr, "(Q,P)\text{-expl}"] \arrow[dd, "(Q,\tilde{v})\text{-expl}"'] & & \Delta^u_{PSE} \arrow[dl, "(I_l,I_n)\text{-expl}"'] \arrow[dd, "(I_l,v)\text{-expl}"] \\
& \Lambda^{uz^2} \in \mathrm{Expl}(\Delta^u) = \mathrm{Expl}(\Delta^u_{PSE}) \arrow[dr, "\text{prolongation}"] & \\
\Lambda^{u\tilde{v}} \in \mathbf{Expl}(\Delta^u) \arrow[rr, leftrightarrow, "\text{EM-equivalence}"] & & \mathbf{\Lambda}^{uv} \in \mathbf{Expl}(\Delta^u_{PSE})
\end{tikzcd}$$

The systems Δ^u and Δ^u_{PSE} above are DACSs and their ex-equivalence is (Q, P)-equivalence of DACSs. The system $\Lambda^{u\tilde{v}}$ and Λ^{uv} at the bottom are control systems and their EM-equivalence is the extended Morse equivalence given in Definition 2.7. Note that the implication that the $(Q, \tilde{v})$-explicitation $\Lambda^{u\tilde{v}}$ of Δ^u is EM-equivalent to the prolongation system $\mathbf{\Lambda}^{uv}$ is a corollary of Theorem 2.9 below since $\mathbf{\Lambda}^{uv} \in \mathbf{Expl}(\Delta^u_{PSE})$, $\Lambda^{u\tilde{v}} \in \mathbf{Expl}(\Delta^u)$, and $\Delta^u_{PSE} \overset{ex}{\sim} \Delta^u$.

Remark 2.6. The above explicitation (via driving variables) procedure can also be applied to more general DAE systems such as DACSs with time delays (see e.g., [1]) and external disturbances (see e.g., [5]). For example, take a DACS of the following form

$$E\dot{x}(t) = Hx(t) + Lu(t) + Tx(t - \tau) + Sd(t), \tag{12}$$

where τ represents a time delay and $d(t)$ is a vector of external disturbances. It is always possible to find an invertible matrix Q such that E_1 of $QE = \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right]$ is of full row rank. Then we denote

$$QH = \left[\begin{smallmatrix} H_1 \\ H_2 \end{smallmatrix}\right], \quad QL = \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right], \quad QT = \left[\begin{smallmatrix} T_1 \\ T_2 \end{smallmatrix}\right], \quad QS = \left[\begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix}\right].$$

Choose B^v such that $\mathrm{Im} B^v = \ker E_1$ and a right inverse $E_1^\dagger$ of E_1, and define

$$A := E_1^\dagger H_1, \quad B^u := E_1^\dagger L_1, \quad M := E_1^\dagger T_1, \quad N := E_1^\dagger S_1, \quad C := H_2, \quad D^u =: L_2, \quad J := T_2, \quad K := S_2.$$

With the above defined matrices, we can attach the following ODECS with time delays and external disturbance to (12):

$$\begin{cases} \dot{x}(t) = Ax(t) + B^u u(t) + B^v v(t) + Mx(t - \tau) + Nd(t) \\ y(t) = Cx(t) + D^u u(t) + Jx(t - \tau) + Kd(t). \end{cases} \tag{13}$$

It is clear that if DACS (12) is *not* time-delayed, i.e. $T = 0$ (hence $M = 0$) and thus $x(t-\tau)$ is absent, then the results of Proposition 2.4 still hold for (12) and (13), meaning that solutions $(x(\cdot), d(\cdot), u(\cdot))$ of (12) are in a one-to-one correspondence with solutions $(x(\cdot), u(\cdot), d(\cdot), v(\cdot))$ of (13) with outputs $y = 0$. While if a delayed term is present, the analysis of solutions is more complicated because for delayed DAE systems, the existence of solutions depends on the initial condition $x(t) = \phi(t)$, for $t \in [-\tau, 0]$ (see some studies on solutions of regular delay DAEs in [13, 18]). A particular case is that if the matrices E and T of (12) satisfy $\ker E \subseteq \ker T$, implying that there are no delayed free variables in the generalized state x, then it is clear that solutions of (12) and those of (13) still have a one-to-one correspondence. We will not give further discussions on solutions of delayed DAE/DACSs since the purpose of this paper is to study canonical forms but the application of the explicitation method to such systems seem to be an interesting subject for further research.

Since the explicitation of Δ^u is a class of ODECSs of the form (2), we give the following definition of equivalence for ODECSs of the form (2). This definition is a natural extension of the Morse equivalence ([28], extended by Molinari [27], see also [15]) of classical ODECSs of the form (3).

Definition 2.7 (extended Morse equivalence and extended Morse transformations)**.** Two ODECSs

$$\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u), \quad \tilde{\Lambda}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^{\tilde{u}})$$

are called extended Morse equivalent, shortly EM-equivalent, denoted by $\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$, if there exist matrices $T_x \in Gl(n, \mathbb{R})$, $T_u \in Gl(m, \mathbb{R})$, $T_v \in Gl(s, \mathbb{R})$, $T_y \in Gl(p, \mathbb{R})$, $F_u \in \mathbb{R}^{m\times n}$, $F_v \in \mathbb{R}^{s\times n}$, $R \in \mathbb{R}^{s\times m}$, $K \in \mathbb{R}^{n\times p}$ such that the system matrices of Λ^{uv} and $\tilde{\Lambda}^{\tilde{u}\tilde{v}}$ satisfy:

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{bmatrix} = \begin{bmatrix} T_x & T_xK \\ 0 & T_y \end{bmatrix} \begin{bmatrix} A & B^u & B^v \\ C & D^u & 0 \end{bmatrix} \begin{bmatrix} T_x^{-1} & 0 & 0 \\ F_uT_x^{-1} & T_u^{-1} & 0 \\ (F_v+RF_u)T_x^{-1} & RT_u^{-1} & T_v^{-1} \end{bmatrix}. \tag{14}$$

An 8-tuple $(T_x, T_u, T_v, T_y, F_u, F_v, R, K)$, acting on the system according to (14), will be called an extended Morse transformation and denoted by EM_{tran}.

The matrices T_x, T_u, T_v and T_y are coordinates transformations in the, respectively, state space $\mathscr{X} = \mathbb{R}^n$, input subspace $\mathscr{U}_u = \mathbb{R}^m$, input subspace $\mathscr{U}_v = \mathbb{R}^s$ and, output space $\mathscr{Y} = \mathbb{R}^p$, where F_u defines a state feedback of u, F_v and R define a feedback of v, K defines an output injection.

Remark 2.8. (i) An extended Morse transformation, whose action is given by (14), includes two kinds of feedback transformations:

$$v = F_vx + Ru + T_v^{-1}\tilde{v} \quad \text{and} \quad u = F_ux + T_u^{-1}\tilde{u}. \tag{15}$$

The vector of driving variables v is "stronger" than the original control vector u since when transforming v we can use both u and x as feedback, but when transforming u we are not allowed to use v. This is expressed by the triangular form of the matrix multiplying on the right in (14).

(ii) Recall the definition of the Morse equivalence and the Morse transformation [28] (and their generalization by Molinari [27] for $D^u \neq 0$, see also [15]): for two ODECSs $\Lambda^u = (A, B^u, C, D^u)$ and

$\tilde{\Lambda}^{\tilde{u}} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{C}, \tilde{D}^{\tilde{u}})$ of the form (3), if

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} \end{bmatrix} = \begin{bmatrix} T_x & T_x K \\ 0 & T_y \end{bmatrix} \begin{bmatrix} A & B^u \\ C & D^u \end{bmatrix} \begin{bmatrix} T_x^{-1} & 0 \\ F_u T_x^{-1} & T_u^{-1} \end{bmatrix},$$

then Λ^u and $\tilde{\Lambda}^{\tilde{u}}$ are called Morse equivalent (shortly M-equivalent) and the Morse transformation (T_x, T_u, T_y, F_u, K) is denoted by M_{tran}. Clearly, M-equivalence is an equivalence relation for ODECSs of the form (3), defined by a 4-tuples (A, B^u, C, D^u) and EM-equivalence is for ODECSs of the form (2), defined by a 5-tuples (A, B^u, B^v, C, D^u). Observe that if the vector of driving variables v is of dimension zero (B^v is absent), then the EM-equivalence reduces to the M-equivalence.

(iii) Recall that we can express an ODECS of the form $\Lambda^{uv} = (A, B^u, B^v, C, D^u)$ as a standard ODECS $\Lambda^w = (A, B^w, C, D^w)$ of the form (3) with one type of controls w, where $w = [u^T, v^T]^T$. Now let

$$F_w = \begin{bmatrix} F_u \\ F_v + RF_u \end{bmatrix}, \quad T_w^{-1} = \begin{bmatrix} T_u^{-1} & 0 \\ RT_u^{-1} & T_v^{-1} \end{bmatrix},$$

then we conclude the following equation from (14) (notice that T_w has a block-triangular structure):

$$\begin{bmatrix} \tilde{A} & \tilde{B}^w \\ \tilde{C} & \tilde{D}^w \end{bmatrix} = \begin{bmatrix} T_x & T_x K \\ 0 & T_y \end{bmatrix} \begin{bmatrix} A & B^w \\ C & D^w \end{bmatrix} \begin{bmatrix} T_x^{-1} & 0 \\ F_w T_x^{-1} & T_w^{-1} \end{bmatrix}, \tag{16}$$

which is exactly the expression of the M-equivalence for systems Λ^w (compare Remark 2.8(ii) above). It implies that the EM-equivalence can be expressed as a form of the M-equivalence with a triangular matrix T_w (input coordinates transformation matrix). This triangular form is a consequence of two kinds of feedback transformation shown in equation (15).

Now we give the main result of this subsection:

Theorem 2.9. *Consider two DACSs $\Delta^u_{l,n,m} = (E, H, L)$ and $\tilde{\Delta}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{H}, \tilde{L})$ as well as two ODECSs $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$ and $\tilde{\Lambda}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^{\tilde{u}})$ satisfying $\Lambda^{uv} \in$ $\mathbf{Expl}(\Delta^u)$ and $\tilde{\Lambda}^{\tilde{u}\tilde{v}} \in \mathbf{Expl}(\tilde{\Delta}^{\tilde{u}})$. Then, $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$ if and only if $\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$.*

The proof will be given in Section 6.1. In the Appendix, we recall the definitions of geometric subspaces for DACSs and ODECSs. More specifically, for a DACS Δ^u, we recall the augmented Wong sequences $\mathscr{V}_i$ and $\mathscr{W}_i$, together with $\hat{\mathscr{W}}_i$ (see [7],[23]); for an ODECS Λ^w, we recall the subspaces sequences $\mathcal{V}_i$ and $\mathcal{W}_i$ (see [36],[35],[2]), whose limits are controlled and conditioned invariant subspaces, respectively, and we introduce a subspaces sequence $\hat{\mathcal{W}}_i$.

Proposition 2.10. *Given $\Delta^u_{l,n,m} = (E, H, L)$ and $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$ (or equivalently, $\Lambda^w_{n,m+s,p} = (A, B^w, C, D^w)$), consider the subspaces $\mathscr{V}_i$, $\mathscr{W}_i$, $\hat{\mathscr{W}}_i$ of Δ^u, given by Definition 7.2 and the subspaces $\mathcal{V}_i$, $\mathcal{W}_i$, $\hat{\mathcal{W}}_i$ of Λ^w, given by Lemma 7.4 in the Appendix. Assume that $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$. Then we have for $i \in \mathbb{N}$,*

$$\mathscr{V}_i(\Delta^u) = \mathcal{V}_i(\Lambda^w), \qquad \mathscr{W}_i(\Delta^u) = \mathcal{W}_i(\Lambda^w),$$

and for $i \in \mathbb{N}^+$,

$$\hat{\mathscr{W}}_i(\Delta^u) = \hat{\mathcal{W}}_i(\Lambda^w).$$

The proof will be given in Section 6.2. Note that Theorem 2.9 and Proposition 2.10 are fundamental results for the remaining part of the paper. The above proposition shows the importance of the notion of (Q,v)-explicitation. Namely, the augmented Wong sequences of any DACS Δ^u and the invariant subspaces of its (Q,v)-explicitation Λ^w coincide (in particular, they are subspaces of the same generalized state-space $\mathscr{X}$). If we use the (Q,P)-explicitation, we need to establish relations between subspaces of different spaces $\mathscr{X}$ and $\mathscr{X}_1$ (see Remark 2.5). Our purpose is to find the **FBCF** of DACSs via explicitation. We have proven in Theorem 2.9 that the ex-fb-equivalence for DACSs corresponds to the EM-equivalence for their explicitations. Thus rather than transforming a DACS Δ^u directly into its **FBCF** under ex-fb-equivalence, we will look for the canonical form for $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$ under EM-equivalence.

3. The Morse triangular form and its extension

In the beginning of this section, we show that the normal form given in [27] (called Morse normal form **MNF** in the present paper) for the 4-tuple ODECS Λ^u, given by equation (3), can be constructed through a Morse triangular form **MTF** that we propose. Although the constructed normal form is the same as the one in [27], we will provide explicit transformations with the help of the invariant subspaces given in Lemma 7.4 of the Appendix, which makes the normalizing procedure simple and transparent.

Proposition 3.1 (Morse triangular form **MTF**). *For an ODECS $\Lambda^u_{n,m,p} = (A, B^u, C, D^u)$, consider the subspaces $\mathcal{V}^*$, $\mathcal{U}^*_u$, $\mathcal{W}^*$, $\mathcal{Y}^*$ given by Definition 7.3 of the Appendix. Choose full rank matrices $T^1_s \in \mathbb{R}^{n\times n_1}$, $T^2_s \in \mathbb{R}^{n\times n_2}$, $T^3_s \in \mathbb{R}^{n\times n_3}$, $T^4_s \in \mathbb{R}^{n\times n_4}$, $T^1_i \in \mathbb{R}^{m\times m_1}$, $T^3_i \in \mathbb{R}^{m\times m_3}$, $T^3_o \in \mathbb{R}^{p\times p_3}$, $T^4_o \in \mathbb{R}^{p\times p_4}$ such that*

$$\begin{aligned}
&\operatorname{Im} T^1_s = \mathcal{V}^* \cap \mathcal{W}^*, && \mathcal{V}^* \cap \mathcal{W}^* \oplus \operatorname{Im} T^2_s = \mathcal{V}^*,\\
&\mathcal{V}^* \cap \mathcal{W}^* \oplus \operatorname{Im} T^3_s = \mathcal{W}^*, && (\mathcal{V}^* + \mathcal{W}^*) \oplus \operatorname{Im} T^4_s = \mathscr{X} = \mathbb{R}^n,\\
&\operatorname{Im} T^1_i = \mathcal{U}^*_u, && \operatorname{Im} T^3_i \oplus \operatorname{Im} T^1_i = \mathscr{U}_u = \mathbb{R}^m,\\
&\operatorname{Im} T^3_o = \mathcal{Y}^*, && \operatorname{Im} T^4_o \oplus \operatorname{Im} T^3_o = \mathscr{Y} = \mathbb{R}^p,
\end{aligned}$$

where $n = n_1 + n_2 + n_3 + n_4$, $m = m_1 + m_3$, $p = p_3 + p_4$. Then

$$T_s = [\, T^1_s \; T^2_s \; T^3_s \; T^4_s \,]^{-1} \in Gl(n,\mathbb{R}), \quad T_i = [\, T^1_i \; T^3_i \,]^{-1} \in Gl(m,\mathbb{R}), \quad T_o = [\, T^3_o \; T^4_o \,]^{-1} \in Gl(p,\mathbb{R}), \tag{17}$$

*and there exist matrices $F_{MT} \in \mathbb{R}^{m\times n}$ and $K_{MT} \in \mathbb{R}^{n\times p}$ such that the Morse transformation $M_{tran} = (T_s, T_i, T_o, F_{MT}, K_{MT})$ brings Λ^u into $\tilde{\Lambda}^{\tilde{u}} = M_{tran}(\Lambda^u)$, represented in the Morse triangular form **MTF**, that is given by $\tilde{\Lambda}^{\tilde{u}} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{C}, \tilde{D}^{\tilde{u}})$, where*

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} \end{bmatrix} = \left[\begin{array}{cccc|cc}
\tilde{A}_1 & \tilde{A}^2_1 & \tilde{A}^3_1 & \tilde{A}^4_1 & \tilde{B}_1 & \tilde{B}^2_1 \\
0 & \tilde{A}_2 & 0 & \tilde{A}^4_2 & 0 & 0 \\
0 & 0 & \tilde{A}_3 & \tilde{A}^4_3 & 0 & \tilde{B}_3 \\
0 & 0 & 0 & \tilde{A}_4 & 0 & 0 \\ \hline
0 & 0 & \tilde{C}_3 & \tilde{C}^4_3 & 0 & \tilde{D}_3 \\
0 & 0 & 0 & \tilde{C}_4 & 0 & 0
\end{array}\right]. \tag{18}$$

*In the above **MTF**, the pair $(\tilde{A}_1, \tilde{B}_1)$ is controllable, the pair $(\tilde{C}_4, \tilde{A}_4)$ is observable and the 4-tuple $(\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$ is prime* [1].

The proof is given in Section 6.3. In the next proposition, we describe a way to transform the above **MTF** into the Morse normal form **MNF**, which is a further simplification of the **MTF**. We will use the same notations as in Proposition 3.1.

Proposition 3.2 (Morse normal form **MNF**)**.** *There exists a feedback transformation matrix $F_{MN} \in \mathbb{R}^{m\times n}$, an output injection matrix $K_{MN} \in \mathbb{R}^{n\times p}$ and a state space coordinate transformation matrix $T_{MN} \in Gl(n, \mathbb{R})$, which can be chosen by **MNF** Algorithm 3.3 below, such that the Morse transformation $M_{tran} = (T_{MN}, I_u, I_y, F_{MN}, K_{MN})$ brings $\tilde{\Lambda}^{\tilde{u}}$ of Proposition 3.1, given by (18), into $\bar{\Lambda}^{\bar{u}} = M_{tran}(\tilde{\Lambda}^{\tilde{u}})$, represented in the Morse normal form **MNF**, that is given by $\bar{\Lambda}^{\bar{u}} = (\bar{A}, \bar{B}^{\bar{u}}, \bar{C}, \bar{D}^{\bar{u}})$, where*

$$\begin{bmatrix} \bar{A} & \bar{B}^{\bar{u}} \\ \bar{C} & \bar{D}^{\bar{u}} \end{bmatrix} = \left[\begin{array}{cccc|cc} \bar{A}_1 & 0 & 0 & 0 & \bar{B}_1 & 0 \\ 0 & \bar{A}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{A}_3 & 0 & 0 & \bar{B}_3 \\ 0 & 0 & 0 & \bar{A}_4 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_3 & 0 & 0 & \bar{D}_3 \\ 0 & 0 & 0 & \bar{C}_4 & 0 & 0 \end{array}\right]. \tag{19}$$

*In the above **MNF**, the pair $(\bar{A}_1, \bar{B}_1)$ is controllable, the pair $(\bar{C}_4, \bar{A}_4)$ is observable, and the 4-tuple $(\bar{A}_3, \bar{B}_3, \bar{C}_3, \bar{D}_3)$ is prime.*

The proof of Proposition 3.2 will be given in Section 6.4 and in that proof, we will use the construction of transformation matrices F_{MN}, K_{MN} and T_{MN}, which is formulated in the following algorithm.

MNF Algorithm 3.3. *Step 1: Given the matrix (18), choose F_{MN} and K_{MN}:*

$$F_{MN} = \begin{bmatrix} F^1_{MN} & 0 & 0 & 0 \\ 0 & 0 & F^2_{MN} & F^3_{MN} \end{bmatrix}, \quad K_{MN} = \begin{bmatrix} K^1_{MN} & 0 \\ 0 & 0 \\ K^2_{MN} & 0 \\ 0 & K^3_{MN} \end{bmatrix},$$

such that the spectra of $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ and $\bar{A}_4$ defined by the equation below are mutually disjoint (notice that F_{MN} and K_{MN} preserve the zero blocks of $\tilde{\Lambda}^{\tilde{u}} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{C}, \tilde{D}^{\tilde{u}})$):

$$\begin{bmatrix} I_n & K_{MN} \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ F_{MN} & I_m \end{bmatrix} = \left[\begin{array}{cccc|cc} \bar{A}_1 & \bar{A}^2_1 & \bar{A}^3_1 & \bar{A}^4_1 & \bar{B}_1 & \bar{B}^2_1 \\ 0 & \bar{A}_2 & 0 & \bar{A}^4_2 & 0 & 0 \\ 0 & 0 & \bar{A}_3 & \bar{A}^4_3 & 0 & \bar{B}_3 \\ 0 & 0 & 0 & \bar{A}_4 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_3 & \bar{C}^4_3 & 0 & \bar{D}_3 \\ 0 & 0 & 0 & \bar{C}_4 & 0 & 0 \end{array}\right].$$

Step 2: Find matrices T^1_{MN}, T^2_{MN}, T^3_{MN}, T^4_{MN}, T^5_{MN} via the following (constrained) Sylvester equations:

$$\begin{gathered} \bar{A}_1 T^1_{MN} - T^1_{MN}\bar{A}_2 = -\bar{A}^2_1, \qquad \bar{A}_2 T^4_{MN} - T^4_{MN}\bar{A}_4 = -\bar{A}^4_2, \\ \bar{A}_1 T^3_{MN} - T^3_{MN}\bar{A}_4 = -\bar{A}^4_1 - \bar{A}^2_1 T^4_{MN} - \bar{A}^3_1 T^5_{MN}; \end{gathered} \tag{20}$$

$$\begin{gathered} \bar{A}_1 T^2_{MN} - T^2_{MN}\bar{A}_3 = -\bar{A}^3_1, \quad T^2_{MN}\bar{B}_3 = -\bar{B}^2_1, \\ \bar{A}_3 T^5_{MN} - T^5_{MN}\bar{A}_4 = -\bar{A}^4_3, \quad \bar{C}_3 T^5_{MN} = -\bar{C}_4. \end{gathered} \tag{21}$$

[1] A control system is called prime if it is M-equivalent to m_3 independent chains of integrators, see [28] and [27].

Step 3: Set

$$T_{MN} = \begin{bmatrix} I & T^1_{MN} & T^2_{MN} & T^3_{MN} \\ 0 & I & 0 & T^4_{MN} \\ 0 & 0 & I & T^5_{MN} \\ 0 & 0 & 0 & I \end{bmatrix}^{-1}.$$

Remark 3.4. It is not surprising that Propositions 3.1 and 3.2 describe results similar to those of Theorem 2.3 and Theorem 2.6 of [9], as we have shown in [15] that there are direct connections between the geometric subspaces (the Wong sequences) of a DAE $\Delta : E\dot{x} = Hx$ and invariant subspaces of a control system $\Lambda = (A, B, C, D) \in \mathrm{Expl}(\Delta)$. There are, however, differences between Propositions 3.1 and 3.2 and results of [9]. In particular, in Theorem 2.6 of [9], one has to solve generalized Sylvester equations, while in Propositions 3.2 we use (constrained) Sylvester equations. In addition, our transformations differ from those proposed in the original paper [29] and [27] for the **MNF** and seem to be more transparent and explicit.

Recall that the explicitation of a DACS Δ^u is a class of ODECSs with two kinds of inputs of the form (2). In the following theorems, we will extend the results of Proposition 3.1 and 3.2 to ODECSs with two kinds of inputs.

Theorem 3.5 (extended Morse triangular form **EMTF**)**.** *For a DACS*

$$\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u),$$

there exists an extended Morse transformation EM_{tran} bringing Λ^{uv} into $EM_{tran}(\Lambda^{uv}) = \tilde{\Lambda}^{\tilde{u}\tilde{v}}$ represented in the extended Morse triangular form ***EMTF****, that is given by $\tilde{\Lambda}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^{\tilde{u}})$, where*

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{bmatrix} = \left[\begin{array}{cccc|cc|cc} \tilde{A}_1 & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} & \tilde{B}^{\tilde{u}}_1 & \tilde{B}^{\tilde{u}}_{12} & \tilde{B}^{\tilde{v}}_1 & \tilde{B}^{\tilde{v}}_{12} \\ 0 & \tilde{A}_2 & 0 & \tilde{A}_{24} & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{A}_3 & \tilde{A}_{34} & 0 & \tilde{B}^{\tilde{u}}_3 & 0 & \tilde{B}^{\tilde{v}}_3 \\ 0 & 0 & 0 & \tilde{A}_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{C}_3 & \tilde{C}_{34} & 0 & \tilde{D}^{\tilde{u}}_3 & 0 & 0 \\ 0 & 0 & 0 & \tilde{C}_4 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{22}$$

In the above ***EMTF****, the pair $(\tilde{A}_1, \tilde{B}^{\tilde{w}}_1)$ is controllable, where $\tilde{B}^{\tilde{w}}_1 = [\tilde{B}^{\tilde{u}}_1, \tilde{B}^{\tilde{v}}_1]$; the pair $(\tilde{C}_4, \tilde{A}_4)$ is observable ; the 4-tuple $(\tilde{A}_3, \tilde{B}^{\tilde{w}}_3, \tilde{C}_3, \tilde{D}^{\tilde{w}}_3)$ is prime, where $\tilde{B}^{\tilde{w}}_3 = [\tilde{B}^{\tilde{u}}_3, \tilde{B}^{\tilde{v}}_3]$, $\tilde{D}^{\tilde{w}}_3 = [\tilde{D}^{\tilde{u}}_3, 0]$.*

Theorem 3.6 (extended Morse normal form **EMNF**)**.** *For $\tilde{\Lambda}^{\tilde{u}\tilde{v}}_{n,m,s,p} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{B}^{\tilde{v}}, \tilde{C}, \tilde{D}^{\tilde{u}})$ in the* ***EMTF****, as given by Theorem 3.5, there exists an extended Morse transformation EM_{tran} bringing $\tilde{\Lambda}^{\tilde{u}\tilde{v}}$ into $\bar{\Lambda}^{\bar{u}\bar{v}} = EM_{tran}(\tilde{\Lambda}^{\tilde{u}\tilde{v}})$ represented in the extended Morse normal form* ***EMNF****, that is given by $\bar{\Lambda}^{\bar{u}\bar{v}}_{n,m,s,p} = (\bar{A}, \bar{B}^{\bar{u}}, \bar{B}^{\bar{v}}, \bar{C}, \bar{D}^{\bar{u}})$, where*

$$\begin{bmatrix} \bar{A} & \bar{B}^{\bar{u}} & \bar{B}^{\bar{v}} \\ \bar{C} & \bar{D}^{\bar{u}} & 0 \end{bmatrix} = \left[\begin{array}{cccc|cc|cc} \bar{A}_1 & 0 & 0 & 0 & \bar{B}^{\bar{u}}_1 & 0 & \bar{B}^{\bar{v}}_1 & 0 \\ 0 & \bar{A}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{A}_3 & 0 & 0 & \bar{B}^{\bar{u}}_3 & 0 & \bar{B}^{\bar{v}}_3 \\ 0 & 0 & 0 & \bar{A}_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_3 & 0 & 0 & \bar{D}^{\bar{u}}_3 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_4 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{23}$$

In the above ***EMNF****, the pair $(\bar{A}_1, \bar{B}^{\bar{w}}_1)$ is controllable, where $\bar{B}^{\bar{w}}_1 = [\bar{B}^{\bar{u}}_1, \bar{B}^{\bar{v}}_1]$; the pair $(\bar{C}_4, \bar{A}_4)$ is observable; the 4-tuple $(\bar{A}_3, \bar{B}^{\bar{w}}_3, \bar{C}_3, \bar{D}^{\bar{w}}_3)$ is prime, where $\bar{B}^{\bar{w}}_3 = [\bar{B}^{\bar{u}}_3, \bar{B}^{\bar{v}}_3]$, $\tilde{D}^{\bar{w}}_3 = [\tilde{D}^{\bar{u}}_3, 0]$.*

The proofs of Theorem 3.5 and Theorem 3.6 are given in Section 6.5.

4. From the extended Morse normal form EMNF to the feedback canonical form FBCF

We show that, with a suitable choice of an extended Morse transformation for each subsystem in the **EMNF** of Theorem 3.6, we can bring the **EMNF** into the extended Morse canonical form **EMCF**. Below the upper indices refer to: c to controllable, nn to non-controllable and non-observable, p to prime, o to observable. If an ODECS $\Lambda^{uv}_{EM} = (A_{EM}, B^u_{EM}, B^v_{EM}, C_{EM}, D^u_{EM})$ is in the **EMCF**, then the matrices $A_{EM}, B^u_{EM}, B^v_{EM}, C_{EM}, D^u_{EM}$ are given by

$$\begin{bmatrix} A_{EM} & B^u_{EM} & B^v_{EM} \\ C_{EM} & D^u_{EM} & 0 \end{bmatrix} = \left[\begin{array}{cccccc|cc|cc} A^{cu} & 0 & 0 & 0 & 0 & 0 & B^{cu} & 0 & 0 & 0 \\ 0 & A^{cv} & 0 & 0 & 0 & 0 & 0 & 0 & B^{cv} & 0 \\ 0 & 0 & A^{nn} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A^{pu} & 0 & 0 & 0 & B^{pu} & 0 & 0 \\ 0 & 0 & 0 & 0 & A^{pv} & 0 & 0 & 0 & 0 & B^{pv} \\ 0 & 0 & 0 & 0 & 0 & A^{o} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C^{pu} & 0 & 0 & 0 & D^{pu} & 0 & 0 \\ 0 & 0 & 0 & 0 & C^{pv} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C^{o} & 0 & 0 & 0 & 0 \end{array}\right], \tag{24}$$

with the matrices and their invariants of the following form:

(i) $A^{cu} = \mathrm{diag}\{A^{cu}_{\epsilon_1}, ..., A^{cu}_{\epsilon_a}\}$, $B^{cu} = \mathrm{diag}\{B^{cu}_{\epsilon_1}, ..., B^{cu}_{\epsilon_a}\}$, $A^{cv} = \mathrm{diag}\{A^{cv}_{\bar\epsilon_b}, ..., A^{cv}_{\bar\epsilon_b}\}$, $B^{cv} = \mathrm{diag}\{B^{cv}_{\bar\epsilon_1}, ..., B^{cv}_{\bar\epsilon_b}\}$, where

$$A^{cu}_{\epsilon} = \begin{bmatrix} 0 & I_{\epsilon-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\epsilon\times\epsilon},\ B^{cu}_{\epsilon} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\epsilon},\ A^{cv}_{\bar\epsilon} = \begin{bmatrix} 0 & I_{\bar\epsilon-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\bar\epsilon\times\bar\epsilon},\ B^{cv}_{\bar\epsilon} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\bar\epsilon}.$$

The integers $\epsilon_1, ..., \epsilon_a \in \mathbb{N}^+$ are the controllability indices of (A^{cu}, B^{cu}), the integers $\bar\epsilon_1, ..., \bar\epsilon_b \in \mathbb{N}^+$ are the controllability indices of (A^{cv}, B^{cv}).

(ii) $A^{nn} \in \mathbb{R}^{n_2\times n_2}$ is unique up to similarity and can always be put in the real Jordan form.

(iii) Both the 4-tuple $(A^{pu}, B^{pu}, C^{pu}, D^{pu})$ and the triple (A^{pv}, B^{pv}, C^{pv}) are prime, and thus controllable and observable. That is,

$$\begin{bmatrix} A^{pu} & B^{pu} \\ C^{pu} & D^{pu} \end{bmatrix} = \left[\begin{array}{c|cc} \hat A^{pu} & \hat B^{pu} & 0 \\ \hline \hat C^{pu} & 0 & 0 \\ 0 & 0 & I_\delta \end{array}\right],$$

where $\begin{bmatrix} \hat A^{pu} & \hat B^{pu} \\ \hat C^{pu} & 0 \end{bmatrix}$ is square and invertible and $\delta = \mathrm{rank}\, \hat D^{pu} \in \mathbb{N}$, and the matrices

$$\hat A^{pu} = \mathrm{diag}\{\hat A^{pu}_{\sigma_1}, ..., \hat A^{pu}_{\sigma_c}\}, \quad \hat B^{pu} = \mathrm{diag}\{\hat B^{pu}_{\sigma_1}, ..., \hat B^{pu}_{\sigma_c}\}, \quad \hat C^{pu} = \mathrm{diag}\{\hat C^{pu}_{\sigma_1}, ..., \hat C^{pu}_{\sigma_c}\},$$
$$A^{pv} = \mathrm{diag}\{A^{pv}_{\bar\sigma_1}, ..., A^{pv}_{\bar\sigma_d}\}, \quad B^{pv} = \mathrm{diag}\{B^{pv}_{\bar\sigma_1}, ..., B^{pv}_{\bar\sigma_d}\}, \quad C^{pv} = \mathrm{diag}\{C^{pv}_{\bar\sigma_1}, ..., C^{pv}_{\bar\sigma_d}\},$$

where

$$\hat A^{pu}_{\sigma} = \begin{bmatrix} 0 & I_{\sigma-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\sigma\times\sigma}, \quad \hat B^{pu}_{\sigma} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\sigma\times 1}, \quad \hat C^{pu}_{\sigma} = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times\sigma},$$
$$A^{pv}_{\bar\sigma} = \begin{bmatrix} 0 & I_{\bar\sigma-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\bar\sigma\times\bar\sigma}, \quad B^{pv}_{\bar\sigma} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\bar\sigma\times 1}, \quad C^{pv}_{\bar\sigma} = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times\bar\sigma}.$$

The integers $\sigma_1, ..., \sigma_c \in \mathbb{N}^+$ are the controllability indices of the pair $(\hat A^{pu}, \hat B^{pu})$ and they are equal to the observability indices of the pair $(\hat C^{pu}, \hat A^{pu})$. The integers $\bar\sigma_1, ..., \bar\sigma_d \in \mathbb{N}^+$ are the controllability indices of the pair (A^{pv}, B^{pv}) and they are equal to the observability indices of the pair (C^{pv}, A^{pv}).

(iv) $A^{o} = \mathrm{diag}\{A^{o}_{\eta_1}, ..., A^{o}_{\eta_e}\}$, $C^{o} = \mathrm{diag}\{C^{o}_{\eta_1}, ..., C^{o}_{\eta_e}\}$, where

$$A^{o}_{\eta} = \begin{bmatrix} 0 & I_{\eta-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\eta\times\eta}, \qquad C^{o}_{\eta} = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times\eta}.$$

The integers $\eta_1, ..., \eta_e \in \mathbb{N}^+$ are the observability indices of the pair (C^{o}, A^{o}).

Theorem 4.1 (extended Morse canonical form **EMCF**)**.** *For any*

$$\Lambda^{uv} = \Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u),$$

there exists an extended Morse transformation EM_{tran} bringing Λ^{uv} into

$$\Lambda^{uv}_{EM} = (A_{EM}, B^u_{EM}, B^v_{EM}, C_{EM}, D^u_{EM}) = EM_{tran}(\Lambda^{uv}),$$

*represented by the extended Morse canonical form **EMCF**.*

The proof will be given in Section 6.6. Throughout if we only consider the differential equation of (2) (meaning (2) without the output y), we denote it as $\Lambda^{uv}_{n,m,s} = (A, B^u, B^v)$. Now we introduce the *driving variables v-reduction* and *implicitation* (compare [15]) to reduce the driving variables v and implicit the **EMCF** to a DACS.

Definition 4.2 (v-reduction and implicitation)**.** For a control system Λ^{uz^2} and its prolongation $\mathbf{\Lambda}^{uv}$, given by (10) and (11), respectively, the inverse operation of prolongation will be called the v-reduction, that is, the v-reduction of $\mathbf{\Lambda}^{uv}$ is Λ^{uz^2}. For an ODECS Λ^{uz^2}, the implicitation of Λ^{uz^2} is a DACS $\mathrm{Impl}(\Lambda^{uz^2})$ constructed by setting the output $y = 0$, that is,

$$\mathrm{Impl}(\Lambda^{uz^2}) : \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot z^1 \\ \dot z^2 \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} u.$$

Remark 4.3. If $\Delta^u = \mathrm{Impl}(\Lambda^{uz^2})$, where Λ^{uz^2} is the v-reduction of $\mathbf{\Lambda}^{uv}$, then $\mathbf{\Lambda}^{uv} \in \mathbf{Expl}(\Delta^u)$.

Then with the help of the above *v-reduction and implicitation* procedure, we can regard the feedback canonical form **FBCF** for DACSs of the form $\Delta^u_{l,n,m} = (E, H, L)$ given in [24] as a corollary of Theorem 4.1. In the following, in order to save space and simplify notations, we denote

$$K_i = \left[\begin{smallmatrix} 0 & I_{i-1} \end{smallmatrix}\right] \in \mathbb{R}^{(i-1)\times i},\ L_i = \left[\begin{smallmatrix} I_{i-1} & 0 \end{smallmatrix}\right] \in \mathbb{R}^{(i-1)\times i},\ N_i = \left[\begin{smallmatrix} 0 & 0 \\ I_{i-1} & 0 \end{smallmatrix}\right] \in \mathbb{R}^{i\times i},\ e_i = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \in \mathbb{R}^{i},$$

where $\beta = (\beta_1, \ldots, \beta_k)$, $|\beta| = \beta_1 + \cdots + \beta_k$, and

$$\begin{aligned} &N_\beta = \mathrm{diag}\left\{N_{\beta_1}, \ldots, N_{\beta_k}\right\} \in \mathbb{R}^{|\beta|\times|\beta|} && K_\beta = \mathrm{diag}\left\{K_{\beta_1}, \ldots, K_{\beta_k}\right\} \in \mathbb{R}^{(|\beta|-k)\times|\beta|}, \\ &L_\beta = \mathrm{diag}\left\{L_{\beta_1}, \ldots, L_{\beta_k}\right\} \in \mathbb{R}^{(|\beta|-k)\times|\beta|}, && \mathcal{E}_\beta = \mathrm{diag}\left\{e_{\beta_1}, \ldots, e_{\beta_k}\right\} \in \mathbb{R}^{|\beta|\times k}, \end{aligned}$$

Theorem 4.4 (feedback canonical form of DACSs [24])**.** *Any DACS $\Delta^u_{l,n,m} = (E, H, L)$ is ex-fb-equivalent to the following feedback canonical form **FBCF**:*

$$\left(\begin{bmatrix} I_{|\epsilon'|} & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{\bar\epsilon'} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & K^T_{\sigma'} & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{\bar\sigma'} & 0 \\ 0 & 0 & 0 & 0 & 0 & L^T_{\eta'} \end{bmatrix}, \begin{bmatrix} N^T_{\epsilon'} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{\bar\epsilon'} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & L^T_{\sigma'} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\bar\sigma'|} & 0 \\ 0 & 0 & 0 & 0 & 0 & K^T_{\eta'} \end{bmatrix}, \begin{bmatrix} \mathcal{E}_{\epsilon'} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathcal{E}_{\sigma'} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right),$$

where $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_{a'}) \in (\mathbb{N}^+)^{a'}$, $\bar\epsilon' = (\bar\epsilon'_1, \ldots, \bar\epsilon'_{b'}) \in (\mathbb{N}^+)^{b'}$, $\sigma' = (\sigma'_1, \ldots, \sigma'_{c'}) \in (\mathbb{N}^+)^{c'}$, $\bar\sigma' = (\bar\sigma'_1, \ldots, \bar\sigma'_{d'}) \in (\mathbb{N}^+)^{d'}$, $\eta' = (\eta'_1, \ldots, \eta'_{e'}) \in (\mathbb{N}^+)^{e'}$ are multi-indices and the matrix A_ρ is given up to similarity (and can always be put into real Jordan form).

Remark 4.5. (i) The above theorem of the **FBCF** of DACSs is a corollary of Theorem 4.1. Indeed, for any DACS $\Delta^u = (E, H, L)$, we can construct an ODECS $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$. Then, by Theorem

4.1, we have $\Lambda^{uv} \overset{EM}{\sim}$ **EMCF**. It is not hard to see that the **FBCF** is the implicitation of the v-reduction (see Definition 4.2) of the **EMCF**. A crucial observation is that **EMCF** $\in$ **Expl**(**FBCF**) (see Remark 4.3). Thus, by Theorem 2.9, we conclude $\Delta^u \overset{ex-fb}{\sim}$ **FBCF** (since $\Lambda^{uv} \overset{EM}{\sim}$ **EMCF**).

(ii) There exists a perfect correspondence between the six subsystems of the **EMCF** and their counterparts of the **FBCF**. Morse specifically,

$$(A^{cu}, B^{cu}) \leftrightarrow (I_{|\epsilon'|}, N^T_{\epsilon'}, \mathcal{E}_{\epsilon'}), \qquad (A^{cv}, B^{cv}) \leftrightarrow (L_{\bar{\epsilon}'}, K_{\bar{\epsilon}'}, 0), \qquad A^{nn} \leftrightarrow (I_{n_\rho}, A_\rho),$$
$$(A^{pu}, B^{pu}, C^{pu}, D^{pu}) \leftrightarrow (K^T_{\sigma'}, L^T_{\sigma'}, \mathcal{E}_{\sigma'}), \quad (A^{pv}, B^{pv}, C^{pv}) \leftrightarrow (N_{\bar{\sigma}'}, I_{|\bar{\sigma}'|}, 0), \quad (C^o, A^o) \leftrightarrow (L^T_{\eta'}, K^T_{\eta'}, 0).$$

(iii) Since the **FBCF** is the implicitation of the v-reduction of the **EMCF**, it is easy to observe that the indices of the **FBCF** and **EMCF** have the following relations: $a = a'$ and $\epsilon_k = \epsilon'_k$ for $k = 1, \ldots, a$; $b = b'$ and $\bar{\epsilon}_k = \bar{\epsilon}'_k$ for $k = 1, \ldots, b$; $n_2 = n_\rho$ and $A^{nn} \approx A_\rho$ (similar matrices); $c + \delta = c'$ and $\sigma'_1 = \sigma'_2 = \cdots = \sigma'_\delta = 1$, $\sigma'_{\delta+1} = \sigma_1 + 1$, $\sigma'_{\delta+2} = \sigma_2 + 1$, …, $\sigma'_{\delta+c} = \sigma_c + 1$; moreover, $d = d'$ and $\bar{\sigma}_k = \bar{\sigma}'_k$ for $k = 1, \ldots, d$; $e = e'$ and $\eta_k + 1 = \eta'_k$ for $k = 1, \ldots, e$.

In an algorithm below, we summarize how to construct the **FBCF** for a given DACS $\Delta^u_{l,n,m} = (E, H, L)$ based on the explicitation procedure.

Algorithm 4.6 the construction of the **FBCF** for linear DACSs via the explicitation

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Initialization: Consider a DACS $\Delta^u_{l,n.m} = (E, H, L)$ with $E \in \mathbb{R}^{l\times n}$, $H \in \mathbb{R}^{l\times n}$, $L \in \mathbb{R}^{l\times m}$.

Step 1: Construct an ODECS Λ^{uv} such that $\Lambda^{uv} \in$ **Expl**(Δ^u) by Definition 2.2:

1: Find Q such that E_1 of $QE = \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right]$ is of full row rank, denote $QH = \left[\begin{smallmatrix} H_1 \\ H_2 \end{smallmatrix}\right]$, $QL = \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$;

2: Set $A = E_1^\dagger H_1$, $B^u = E_1^\dagger L_1$, $C = H_2$, $D^u = L_2$ and find B^v such that $\operatorname{Im} B^v = \ker E_1 = \ker E$;

3: Set $\Lambda^{uv} = (A, B^u, B^v, C, D^v)$, then we have $\Lambda^{uv} \in$ **Expl**(Δ^u).

Step 2: Find EM_{tran} such that $\tilde{\Lambda}^{\tilde{u}\tilde{v}} = EM_{tran}(\Lambda^{uv})$ is in the **EMTF** by Theorem 3.5:

4: Calculate the subspaces $\mathcal{V}^*$, $\mathcal{U}^*_u$, $\mathcal{W}^*$, $\mathcal{Y}^*$ for $\Lambda^w = \Lambda^{uv}$ by Lemma 7.4;

5: Construct T_s, T_o by (17) and T_w by (37);

6: Find $K_{MT} = T_s^{-1} K T_o$ and $F_{MT} = T_i^{-1} F T_s$ by (30) and (31);

7: Set $T_x = T_s$, $T_y = T_o$, $F_w = F_{MT}$, $K_w = K_{MT}$ and $M_{trans} = (T_x, T_w, T_y, F_w, K_w)$, then we have $\tilde{\Lambda}^{\tilde{w}} = M_{trans}(\Lambda^w)$ is in the **MTF**, i.e., $\exists\, EM_{tran}$: $\tilde{\Lambda}^{\tilde{u}\tilde{v}} = EM_{tran}(\Lambda^{uv})$ is in the **EMTF**.

Step 3: Find EM_{tran} such that $\bar{\Lambda}^{\bar{u}\bar{v}} = EM_{tran}(\tilde{\Lambda}^{\tilde{u}\tilde{v}})$ is in the **EMNF** by Theorem 3.6:

8: Construct F_{MN}, K_{MN}, T_{MN} for $\tilde{\lambda}^{\tilde{w}}$ by the **MNF Algorithm 3.3**.

9: Set $M_{tran} = (T_{MN}, I_u, I_y, F_{MN}, K_{MN})$, then we have $\bar{\Lambda}^{\bar{w}} = M_{tran}(\tilde{\Lambda}^{\tilde{w}})$ is in the **MNF**, i.e., $\exists\, EM_{tran}$ such that $\bar{\Lambda}^{\bar{u}\bar{v}} = EM_{tran}(\tilde{\Lambda}^{\tilde{u}\tilde{v}})$ is in the **EMNF**.

Step 4: By the procedure shown in the proof of Theorem 4.1, bring $\bar{\Lambda}^{\bar{u}\bar{v}}$ into the **EMCF**.

Step 5: By Definition 4.2, find the implicitation of the v-reduction of $\bar{\Lambda}^{\bar{u}\bar{v}}$, denoted by $\bar{\Delta}^{\bar{u}}$.

Result: $\bar{\Delta}^{\bar{u}}$ is in the **FBCF** and $\Delta^u \overset{ex-fb}{\sim} \bar{\Delta}^{\bar{u}}$.

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5. Example

In this section, we illustrate the construction of Algorithm 4.6 by an example taken from [9]. Consider the following mathematical model of an electrical circuit (see Fig. 1.1 of [9]), which is a DACS of the form $E\dot{x} = Hx + Lu$:

$$\begin{bmatrix} 0&0&0&0&L&0&0&0&0&0&0&0&0&0\\ 0&0&-Ca&Ca&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&-1&0&0&0&0&0&R_G&0&0&0&0&0&0\\ 0&1&-1&0&0&0&0&0&R_F&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&R&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&-1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1&0&1&-1&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&1&1\\ 0&0&-1&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\1&0\\0&0\\0&0\\0&0\\0&1 \end{bmatrix} \begin{bmatrix} I\\ V \end{bmatrix},$$

where $u = [I, V]^T$ is the control vector, L, Ca, R, R_G, R_F are real scalars (all assumed to be nonzero). In [9], only the matrix pencil $sE - H$ is transformed into a quasi-Kronecker form. Below, we will transform [2] the whole DACS into its **FBCF** via Algorithm 4.6.

Step 1: Find an ODECS $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$, which we take as

$$\Lambda^{uv}: \begin{cases} \dot{x} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1/Ca&0\\ 1/L&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0 \end{bmatrix} u + \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} v \\ \\ y = \begin{bmatrix} 0&-1&0&0&0&0&0&R_G&0&0&0&0&0&0\\ 0&1&-1&0&0&0&0&0&R_F&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&R&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&-1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1&0&1&-1&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&1&1\\ 0&0&-1&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\1&0\\0&0\\0&0\\0&0\\0&1 \end{bmatrix} u. \end{cases}$$

Step 2: Calculate the subspaces $\mathcal{V}^*$, $\mathcal{U}_w^*$, $\mathcal{U}_v^*$, $\mathcal{W}^*$, $\mathcal{Y}^*$ of $\Lambda^w = (A, B^w, C, D^w)$ by Lemma 7.4 of the Appendix. They are $\mathcal{W}^* = \mathscr{X} = \mathbb{R}^{14}$, $\mathcal{Y}^* = \mathscr{Y} = \mathbb{R}^{11}$ and

$$\mathcal{V}^* = \operatorname{Im} \begin{bmatrix} R_G&0&0&0&0\\ R_G&0&0&0&0\\ R_F+R_G&0&0&0&0\\ 0&R&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&0&0&-1&1\\ -1&-1&0&1&-1 \end{bmatrix}, \ \mathcal{U}_w^* = \operatorname{Im} \begin{bmatrix} 0&0&0\\ 0&0&0\\ R*R_G&0&0\\ R*R_G&0&0\\ R*(R_F+R_G)&0&0\\ 0&0&0\\ 0&0&0\\ R&0&0\\ R&0&0\\ R_F+R_G&0&0\\ 0&1&0\\ 0&0&1\\ R&-1&1\\ -(R+R_F+R_G)&1&-1 \end{bmatrix}, \ \mathcal{U}_v^* = \operatorname{Im} \begin{bmatrix} R*R_G&0&0\\ R*R_G&0&0\\ R*(R_F+R_G)&0&0\\ 0&0&0\\ 0&0&0\\ R&0&0\\ R&0&0\\ R_F+R_G&0&0\\ 0&1&0\\ 0&0&1\\ R&-1&1\\ -(R+R_F+R_G)&1&-1 \end{bmatrix}.$$

By the proof of Theorem 3.5 and Proposition 3.1, we can choose the following transformation

[2] The calculations of the invariant subspaces and the transformation matrices in the example are implemented by Matlab and the source code are available on the webpage of the first author.

matrices: $T_y = I_{11}$, $K_{MT} = 0_{14\times 11}$,

$$T_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{R_F+R_G}{R_G} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/R_G & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/R_G & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{R} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1/R_G & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 1/R_G & 0 & 0 & \frac{-1}{R} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \end{bmatrix}, \quad T_w = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(R_F+R_G)/R_G & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/R_G & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/R_G & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(R_F+R_G)/R{*}R_G & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/R_G & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & (R+R_F+R_G)/(R{*}R_G) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \end{bmatrix},$$

$$F_{MT} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (R_F+R_G)/R_G & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/(Ca{*}R{*}R_G) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/(Ca{*}R) & 1/(Ca{*}R) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/(Ca{*}R{*}R_G) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/(Ca{*}R) & -1/(Ca{*}R) & 0 & 0 \end{bmatrix}.$$

Then the Morse transformation $M_{trans}(T_s, T_w, T_y, F_{MT}, K_{MT})$ brings Λ^w into $\tilde{\Lambda}^{\tilde{w}} = (\tilde{A}, \tilde{B}^{\tilde{w}}, \tilde{C}, \tilde{D}^{\tilde{w}})$, which is in the **EMTF**, where

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B}^{\tilde{w}} \\ \hline \tilde{C} & \tilde{D}^{\tilde{w}} \end{array}\right] = \left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \hline \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{array}\right] = \left[\begin{array}{cc|c|cc} \tilde{A}_1 & \tilde{A}_{13} & 0 & \tilde{B}_1^{\tilde{v}} & \tilde{B}_{12}^{\tilde{v}} \\ 0 & \tilde{A}_3 & 0 & 0 & \tilde{B}_3^{\tilde{v}} \\ \hline 0 & \tilde{C}_3 & \tilde{D}_3^{\tilde{u}} & 0 & 0 \end{array}\right] =$$

$$\left[\begin{array}{cccccccccccccc|cc|cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/(Ca{*}R_G) & 0 & 0 & -1/Ca & 1/Ca & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/Ca & 0 & 0 & 0 & (R_F+R_G)/R_G & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/(Ca{*}R) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/R & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/(Ca{*}R) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/R & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & R_G & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & R_F & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

Step 3: By **MNF** Algorithm 3.3, set

$$K_{MN} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/(Ca{*}R_G) & 0 & 0 & 0 & -1/Ca & 1/(Ca{*}R_G) & 0 & 1/Ca & -1/Ca & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_{MN} = 0_{14\times 14}.$$

Then find T_{MN}^2 via the following constrained Sylvester equation,

$$\bar{A}_1 T_{MN}^2 - T_{MN}^2 \bar{A}_3 = -\bar{A}_1, \quad T_{MN}^2 \bar{B}_3^{\bar{w}} = -\bar{B}_{12}^{\bar{w}},$$

where $\bar{A} = \tilde{A} + K_{MN}\tilde{C}$, $\bar{B}^{\bar{w}} = \tilde{B}^{\tilde{w}} + K_{MN}\tilde{D}^{\tilde{w}}$. The above equation is solvable and the solution is

$$T_{MN}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the Morse transformation $M_{tran} = (T_{MN}, I_{14}, I_{11}, F_{MN}, K_{MN})$, where $T_{MN} = \begin{bmatrix} I & T_{MN}^2 \\ 0 & I \end{bmatrix}$, brings $\tilde{\Lambda}^{\tilde{w}}$ into $\bar{\Lambda}^{\bar{w}} = (\bar{A}, \bar{B}^{\bar{w}}, \bar{C}, \bar{D}^{\bar{w}})$, which is in the **EMNF**, where

$$\left[\begin{array}{c|c}\bar{A} & \bar{B}^{\bar{w}}\\ \hline \bar{C} & \bar{D}^{\bar{w}}\end{array}\right]=\left[\begin{array}{c|c|c}\bar{A} & \bar{B}^{\bar{u}} & \bar{B}^{\bar{v}}\\ \hline \bar{C} & \bar{D}^{\bar{u}} & 0\end{array}\right]=\left[\begin{array}{cc|c|cc}\bar{A}_{11} & 0 & 0 & \bar{B}^{\bar{v}}_{11} & 0\\ 0 & \bar{A}_{33} & 0 & 0 & \bar{B}^{\bar{v}}_{32}\\ \hline 0 & \bar{C}_{13} & \bar{D}^{\bar{u}}_{12} & 0 & 0\end{array}\right]=$$

$$\left[\begin{array}{cccccccccccccc|cc|cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1/(Ca*R_G) & 0 & 0 & -1/Ca & 1/Ca & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (R_F+R_G)/R_G & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1/L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/(Ca*R) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/R & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/(Ca*R) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/R & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
\hline
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & R_G & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & R_F & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

Step 4: Transform each subsystem of $\bar{\Lambda}^{\bar{w}}$ into its canonical form as in Theorem 4.1 to obtain

$$\textbf{EMCF}:\left[\begin{array}{cc|c|cc}A^{cv} & 0 & 0 & B^{cv} & 0\\ 0 & A^{pv} & 0 & 0 & B^{pv}\\ \hline 0 & 0 & D^{pu} & 0 & 0\\ 0 & \bar{C}^{pv} & 0 & 0 & 0\end{array}\right]=\left[\begin{array}{cccccccccccccc|cc|cccccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

The **EMCF** indices are $\bar{\epsilon}_1 = 2, \bar{\epsilon}_2 = 2, \bar{\epsilon}_3 = 1$, $\delta = 2$, $\bar{\sigma}_1 = \bar{\sigma}_2 =, \ldots, = \bar{\sigma}_9 = 1$. Note that n_2, a, c, e are all zeros and we have 3 subsystems only.

Step 5: Using the v-reduction and implicitation of Definition 4.2, we get the following DACS from the above **EMCF**:

$$\left[\begin{array}{cccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\dot{z}=\left[\begin{array}{cccccccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]z+\left[\begin{array}{cc}
0 & 0\\
0 & 0\\
1 & 0\\
0 & 1\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & 0
\end{array}\right]\tilde{u},$$

where z and $\tilde{u}$ is the new "generalized" state and the new input, respectively. Obviously, the above DACS is in the **FBCF** with indices $\bar{\epsilon}'_1 = 2, \bar{\epsilon}'_2 = 2, \bar{\epsilon}'_3 = 1$, $\sigma'_1 = \sigma'_2 = 1$, $\bar{\sigma}'_1 = \bar{\sigma}'_2 =, \ldots, = \bar{\sigma}'_9 = 1$. Moreover, $a' = n_\rho = e' = 0$, $c' = \delta = 2$.

6. Proofs of the results

6.1. Proofs of Proposition 2.3, Proposition 2.4 and Theorem 2.9

Proof of Proposition 2.3. If. Suppose that Λ^{uv} and $\tilde{\Lambda}^{u\tilde{v}}$ are equivalent via a transformation given by (9). First, $\operatorname{Im}\tilde{B}^{\tilde{v}} \overset{(9)}{=} \operatorname{Im}B^vT_v^{-1} = \ker E_1 = \ker E$ implies that $\tilde{B}^{\tilde{v}}$ is another choice such that $\operatorname{Im}\tilde{B}^{\tilde{v}} = \ker E$. Observe that

$$\tilde{\Lambda}^{u\tilde{v}} : \begin{cases} \dot{x} = \tilde{A}x + \tilde{B}^u u + \tilde{B}^{\tilde{v}}\tilde{v} \overset{(9)}{=} (A + KC + B^vF_v)x + (B^u + KD^u + B^vR)u + B^vT_v^{-1}\tilde{v} \\ \tilde{y} = \tilde{C}x + \tilde{D}^u u \overset{(9)}{=} T_yCx + T_yD^u u. \end{cases}$$

Then pre-multiply the differential part of $\tilde{\Lambda}^{u\tilde{v}}$ by E_1, to get (notice that $A = E_1^{\dagger}H_1$, $B^u = E_1^{\dagger}L_1$, $\operatorname{Im}B^v = \ker E_1$ and $C = H_2$, $D^u = L_2$)

$$\begin{cases} E_1\dot{x} = (H_1 + E_1KH_2)x + (L_1 + E_1KL_2)u \\ \quad\tilde{y} = T_yH_2x + T_yL_2u. \end{cases}$$

Thus $\tilde{\Lambda}^{u\tilde{v}}$ is an $(I_l, \tilde{v})$-explicitation of the following DACS:

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix}\dot{x} = \begin{bmatrix} H_1 + E_1KH_2 \\ T_yH_2 \end{bmatrix}x + \begin{bmatrix} L_1 + E_1KL_2 \\ T_yL_2 \end{bmatrix}u.$$

Since the above DACS can be transformed from Δ^u via $\tilde{Q} = Q'Q$, where $Q' = \begin{bmatrix} I_q & E_1K \\ 0 & T_y \end{bmatrix}$, it proves that $\tilde{\Lambda}^{u\tilde{v}}$ is a $(\tilde{Q}, \tilde{v})$-explicitation of Δ^u corresponding to the choice of invertible matrix $\tilde{Q}$. Finally, by $E_1\tilde{A} = H_1 + E_1KH_2$, $E_1\tilde{B}^u = L_1 + E_1KL_2$, we get $\tilde{A} = \tilde{E}_1^{\dagger}(H_1 + KH_2)$ and $\tilde{B}^u = \tilde{E}_1^{\dagger}(L_1 + KL_2)$ for another choice of right inverse $\tilde{E}_1^{\dagger}$ of E_1.

Only if. Suppose that $\tilde{\Lambda}^{u\tilde{v}} \in \mathbf{Expl}(\Delta^u)$ via $\tilde{Q}$, $\tilde{E}_1^{\dagger}$ and $\tilde{B}^{\tilde{v}}$. First, by $\operatorname{Im}\tilde{B}^{\tilde{v}} = \ker E = \operatorname{Im}B^v$, there exists an invertible matrix T_v^{-1} such that $\tilde{B}^{\tilde{v}} = B^vT_v^{-1}$. Moreover, since $E_1^{\dagger}$ is a right inverse of E_1 if and only if any solution $\dot{x}$ of $E_1\dot{x} = w$ is given by $E_1^{\dagger}w$, we have $E_1E_1^{\dagger}(H_1x + L_1u) = H_1x + L_1u$ and $E_1\tilde{E}_1^{\dagger}(H_1x + L_1u) = H_1x + L_1u$. It follows that $E_1(\tilde{E}_1^{\dagger} - E_1^{\dagger})(H_1x + L_1u) = 0$, so $(\tilde{E}_1^{\dagger} - E_1^{\dagger})H_1 \in \ker E_1$, $(\tilde{E}_1^{\dagger} - E_1^{\dagger})L_1 \in \ker E_1$. Since $\ker E_1 = \operatorname{Im}B^v$, it follows that $(\tilde{E}_1^{\dagger} - E_1^{\dagger})H_1 = B^vF_v$ and $(\tilde{E}_1^{\dagger} - E_1^{\dagger})L_1 = B^vR$ for suitable F_v and R. Furthermore, since Q is such that E_1 of $QE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}$ is of full row rank, it follows that any other $\tilde{Q}$, such that $\tilde{E}_1$ of $\tilde{Q}E = \begin{bmatrix} \tilde{E}_1 \\ 0 \end{bmatrix}$ is full row rank, must be of the form $\tilde{Q} = Q'Q$, where $Q' = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix}$. Thus via $\tilde{Q}$, Δ^u is ex-equivalent to

$$Q'\begin{bmatrix} E_1 \\ 0 \end{bmatrix}\dot{x} = Q'\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} + Q'\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}u \Rightarrow \begin{bmatrix} Q_1E_1 \\ 0 \end{bmatrix}\dot{x} = \begin{bmatrix} Q_1H_1 + Q_2H_2 \\ Q_4H_2 \end{bmatrix} + \begin{bmatrix} Q_1L_1 + Q_2L_2 \\ Q_4L_2 \end{bmatrix}u.$$

We obtain the following equations, using $\tilde{E}_1^{\dagger}$ and $\tilde{B}^{\tilde{v}}$, and based on the right-hand side of the above:

$$\begin{cases} \dot{x} = (\tilde{E}_1^{\dagger}H_1 + \tilde{E}_1^{\dagger}Q_1^{-1}Q_2H_2)x + (\tilde{E}_1^{\dagger}L_1 + \tilde{E}_1^{\dagger}Q_1^{-1}Q_2L_2)u + \tilde{B}^{\tilde{v}}v \\ \quad = (E_1^{\dagger}H_1 + B^vF_v + E_1^{\dagger}Q_1^{-1}Q_2C)x + (E_1^{\dagger}H_1 + B^vR + E_1^{\dagger}Q_1^{-1}Q_2D^u)u + B^vT_v^{-1}\tilde{v} \\ 0 = Q_4H_2 + Q_4L_2 = Q_4Cx + Q_4D^u. \end{cases}$$

Thus the explicitation of Δ^u via $\tilde{Q}$, $\tilde{E}_1^{\dagger}$ and $\tilde{B}^{\tilde{v}}$ is

$$\tilde{\Lambda}^{u\tilde{v}} : \begin{cases} \dot{x} = Ax + K(Cx + D^u u) + B^v(F_vx + Ru + T_v^{-1}\tilde{v}) = \tilde{A}x + \tilde{B}^u u + \tilde{B}^{\tilde{v}}\tilde{v} \\ \tilde{y} = T_yCx + T_yD^u u = \tilde{C}x + \tilde{D}^u u. \end{cases}$$

where $K = E_1^{\dagger}Q_1^{-1}Q_2$, $T_y = Q_4$. Now we can see that Λ^{uv} and $\tilde{\Lambda}^{u\tilde{v}}$ are equivalent via transformations listed in (9). $\square$

Proof of Proposition 2.4. Consider equation (5) of the (Q,v)-explicitation procedure. Since Q-transformations preserve solutions of Δ^u, equation (5) resulting from a Q-transformation of Δ^u has the same solutions as Δ^u. Thus we need to prove that equations (5) and (8) have corresponding solutions for any choices of $E_1^\dagger$ and B^v. Moreover, the second equation $0 = H_2x + L_2u$ of (5) coincides with $0 = Cx + D^u u$ of (8) (since $C = H_2$ and $D^u = L_2$). So we only need to prove that $(x(t), u(t))$ with $x(t) \in \mathcal{C}^1$ and $u(t) \in \mathcal{C}^0$ is a solution of (5a) if and only if there exists $v(t) \in \mathcal{C}^0$ such that $(x(t), u(t), v(t))$ is a solution of (7) independently of the choice of $E_1^\dagger$, defining $A = E_1^\dagger H$ and $B^u = E_1^\dagger L_1$, and of the choice of B^v satisfying $\operatorname{Im} B^v = \ker E_1$.

If. Suppose that $(x(t), u(t), v(t))$ is a solution of (7). Then we have $\dot{x}(t) = Ax(t) + B^u u(t) + B^v v(t)$. Pre-multiplying the last equation by E_1, we conclude (recall that $A = E_1^\dagger H_1$, $B^u = E_1^\dagger L_1$, $\ker E_1 = \operatorname{Im} B_v$) that $E_1\dot{x}(t) = H_1x(t) + L_1u(t)$, which proves that $(x(t), u(t))$ is a solution of (5a).

Only if. Suppose that $(x(t), u(t))$ is a solution of (5a). Rewrite $E_1\dot{x}$ as $\left[\begin{smallmatrix} E_1^1 & E_1^2 \end{smallmatrix}\right]\left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right]$, where $E_1^1 \in \mathbb{R}^{q\times q}$ and $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$. Then, without loss of generality, we assume that the matrix E_1^1 is invertible (if not, we permute the components of x such that the first q columns of E_1 are independent). Thus, a choice of the right inverse of E_1 is $E_1^\dagger = \left[\begin{smallmatrix} (E_1^1)^{-1} \\ 0 \end{smallmatrix}\right]$ (since $\left[\begin{smallmatrix} E_1^1 & E_1^2 \end{smallmatrix}\right]\left[\begin{smallmatrix} (E_1^1)^{-1} \\ 0 \end{smallmatrix}\right] = I_q$), which gives the matrices A, B^u, B^v of (7) to be, respectively,

$$A := E_1^\dagger H_1 = \left[\begin{smallmatrix} (E_1^1)^{-1}H_1 \\ 0 \end{smallmatrix}\right], \quad B^u := E_1^\dagger L_1 = \left[\begin{smallmatrix} (E_1^1)^{-1}L_1 \\ 0 \end{smallmatrix}\right], \quad B^v := \left[\begin{smallmatrix} -(E_1^1)^{-1}E_1^2 \\ I_s \end{smallmatrix}\right].$$

Let $v(t) = \dot{x}_2(t)$, then $v \in \mathcal{C}^0$ and it is clear that if $(x(t), u(t)) = ((x_1(t), x_2(t)), u(t))$ is a solution of (5a), then $(x(t), u(t), v(t))$ solves (7) with (A, B^u, B^v) as above, since

$$\left[\begin{smallmatrix} E_1^1 & E_1^2 \end{smallmatrix}\right]\left[\begin{smallmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{smallmatrix}\right] = H_1x_1(t) + L_1u(t) \Rightarrow \dot{x}_1(t) = (E_1^1)^{-1}H_1x(t) + (E_1^1)^{-1}L_1u(t) - (E_1^1)^{-1}E_1^2\dot{x}_2(t).$$

Notice that if we choose another right inverse $\tilde{E}_1^\dagger$ of E_1 and another matrix $\tilde{B}^v$ such that $\operatorname{Im} \tilde{B}^v = \ker E_1$, then by Proposition 2.3, equation (7) becomes

$$\dot{x} = \tilde{A}x + \tilde{B}^u u + \tilde{B}^{\tilde{v}}\tilde{v} \Leftrightarrow \dot{x} = Ax + B^u u + B^v(F_v x + Ru + T_v^{-1}\tilde{v}).$$

We thus conclude that there exists $\tilde{v}(t) = -T_vF_vx(t) - T_vRu(t) + T_vv(t) = -T_vF_vx(t) - T_vRu(t) + T_v\dot{x}_2(t)$ such that $(x(t), u(t), \tilde{v}(t))$ solves equation (7). Therefore, Δ^u has corresponding solutions with any (Q,v)-explicitation independently of the choice of Q, $E_1^\dagger$ and B^v. $\square$

Proof of Theorem 2.9. Without loss of generality, we assume that the system matrices of $\Delta^u = (E, H, L)$ and $\tilde{\Delta}^{\tilde{u}} = (\tilde{E}, \tilde{H}, \tilde{L})$ are of the following form:

$$E = \left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right], \quad H = \left[\begin{smallmatrix} H_1 \\ H_2 \end{smallmatrix}\right], \quad L = \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right], \quad \tilde{E} = \left[\begin{smallmatrix} I_{\tilde{q}} & 0 \\ 0 & 0 \end{smallmatrix}\right], \quad \tilde{H} = \left[\begin{smallmatrix} \tilde{H}_1 \\ \tilde{H}_2 \end{smallmatrix}\right], \quad L = \left[\begin{smallmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{smallmatrix}\right],$$

where $H_1 \in \mathbb{R}^{q\times n}$, $L_1 \in \mathbb{R}^{q\times m}$, $\tilde{H}_1 \in \mathbb{R}^{\tilde{q}\times n}$, $\tilde{L}_1 \in \mathbb{R}^{\tilde{q}\times m}$, $q = \operatorname{rank} E$, $\tilde{q} = \operatorname{rank} \tilde{E}$. Since if not, we can always find $Q, \tilde{Q} \in Gl(l, \mathbb{R})$, $P, \tilde{P} \in Gl(n, \mathbb{R})$ such that

$$(QEP^{-1}, QHP^{-1}, QL) \ \text{ and } \ (\tilde{Q}\tilde{E}\tilde{P}^{-1}, \tilde{Q}\tilde{H}\tilde{P}^{-1}, \tilde{Q}\tilde{L})$$

are of the above desired form and it is easily seen that the ex-fb-equivalence of (E, H, L) and $(\tilde{E}, \tilde{H}, \tilde{L})$

is equivalent to (implied by and implying) that of (QEP^{-1}, QHP^{-1}, QL) and $(\tilde{Q}\tilde{E}\tilde{P}^{-1}, \tilde{Q}\tilde{H}\tilde{P}^{-1}, \tilde{Q}\tilde{L})$. Thus we can use the above system matrices to represent Δ^u and $\tilde{\Delta}^{\tilde{u}}$ in the remaining part of proof.

By the assumptions that $\Lambda^{uv} \in \mathbf{Expl}(\Delta^u)$ and $\tilde{\Lambda}^{\tilde{u}\tilde{v}} \in \mathbf{Expl}(\tilde{\Delta}^{\tilde{u}})$, we have

$$\left[\begin{smallmatrix} A & B^u & B^v \\ C & D^u & 0 \end{smallmatrix}\right] = \left[\begin{array}{c|c|c} H_1 & L_1 & 0 \\ 0 & 0 & I_{n-q} \\ \hline H_2 & L_2 & 0 \end{array}\right], \quad \left[\begin{smallmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} & \tilde{B}^{\tilde{v}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} & 0 \end{smallmatrix}\right] = \left[\begin{array}{c|c|c} \tilde{H}_1 & \tilde{L}_1 & 0 \\ 0 & 0 & I_{n-\tilde{q}} \\ \hline \tilde{H}_2 & \tilde{L}_2 & 0 \end{array}\right]. \tag{25}$$

We have chosen Λ^{uv} and $\tilde{\Lambda}^{\tilde{u}\tilde{v}}$ as above for convenience, any other choice based on the explicitation procedure could have been made. Since any two ODECSs in an explicitation class are EM-equivalent, the choice of a (Q, v)-explicitation makes no difference when proving EM-equivalence. Therefore, we will use the system matrices in (25) for the following proof.

If. Suppose $\Lambda^{uv} \overset{EM}{\sim} \tilde{\Lambda}^{\tilde{u}\tilde{v}}$. Then there exist transformation matrices T_x, T_u, T_v, T_y, F_u, F_v, R, K such that (14) holds. Substituting the system matrices of (25) into (14), we have

$$\left[\begin{array}{c|c|c} \tilde{H}_1 & \tilde{L}_1 & 0 \\ 0 & 0 & I_{n-q} \\ \hline \tilde{H}_2 & \tilde{L}_2 & 0 \end{array}\right] = \begin{bmatrix} T_x & T_x K \\ 0 & T_y \end{bmatrix} \left[\begin{array}{c|c|c} H_1 & L_1 & 0 \\ 0 & 0 & I_{n-\tilde{q}} \\ \hline H_2 & L_2 & 0 \end{array}\right] \begin{bmatrix} T_x^{-1} & 0 & 0 \\ F_u T_x^{-1} & T_u^{-1} & 0 \\ (F_v + RF_u)T_x^{-1} & RT_u^{-1} & T_v^{-1} \end{bmatrix}. \tag{26}$$

Represent $T_x = \left[\begin{smallmatrix} T_x^1 & T_x^2 \\ T_x^3 & T_x^4 \end{smallmatrix}\right]$, where $T_x^1 \in \mathbb{R}^{q\times q}$. By $\tilde{B}^{\tilde{v}} = T_x B^v T_v^{-1}$, we get $\left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right] = \left[\begin{smallmatrix} T_x^1 & T_x^2 \\ T_x^3 & T_x^4 \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right] T_v^{-1}$, hence it can be deduced that $q = \tilde{q}$ and $T_x^2 = 0$. Moreover, $T_x^4 T_v^{-1} = I$ implies that T_x^4 is invertible. Thus by the invertibility of T_x, we have T_x^1 is invertible as well.

Subsequently, premultiply equation (26) by $\left[\begin{smallmatrix} (T_x^1)^{-1} & 0 & 0 \\ 0 & 0 & I_{l-q} \end{smallmatrix}\right]$ and we get

$$\begin{bmatrix} (T_x^1)^{-1} & 0 \\ 0 & I_{l-q} \end{bmatrix} \left[\begin{array}{c|c|c} \tilde{H}_1 & \tilde{L}_1 & 0 \\ \tilde{H}_2 & \tilde{L}_2 & 0 \end{array}\right] = \begin{bmatrix} I_q & K_1 \\ 0 & T_y \end{bmatrix} \left[\begin{array}{c|c|c} H_1 & L_1 & 0 \\ H_2 & L_2 & 0 \end{array}\right] \begin{bmatrix} T_x^{-1} & 0 & 0 \\ F_u T_x^{-1} & T_u^{-1} & 0 \\ (F_v + RF_u)T_x^{-1} & RT_u^{-1} & T_v^{-1} \end{bmatrix},$$

where $K_1 = \left[\begin{smallmatrix} I_q & (T_x^1)^{-1} T_x^2 \end{smallmatrix}\right] K$. It follows that

$$\left[\begin{array}{c|c} \tilde{H}_1 & \tilde{L}_1 \\ \tilde{H}_2 & \tilde{L}_2 \end{array}\right] = \begin{bmatrix} T_x^1 & T_x^1 K_1 \\ 0 & T_y \end{bmatrix} \left[\begin{array}{c|c} H_1 & L_1 \\ H_2 & L_2 \end{array}\right] \begin{bmatrix} T_x^{-1} & 0 \\ F_u T_x^{-1} & T_u^{-1} \end{bmatrix}.$$

Thus $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$ via

$$Q = \begin{bmatrix} T_x^1 & T_x^1 K_1 \\ 0 & T_y \end{bmatrix}, \quad P = T_x, \quad F = F_u, \quad G = T_u^{-1}.$$

Only if. Suppose $\Delta^u \overset{ex-fb}{\sim} \tilde{\Delta}^{\tilde{u}}$. Then there exist invertible matrices Q, P, and matrices F, G of appropriate sizes such that equation (4) holds. Represent $Q = \left[\begin{smallmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{smallmatrix}\right]$, where $Q_1 \in \mathbb{R}^{q\times q}$, and $P^{-1} = \left[\begin{smallmatrix} P_1 & P_2 \\ P_3 & P_4 \end{smallmatrix}\right]$, where $P_1 \in \mathbb{R}^{q\times q}$. Then by

$$\tilde{E} = QEP^{-1} \Rightarrow \left[\begin{smallmatrix} I_{\tilde{q}} & 0 \\ 0 & 0 \end{smallmatrix}\right] = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \left[\begin{smallmatrix} I_q & 0 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} P_1 & P_2 \\ P_3 & P_4 \end{smallmatrix}\right],$$

we immediately get $q = \tilde{q}$ and $Q_1 P_1 = I$, $Q_1 P_2 = 0$, $Q_3 P_1 = 0$, which implies that Q_1, P_1 are invertible matrices, $P_2 = 0$, and $Q_3 = 0$. Thus by the invertibility of Q and P, we have Q_4 and P_4 are invertible matrices as well. Then by equation (4), we get

$$\left[\begin{array}{c|c} \tilde{H}_1 & \tilde{L}_1 \\ \tilde{H}_2 & \tilde{L}_2 \end{array}\right] = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \left[\begin{array}{c|c} H_1 & L_1 \\ H_2 & L_2 \end{array}\right] \begin{bmatrix} P^{-1} & 0 \\ FP^{-1} & G \end{bmatrix},$$

which implies that the following equation holds:

$$\left[\begin{array}{c|c|c} \tilde{H}_1 & \tilde{L}_1 & 0 \\ 0 & 0 & I_{n-q} \\ \hline \tilde{H}_2 & \tilde{L}_2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} Q_1 & 0 & Q_2 \\ X & P_4^{-1} & 0 \\ \hline 0 & 0 & Q_4 \end{array}\right] \left[\begin{array}{c|c|c} H_1 & L_1 & 0 \\ 0 & 0 & I_{n-\tilde{q}} \\ \hline H_2 & L_2 & 0 \end{array}\right] \begin{bmatrix} P^{-1} & 0 & 0 \\ FP^{-1} & G & 0 \\ Y & Z & P_4 \end{bmatrix},$$

where $X=-P_4^{-1}P_3P_1^{-1}$, $Y=(P_3P_1^{-1}H_1+P_3P_1^{-1}L_1F)P^{-1}$, $Z=P_3P_1^{-1}L_1G$. So $\Lambda^{uv}\overset{EM}{\sim}\tilde{\Lambda}^{\tilde{u}\tilde{v}}$ via

$$\begin{aligned}
&T_x=P, \quad T_u=G^{-1}, && T_v=P_4^{-1}, && T_y=Q_4,\\
&F_u=F, \quad F_v=P_3P_1^{-1}H_1, && R=P_3P_1^{-1}L_1, && K=\left[\begin{smallmatrix}P_1Q_2\\ P_3Q_2\end{smallmatrix}\right].
\end{aligned}$$

□

6.2. Proof of Proposition 2.10

Proof. Without loss of generality, we may assume that $\Delta^u_{l,n,m}=(E,H,L)$ is of the following form:

$$\left[\begin{smallmatrix}I_q & 0\\ 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\dot{x}_1\\ \dot{x}_2\end{smallmatrix}\right]=\left[\begin{smallmatrix}H_1 & H_2\\ H_3 & H_4\end{smallmatrix}\right]\left[\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right]+\left[\begin{smallmatrix}L_1\\ L_2\end{smallmatrix}\right]u,$$

where $q=\operatorname{rank}E$ and $H_1\in\mathbb{R}^{q\times q}$, $H_2\in\mathbb{R}^{q\times(n-q)}$, $H_3\in\mathbb{R}^{p\times q}$, $H_4\in\mathbb{R}^{p\times(n-q)}$, $L_1\in\mathbb{R}^{q\times m}$, $L_2\in\mathbb{R}^{p\times m}$, where $p=l-q$. Since if not, we can always find $Q\in Gl(l,\mathbb{R})$, $P\in Gl(n,\mathbb{R})$ such that $\tilde{\Delta}^{\tilde{u}}=(QEP^{-1},QHP^{-1},QL)$ is of the above form. Then, it is not hard to check that $\mathscr{V}_i(\tilde{\Delta}^{\tilde{u}})=P\mathscr{V}_i(\Delta^u)$, $\mathscr{W}_i(\tilde{\Delta}^{\tilde{u}})=P\mathscr{W}_i(\Delta^u)$, $\hat{\mathscr{W}}_i(\tilde{\Delta}^{\tilde{u}})=P\hat{\mathscr{W}}_i(\Delta^u)$. Moreover, for two ODECSs $\Lambda^w=\Lambda^{uv}\in\mathbf{Expl}(\Delta^u)$, $\tilde{\Lambda}^{\tilde{w}}=\tilde{\Lambda}^{\tilde{u}\tilde{v}}\in\mathbf{Expl}(\tilde{\Delta}^{\tilde{u}})$, we can verify that $\mathcal{V}_i(\tilde{\Lambda}^{\tilde{w}})=P\mathcal{V}_i(\Lambda^w)$, $\mathcal{W}_i(\tilde{\Lambda}^{\tilde{w}})=P\mathcal{W}_i(\Lambda^w)$, $\hat{\mathcal{W}}_i(\tilde{\Lambda}^{\tilde{w}})=P\hat{\mathcal{W}}_i(\Lambda^w)$. Therefore, in order to show that the relations of the subspaces (as claimed in Proposition 2.10) hold, replacing Δ^u by $\tilde{\Delta}^{\tilde{u}}$ makes no difference and thus we will assume that Δ^u is of the above form in what follows.

The following system, denoted $\Lambda^w=\Lambda^{uv}$, is a (Q,v)-explicitation of Δ^u,

$$\Lambda^w=\Lambda^{uv}:\begin{cases}\left[\begin{smallmatrix}\dot{x}_1\\ \dot{x}_2\end{smallmatrix}\right]=\left[\begin{smallmatrix}H_1 & H_2\\ 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right]+\left[\begin{smallmatrix}L_1\\ 0\end{smallmatrix}\right]u+\left[\begin{smallmatrix}0\\ I_{n-q}\end{smallmatrix}\right]v\\ \quad y\quad=H_3x_1+H_4x_2+L_2u.\end{cases}\tag{27}$$

Firstly, we calculate $\mathcal{V}_i(\Lambda^w)$ through equation (45) of the Appendix:

$$\begin{aligned}
\mathcal{V}_{i+1}(\Lambda^w)&=\left[\begin{smallmatrix}A\\ C\end{smallmatrix}\right]^{-1}\left(\left[\begin{smallmatrix}I\\ 0\end{smallmatrix}\right]\mathcal{V}_i(\Lambda^w)+\operatorname{Im}\left[\begin{smallmatrix}B^w\\ D^w\end{smallmatrix}\right]\right)=\left[\begin{smallmatrix}H_1 & H_2\\ 0 & 0\\ H_3 & H_4\end{smallmatrix}\right]^{-1}\left(\left[\begin{smallmatrix}\mathcal{V}_i(\Lambda^w)\\ 0\end{smallmatrix}\right]+\operatorname{Im}\left[\begin{smallmatrix}L_1 & 0\\ 0 & I_{n-q}\\ L_2 & 0\end{smallmatrix}\right]\right)\\
&=\left[\begin{smallmatrix}H_1 & H_2\\ H_3 & H_4\end{smallmatrix}\right]^{-1}\left(\left[\begin{smallmatrix}[I_q,0]\mathcal{V}_i(\Lambda^w)\\ 0\end{smallmatrix}\right]+\operatorname{Im}\left[\begin{smallmatrix}L_1 & 0\\ L_2 & 0\end{smallmatrix}\right]\right)=H^{-1}(E\mathcal{V}_i(\Lambda^w)+\operatorname{Im}L).
\end{aligned}$$

Comparing the above expression with equation (42) of the Appendix, it is easily seen that the subspace sequences $\mathcal{V}_{i+1}(\Lambda^w)$ and $\mathscr{V}_{i+1}(\Delta^u)$ are calculated in the same way. Since $\mathscr{V}_0(\Delta^u)=\mathcal{V}_0(\Lambda^w)=\mathbb{R}^n$, we conclude that $\mathscr{V}_i(\Delta^u)=\mathcal{V}_i(\Lambda^w)$ for $i\in\mathbb{N}$.

Then calculate $\mathscr{W}_{i+1}(\Delta^u)$ via equation (43) of the Appendix:

$$\begin{aligned}
\mathscr{W}_{i+1}(\Delta^u)&=E^{-1}(H\mathscr{W}_i(\Delta^u)+\operatorname{Im}L)=\left[\begin{smallmatrix}I_q & 0\\ 0 & 0\end{smallmatrix}\right]^{-1}\left(\left[\begin{smallmatrix}H_1 & H_2\\ H_3 & H_4\end{smallmatrix}\right]\mathscr{W}_i(\Delta^u)+\operatorname{Im}\left[\begin{smallmatrix}L_1\\ L_2\end{smallmatrix}\right]\right)\\
&=\left[\begin{smallmatrix}I_q & 0\\ 0 & 0\end{smallmatrix}\right]^{-1}\left(\left[\begin{smallmatrix}H_1 & H_2 & L_1 & 0\\ H_3 & H_4 & L_2 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathscr{W}_i(\Delta^u)\\ \mathscr{U}_w\end{smallmatrix}\right]\right)\\
&=\left[\begin{smallmatrix}H_1 & H_2 & L_1 & 0\\ 0 & 0 & 0 & 0\end{smallmatrix}\right]\left(\left[\begin{smallmatrix}\mathscr{W}_i(\Delta^u)\\ \mathscr{U}_w\end{smallmatrix}\right]\cap\ker\left[\begin{smallmatrix}H_3 & H_4 & L_2 & 0\end{smallmatrix}\right]\right)+\operatorname{Im}\left[\begin{smallmatrix}0\\ I_{n-q}\end{smallmatrix}\right].
\end{aligned}$$

In the above formula, according to the special form of E, we directly calculate the preimage. Moreover, we can express

$$\left[\begin{smallmatrix}0\\ I_{n-q}\end{smallmatrix}\right]=\left[\begin{smallmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & I_{n-q}\end{smallmatrix}\right]\left(\left[\begin{smallmatrix}\mathscr{W}_i(\Delta^u)\\ \mathscr{U}_w\end{smallmatrix}\right]\cap\ker\left[\begin{smallmatrix}H_3 & H_4 & L_2 & 0\end{smallmatrix}\right]\right).$$

It follows that

$$\mathscr{W}_{i+1}(\Delta^u)=\left[\begin{smallmatrix}H_1 & H_2 & L_1 & 0\\ 0 & 0 & 0 & I_{n-q}\end{smallmatrix}\right]\left(\left[\begin{smallmatrix}\mathscr{W}_i(\Delta^u)\\ \mathscr{U}_w\end{smallmatrix}\right]\cap\ker\left[\begin{smallmatrix}H_3 & H_4 & L_2 & 0\end{smallmatrix}\right]\right)$$

$$= \begin{bmatrix} A & B^w \end{bmatrix} \left(\begin{bmatrix} \mathscr{W}_i(\Delta^u) \\ \mathscr{U}_w \end{bmatrix} \cap \ker \begin{bmatrix} C & D^w \end{bmatrix} \right).$$

It is seen from the above equation and (47) of Appendix that the subspace sequences $\mathcal{W}_{i+1}(\Lambda^w)$ and $\mathscr{W}_{i+1}(\Delta^u)$ are calculated in the same way. Since the initial conditions $\mathcal{W}_0(\Lambda^w) = \mathscr{W}_0(\Delta^u) = \{0\}$, we conclude that $\mathcal{W}_{i+1}(\Lambda^w) = \mathscr{W}_{i+1}(\Delta^u)$ for all $i \in \mathbb{N}$.

Then from (43) and (44), it is seen that the subspaces sequences $\mathscr{W}_i$ and $\hat{\mathscr{W}}_i$ are calculated in the same form, their difference comes from their initial conditions only. Similarly, from (47) and (49), it is seen that $\mathcal{W}_i$ and $\hat{\mathcal{W}}_i$ have different initial conditions but evolve in the same way. Thus, by $\hat{\mathcal{W}}_1(\Lambda^w) = \hat{\mathscr{W}}_1(\Delta^u) = \ker E = \operatorname{Im} B^v$, we get $\hat{\mathcal{W}}_i(\Lambda^w) = \hat{\mathscr{W}}_i(\Delta^u)$ for all $i \in \mathbb{N}^+$. □

6.3. Proof of Proposition 3.1

Proof. Observe that the transformation matrix T_s decomposes the state space $\mathscr{X}$ of Λ^u into $\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \mathscr{X}_3 \oplus \mathscr{X}_4$, where $\mathscr{X}_1 = \mathcal{V}^* \cap \mathcal{W}^*$, $\mathscr{X}_1 \oplus \mathscr{X}_2 = \mathcal{V}^*$, $\mathscr{X}_1 \oplus \mathscr{X}_3 = \mathcal{W}^*$, $(\mathcal{V}^* + \mathcal{W}^*) \oplus \mathscr{X}_4 = \mathscr{X}$. The transformation matrix T_i decomposes the input space $\mathscr{U}_u$ into $\mathscr{U}_u = \mathscr{U}_1 \oplus \mathscr{U}_2$, where $\mathscr{U}_1 = \mathcal{U}_u^*$, $\mathscr{U}_1 \oplus \mathscr{U}_2 = \mathscr{U}_u$. The transformation matrix T_o decomposes the output space $\mathscr{Y}$ into $\mathscr{Y} = \mathscr{Y}_1 \oplus \mathscr{Y}_2$, where $\mathscr{Y}_1 = \mathcal{Y}^*$, $\mathscr{Y}_1 \oplus \mathscr{Y}_2 = \mathscr{Y}$. Let $\Lambda' = (A', B', C', D') = M_{tran}(\Lambda^u)$, where M_{tran} is the Morse transformation $M_{tran} = (T_s, T_i, T_o, 0, 0)$. Then consider the following equation and subspaces:

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} T_s & 0 \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B^u \\ C & D^u \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ 0 & T_i^{-1} \end{bmatrix} = \left[\begin{array}{cccc|cc} A_1^1 & A_1^2 & A_1^3 & A_1^4 & B_1^1 & B_1^2 \\ A_2^1 & A_2^2 & A_2^3 & A_2^4 & B_2^1 & B_2^2 \\ A_3^1 & A_3^2 & A_3^3 & A_3^4 & B_3^1 & B_3^2 \\ A_4^1 & A_4^2 & A_4^3 & A_4^4 & B_4^1 & B_4^2 \\ \hline C_3^1 & C_3^2 & C_3^3 & C_3^4 & D_3^1 & D_3^2 \\ C_4^1 & C_4^2 & C_4^3 & C_4^4 & D_4^1 & D_4^2 \end{array}\right], \quad \begin{array}{l} \mathcal{V}^*(\Lambda') : \begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{W}^*(\Lambda') : \begin{bmatrix} * \\ 0 \\ * \\ 0 \end{bmatrix}, \\ \mathcal{U}_u{}^*(\Lambda') : \begin{bmatrix} * \\ 0 \end{bmatrix}, \quad \mathcal{Y}^*(\Lambda') : \begin{bmatrix} * \\ 0 \end{bmatrix}. \end{array}$$

Now, applying (46), for $i = n$, to both Λ' and the dual system of Λ' (see Appendix), we have

$$\begin{bmatrix} B' \\ D' \end{bmatrix} \mathcal{U}_u{}^* \subseteq \begin{bmatrix} \mathcal{V}^* \\ 0 \end{bmatrix}, \quad \begin{bmatrix} (C')^T \\ (D')^T \end{bmatrix} (\mathcal{Y}^*)^\perp \subseteq \begin{bmatrix} (\mathcal{W}^*)^\perp \\ 0 \end{bmatrix}.$$

It follows that B_3^1, B_4^1, C_4^1, C_4^3, D_3^1, D_4^1, D_2^4 are all zero.

Then applying (45) for $i = n$, to both Λ' and its dual system, we have

$$\begin{bmatrix} A'\mathcal{V}^* \\ C'\mathcal{V}^* \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{V}^* \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} B' \\ D' \end{bmatrix}, \tag{28}$$

$$\begin{bmatrix} (A')^T(\mathcal{W}^*)^\perp \\ (B')^T(\mathcal{W}^*)^\perp \end{bmatrix} \subseteq \begin{bmatrix} (\mathcal{W}^*)^\perp \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} (C')^T \\ (D')^T \end{bmatrix}. \tag{29}$$

The lower parts of equations (28) and (29) give $C'\mathcal{V}^* \subseteq \operatorname{Im} D'$ and $(B')^T(\mathcal{W}^*)^\perp \subseteq \operatorname{Im}(D')^T$, which implies that B_2^1 and C_2^4 are zero. On the other hand, equation (28) gives that

$$\operatorname{Im} \begin{bmatrix} A_3^1 \\ A_4^1 \\ C_3^1 \end{bmatrix} \subseteq \operatorname{Im} \begin{bmatrix} B_3^2 \\ B_4^2 \\ D_3^2 \end{bmatrix} \quad \text{and} \quad \operatorname{Im} \begin{bmatrix} A_3^2 \\ A_4^2 \\ C_3^2 \end{bmatrix} \subseteq \operatorname{Im} \begin{bmatrix} B_3^2 \\ B_4^2 \\ D_3^2 \end{bmatrix},$$

implying that there exist matrices $F_1 \in \mathbb{R}^{m_3 \times n_1}$ and $F_2 \in \mathbb{R}^{m_3 \times n_2}$ such that

$$\begin{bmatrix} A_3^1 \\ A_4^1 \\ C_3^1 \end{bmatrix} = - \begin{bmatrix} B_3^2 \\ B_4^2 \\ D_3^2 \end{bmatrix} F_1 \quad \text{and} \quad \begin{bmatrix} A_3^2 \\ A_4^2 \\ C_3^2 \end{bmatrix} = - \begin{bmatrix} B_3^2 \\ B_4^2 \\ D_3^2 \end{bmatrix} F_2. \tag{30}$$

Then setting $F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F_1 & F_2 & 0 & 0 \end{bmatrix}$, we have

$$\begin{bmatrix} T_s & 0 \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B^u \\ C & D^u \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ T_i^{-1}F & T_i^{-1} \end{bmatrix} = \left[\begin{array}{cccc|cc} A_1^1+B_1^2F_1 & A_1^2+B_1^2F_2 & A_1^3 & A_1^4 & B_1^1 & B_1^2 \\ A_2^1+B_2^2F_1 & A_2^2+B_2^2F_1 & A_2^3 & A_2^4 & 0 & B_2^2 \\ 0 & 0 & A_3^3 & A_3^4 & 0 & B_3^2 \\ 0 & 0 & A_4^3 & A_4^4 & 0 & B_4^2 \\ \hline 0 & 0 & C_3^3 & C_3^4 & 0 & D_3^2 \\ 0 & 0 & 0 & C_4^4 & 0 & 0 \end{array}\right].$$

Since $\mathcal{W}^*$ is feedback invariant, equation (29) also holds for the above transformed system. Thus the upper part of (29) becomes

$$(A' + B'F)^T (\mathcal{W}^*(\Lambda'))^\perp \subseteq (\mathcal{W}^*(\Lambda'))^\perp + \operatorname{Im}(C')^T,$$

which gives that $(A_2^1 + B_1^2 F_1)^T = 0$,

$$\operatorname{Im} \begin{bmatrix} (A_2^3)^T \\ (B_2^2)^T \end{bmatrix} \subseteq \operatorname{Im} \begin{bmatrix} (C_1^3)^T \\ (D_1^2)^T \end{bmatrix} \quad \text{and} \quad \operatorname{Im} \begin{bmatrix} (A_4^3)^T \\ (B_4^2)^T \end{bmatrix} \subseteq \operatorname{Im} \begin{bmatrix} (C_3^3)^T \\ (D_3^2)^T \end{bmatrix}.$$

It follows that there exist $K_1 \in \mathbb{R}^{n_2 \times p_3}$ and $K_2 \in \mathbb{R}^{n_4 \times p_3}$ such that

$$\begin{bmatrix} (A_2^3)^T \\ (B_2^2)^T \end{bmatrix} = - \begin{bmatrix} (C_3^3)^T \\ (D_3^2)^T \end{bmatrix} K_1^T \quad \text{and} \quad \begin{bmatrix} (A_4^3)^T \\ (B_4^2)^T \end{bmatrix} = - \begin{bmatrix} (C_3^3)^T \\ (D_3^2)^T \end{bmatrix} K_2^T. \tag{31}$$

Let $K = \begin{bmatrix} 0 & K_1^T & 0 & K_2^T \\ 0 & 0 & 0 & 0 \end{bmatrix}^T$, which implies that

$$\begin{bmatrix} T_s & KT_o \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_s^{-1} & 0 \\ T_i^{-1}F & T_i^{-1} \end{bmatrix} = \left[\begin{array}{cccc|cc} A_1^1+B_1^2F_1 & A_1^2+B_1^2F_2 & A_1^3 & A_1^4 & B_1^1 & B_1^2 \\ 0 & A_2^2+B_2^2F_1 & 0 & A_2^4+K_1C_1^4 & 0 & 0 \\ 0 & 0 & A_3^3 & A_3^4 & 0 & B_3^2 \\ 0 & 0 & 0 & A_4^4+K_2C_3^4 & 0 & 0 \\ \hline 0 & 0 & C_3^3 & C_3^4 & 0 & D_3^2 \\ 0 & 0 & 0 & C_4^4 & 0 & 0 \end{array}\right].$$

Now it is seen that there exist $K_{MT} = T_s^{-1} K T_o$ and $F_{MT} = T_i^{-1} F T_s$ such that $\tilde{\Lambda}^{\tilde{u}} = (\tilde{A}, \tilde{B}^{\tilde{u}}, \tilde{C}, \tilde{D}^{\tilde{u}})$ has the form (18), where

$$\begin{bmatrix} \tilde{A} & \tilde{B}^{\tilde{u}} \\ \tilde{C} & \tilde{D}^{\tilde{u}} \end{bmatrix} = \begin{bmatrix} T_s & T_s K_{MT} \\ 0 & T_o \end{bmatrix} \begin{bmatrix} A & B^u \\ C & D^u \end{bmatrix} \begin{bmatrix} {T_s}^{-1} & 0 \\ F_{MT} {T_s}^{-1} & {T_i}^{-1} \end{bmatrix}.$$

The system matrices of $\tilde{\Lambda}^u$, see (18), are $\tilde{A}_1 = A_1^1 + B_1^2 F_1$, $\tilde{A}_1^2 = A_1^2$, $\tilde{A}_1^3 = A_1^3$, $\tilde{A}_1^4 = A_1^4$, $\tilde{B}_1 = B_1^1$, $\tilde{B}_1^2 = B_1^2$, $\tilde{A}_2 = A_2^2 + B_2^2 F_1$, $\tilde{A}_2^4 = A_2^4 + K_1 C_1^4$, $\tilde{A}_3 = A_3^3$, $\tilde{A}_3^4 = A_3^4$, $\tilde{B}_3 = B_3^2$, $\tilde{A}_4 = A_4^4 + K_2 C_1^4$, $\tilde{C}_3 = C_3^3$, $\tilde{C}_3^4 = C_3^4$, $\tilde{D}_3 = D_3^2$, $\tilde{C}_4 = C_4^4$.

Now we will show that $(\tilde{A}_1, \tilde{B}_1)$ is controllable. By Lemma 4 of [27] applied to $\tilde{\Lambda}^{\tilde{u}}$, we get

$$\mathcal{W}_i(\tilde{\Lambda}^{\tilde{u}}|_{\mathcal{U}_u^*}) = \mathcal{W}_i(\tilde{\Lambda}^{\tilde{u}}) \cap \mathcal{V}^*(\tilde{\Lambda}^{\tilde{u}}), \tag{32}$$

where $\mathcal{W}_i(\tilde{\Lambda}^{\tilde{u}}|_{\mathcal{U}_u^*})$ denotes the subspace $\mathcal{W}_i$ when the input is restricted to $\mathcal{U}_u^*$. Use the system matrices (18) to calculate $\mathcal{W}_i(\tilde{\Lambda}^{\tilde{u}}|_{\mathcal{U}_u^*})$ and $\mathcal{W}_i(\tilde{\Lambda}^{\tilde{u}}) \cap \mathcal{V}^*(\tilde{\Lambda}^{\tilde{u}})$, which gives

$$\mathcal{W}_n(\tilde{\Lambda}^{\tilde{u}}|_{\mathcal{U}_u^*}) = \mathscr{B}_1 + \tilde{A}_1 \mathscr{B}_1 + \cdots + (\tilde{A}_1)^{n-1} \mathscr{B}_1 \overset{(32)}{=} \mathcal{W}_n(\tilde{\Lambda}^{\tilde{u}}) \cap \mathcal{V}^*(\tilde{\Lambda}^{\tilde{u}}), \tag{33}$$

where $\mathscr{B}_1 = \operatorname{Im} \begin{bmatrix} \tilde{B}_1 & 0 & 0 & 0 \end{bmatrix}^T$. We can see from the above equation that the reachability space of $(\tilde{A}_1, \tilde{B}_1)$ is $\mathcal{W}^*(\tilde{\Lambda}^{\tilde{u}}) \cap \mathcal{V}^*(\tilde{\Lambda}^{\tilde{u}}) = \mathscr{X}_1$, which implies that $(\tilde{A}_1, \tilde{B}_1)$ is controllable. Since the proof of the observability of $(\tilde{C}_4, \tilde{A}_4)$ is completely dual to the above proof, we omit that part.

Subsequently, we prove that the system $\Lambda^3 = (\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$, given by (18), is prime. Using the system matrices of $\tilde{\Lambda}^{\tilde{u}}$ to calculate $\mathcal{W}^*(\tilde{\Lambda}^{\tilde{u}}|_{(\mathcal{U}_u^*)^\perp})$, we get

$$\mathcal{W}^*(\tilde{\Lambda}^{\tilde{u}}|_{(\mathcal{U}_u^*)^\perp}) = \mathscr{R} \times \{0\} \times \mathcal{W}^*(\tilde{\Lambda}^3) \times \{0\},$$

where $\mathscr{R}$ denotes a subspace whose explicit form is irrelevant. From $\mathcal{W}^*(\tilde{\Lambda}^{\tilde{u}}) = \mathcal{W}^*(\tilde{\Lambda}^{\tilde{u}}|_{\mathcal{U}_u^*}) \oplus \mathcal{W}^*(\tilde{\Lambda}^{\tilde{u}}|_{(\mathcal{U}_u^*)^\perp})$ and equation (33), we can deduce that $\mathcal{W}^*(\tilde{\Lambda}^3) = \mathscr{X}(\tilde{\Lambda}^3) = \mathscr{X}_3(\tilde{\Lambda}^{\tilde{u}})$. Moreover, by a direct calculation, we get

$$\mathcal{Y}^*(\tilde{\Lambda}^3) = \mathscr{Y}(\tilde{\Lambda}^3) = \tilde{C}_3 \mathcal{W}^*(\tilde{\Lambda}^3) + \tilde{D}_3 \mathscr{U}_w(\tilde{\Lambda}^3), \quad \mathcal{V}^*(\tilde{\Lambda}^3) = 0, \quad \mathcal{U}_u^*(\tilde{\Lambda}^3) = 0.$$

Finally, by Theorem 10 of [27], we conclude that $\tilde{\Lambda}^3 = (\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$ is prime. □

6.4. Proof of Proposition 3.2

Proof. First, by **MNF** Algorithm 3.3 and a direct calculation, we have

$$\begin{aligned}
\bar{A}_1 &= \tilde{A}_1 + \tilde{B}_1 F^1_{MN}, & \bar{A}^3_1 &= \tilde{A}^3_1 + \tilde{B}^2_1 F^2_{MN} + K^1_{MN}\tilde{C}_3 + K^1_{MN}\tilde{D}_3 F^2_{MN},\\
\bar{A}_4 &= \tilde{A}_4 + K^3_{MN}\tilde{C}^4_3, & \bar{A}_3 &= \tilde{A}_3 + K^2_{MN}\tilde{C}_3 + \tilde{B}_3 F^2_{MN} + K^2_{MN}\tilde{D}_3 F^2_{MN},\\
\bar{B}_3 &= \tilde{B}_3 + K^2_{MN}\tilde{D}_3, & \bar{A}^4_1 &= \tilde{A}^4_1 + \tilde{B}^2_1 F^3_{MN} + K^1_{MN}\tilde{C}_3 + K^1_{MN}\tilde{D}_3 F^3_{MN},\\
\bar{B}^2_1 &= \tilde{B}^2_1 + K^1_{MN}\tilde{D}_3, & \bar{A}^4_3 &= \tilde{A}^4_3 + \tilde{B}_3 F^3_{MN} + K^2_{MN}\tilde{C}^4_3 + K^2_{MN}\tilde{D}_3 F^3_{MN},\\
\bar{C}_3 &= \tilde{C}_3 + \tilde{D}_3 F^2_{MN}, & \bar{C}^4_3 &= \tilde{C}^4_3 + \tilde{D}_3 F^3_{MN}.
\end{aligned}$$

We will show that we can always assume $\tilde{D}_3 = 0$. To this end, we can find a change of coordinates in the input and output spaces to obtain $\tilde{D}_3 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I_\delta \end{smallmatrix}\right]$. Then by suitable choice of feedback and output injection transformation, the 5-tuple $(\tilde{B}^2_1, \tilde{B}_3, \tilde{C}_3, \tilde{C}^4_3, \tilde{D}_3)$ can be brought into the following form:

$$\left[\begin{array}{cc|c} * & * & \tilde{B}^2_1 \\ * & * & \tilde{B}_3 \\ \hline \tilde{C}_3 & \tilde{C}^4_3 & \tilde{D}_3 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} * & * & \hat{B}^2_1 & 0 \\ * & * & \hat{B}_3 & 0 \\ \hline \hat{C}_3 & \hat{C}^4_3 & 0 & 0 \\ 0 & 0 & 0 & I_\delta \end{array}\right].$$

The zero columns of $\hat{B}$ and the zero rows of $\hat{C}$ which correspond to the static relations $y_i = u_i$, $1 \le i \le \sigma$, we will be kept unchanged. Now, by neglecting the zero columns of $\hat{B}$ and the zero rows of $\hat{C}$, we may assume that

$$\left[\begin{array}{cc|c} * & * & \tilde{B}^2_1 \\ * & * & \tilde{B}_3 \\ \hline \tilde{C}_3 & \tilde{C}^4_3 & \tilde{D}_3 \end{array}\right] = \left[\begin{array}{cc|c} * & * & \hat{B}^2_1 \\ * & * & \hat{B}_3 \\ \hline \hat{C}_3 & \hat{C}^4_3 & 0 \end{array}\right],$$

i.e., $\tilde{D}_3$-matrix is $\hat{D}_3 = 0$.

Now with the assumption $\tilde{D}_3 = 0$, we show that the constrained Sylvester equations of (21) can be reduced to normal Sylvester equations by a suitable choice of F_{MN} and K_{MN}. We claim that the following matrix equation

$$\tilde{B}^2_1 = -\hat{T}^2_{MN}\tilde{B}_3 \tag{34}$$

is solvable for $\hat{T}^2_{MN}$. This claim can be proved by observing that

$$\begin{bmatrix} \tilde{B}^{\tilde{u}}(\mathcal{U}^*_u)^\perp \\ \tilde{D}^{\tilde{u}}(\mathcal{U}^*_u)^\perp \end{bmatrix} \cap \begin{bmatrix} \mathcal{V}^* \\ 0 \end{bmatrix} = 0. \tag{35}$$

Note that the above equation is a consequence of the definition of $\mathcal{U}^*_u$ (see equation (46)). Now by (35), we have

$$\operatorname{Im}\left(\operatorname{col}\left[\, \tilde{B}^2_1 \;\, 0 \;\, \tilde{B}_3 \;\, 0 \;\, \tilde{D}_3 \;\, 0 \,\right]\right) \cap \begin{bmatrix} \mathcal{V}^* \\ 0 \end{bmatrix} = 0.$$

Since $\tilde{D}_3$ is already zero, the above equation implies that (34) is solvable for $\hat{T}^2_{MN}$. Consequently, substitute (34) into the upper equations of (21) and we get

$$\bar{A}_1\bar{T}^2_{MN} - \bar{T}^2_{MN}\bar{A}_3 = -\bar{A}^3_1 + \bar{A}^1\hat{T}^2_{MN} - \hat{T}^2_{MN}\bar{A}_3, \quad \bar{T}^2_{MN}\bar{B}_3 = 0, \tag{36}$$

where $\bar{T}^2_{MN} = T^2_{MN} + \hat{T}^2_{MN}$.

Furthermore, since $(\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$ is prime (a consequence of Proposition 3.1), we can always assume $\tilde{B}_3 = [I_{m_3}, 0]^T$ and $\tilde{C}_3 = [I_{p_3}, 0]$ (if not, use coordinates transformations such that $\tilde{B}_3$ and $\tilde{C}_3$ are of that form), where $m_3 = \operatorname{rank}\tilde{B}_3 = \dim(\mathcal{U}^*_u)^\perp = p_3 = \operatorname{rank}\tilde{C}_3 = \dim \mathcal{Y}^*$. Then, it is possible to choose K^1_{MN}, K^2_{MN}, F^2_{MN} such that the 4-tuple $(\bar{A}^3_1, \bar{A}_3, \bar{B}^2_1, \bar{C}_3)$ is transformed into the following form:

$$\left[\begin{array}{c|c} \bar{A}_1^3 & \\ \hline \bar{A}_3 & \bar{B}_3 \\ \hline \tilde{C}_3 & \end{array}\right] = \left[\begin{array}{cc|c} 0 & \bar{A}_1^{3\prime} & \\ \hline 0 & 0 & I_{m_3} \\ 0 & \bar{A}_3' & 0 \\ \hline I_{p_3} & 0 & \end{array}\right].$$

Thus $\bar{T}_{MN}^2$ in equation (36) is of the form $\bar{T}_{MN}^2 = \left[\, 0 \ \hat{T}_{MN}^2 \,\right]$ because $\bar{T}_{MN}^2 \bar{B}_3 = 0$. Hence, solving $\bar{T}_{MN}^2$ via equation (36) is equivalent to solving $\hat{T}_{MN}^2$ via

$$\bar{A}_1 \left[\, 0 \ \hat{T}_{MN}^2 \,\right] - \left[\, 0 \ \hat{T}_{MN}^2 \,\right] \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_3' \end{bmatrix} = \left[\, 0 \ \hat{A}_1^{3\prime} \,\right].$$

Therefore, the upper part of the constrained Sylvester equations of (21) can be reduced to the above normal Sylvester equation. The reduction of the lower part of (21) to a normal Sylvester equation follows dually from the above result and we will omit that proof.

Moreover, from Proposition 3.1, we have that the pair $(\tilde{A}_1, \tilde{B}_1)$ is controllable and the pair $(\tilde{C}_4, \tilde{A}_4)$ is observable. By the standard matrix theory, we can choose F_{MN} and K_{MN} such that the spectra of $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3'$,and $\bar{A}_4^4$ are mutually disjoint (that of A_2 is fixed but, the three others can be made arbitrary). Then there exist unique solutions for T_{MN}^1, T_{MN}^2, T_{MN}^3, T_{MN}^4, T_{MN}^5 in (20) and (21). Furthermore, it is not hard to see that the state coordinates transformation matrix T_{MN} brings $\tilde{\Lambda}^{\tilde{u}}$ into $\bar{\Lambda}^{\bar{u}}$. Feedback transformations preserve controllability, so the controllability of $(\tilde{A}_1, \tilde{B}_1)$ implies the controllability of $(\bar{A}_1, \bar{B}_1)$; output injection preserves observability, so the observability of $(\tilde{C}_4, \tilde{A}_4)$ implies the observability of $(\bar{C}_4, \bar{A}_4)$. The fact that the 4-tuple $(\bar{A}_3, \bar{B}_3, \bar{C}_3, \bar{D}_3)$ is prime is inherited from the fact that $(\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3)$ is prime since $(\tilde{A}_3, \tilde{B}_3, \tilde{C}_3, \tilde{D}_3) \overset{M}{\sim} (\bar{A}_3, \bar{B}_3, \bar{C}_3, \bar{D}_3)$ (see this property of prime systems in [27]). □

6.5. Proofs of Theorem 3.5 and Theorem 3.6

Proof of Theorem 3.5 . Recall Remark 2.8(iii) that there exists an extended Morse transformation EM_{tran} such that $\tilde{\Lambda}^{\tilde{u}\tilde{v}} = EM_{tran}(\Lambda^{uv})$ is of the **EMTF** if and only if there exists a Morse transformation M_{tran} with a triangular (and not just any) input coordinates transformation bringing $\Lambda_{n,m+s,p}^w = (A, B^w, C, D^w)$ into the **MTF**. Now we use the result of Proposition 3.1 for Λ^w with a more subtle way to construct the input coordinates transformation matrix T_w. More specifically, set $T_x = T_s$, $T_y = T_o$, $F_w = F_{MT}$, $K_w = K_{MT}$ as in Proposition 3.1 and define

$$T_w = \left[\, T_u^1 \ T_u^3 \ T_v^1 \ T_v^3 \,\right]^{-1} \in \mathbb{R}^{(m+s)\times(m+s)}, \tag{37}$$

where $T_u^1 \in \mathbb{R}^{(m+s)\times m_1}$, $T_u^3 \in \mathbb{R}^{(m+s)\times m_3}$, $T_v^1 \in \mathbb{R}^{(m+s)\times s_1}$, $T_v^3 \in \mathbb{R}^{(m+s)\times s_3}$ with $m_1 + m_3 = m$, $s_1 + s_3 = s$ are full rank matrices such that

$$\begin{aligned} &\operatorname{Im} T_v^1 = \mathcal{U}_v^*, && \operatorname{Im} T_v^1 \oplus \operatorname{Im} T_v^3 = \mathscr{U}_v, \\ &\operatorname{Im} T_u^1 \oplus \operatorname{Im} T_v^1 = \mathcal{U}_{uv}^* = \mathcal{U}_w^*, && \operatorname{Im} T_u^1 \oplus \operatorname{Im} T_u^3 \oplus \operatorname{Im} T_v^1 \oplus \operatorname{Im} T_v^3 = \mathscr{U}_{uv} = \mathscr{U}_w, \end{aligned}$$

where $\mathcal{U}_v^*$ is $\mathcal{U}_{uv}^*$ when the input $w = [u^T \ v^T]^T$ is restricted to v (i.e., we put $u = 0$). Notice that T_w has a triangular form since $\operatorname{Im} T_v^1 \oplus \operatorname{Im} T_v^3 = \mathscr{U}_v$ and thus preserves $\mathscr{U}_u$. Now the Morse transformation $M_{trans} = (T_x, T_w, T_y, F_w, K_w)$ brings Λ^w into the desired form of (22). Hence, it proves that there exists an EM_{tran} transforming Λ^{uv} into the **EMTF**. The claims that $(\tilde{A}_1, \tilde{B}_1^{\tilde{w}})$ is controllable,

$(\tilde{C}_4, \tilde{A}_4)$ is observable and $(\tilde{A}_3, \tilde{B}_3^{\tilde{w}}, \tilde{C}_3, \tilde{D}_3^{\tilde{w}})$ is prime are inherited from the corresponding results of Proposition 3.1. □

Proof of Theorem 3.6. There exists an EM_{tran} such that $\bar{\Lambda}^{\bar{u}\bar{v}} = EM_{tran}(\tilde{\Lambda}^{\tilde{u}\tilde{v}})$ is in the **EMNF** if and only if there exists a Morse transformation M_{tran} with a triangular input transformation matrix T_w bringing the system $\tilde{\Lambda}^{\tilde{w}}$, given in **MTF**, into the **MNF**. Then as shown in Proposition 3.2, the input coordinates transformation matrix of the Morse transformation, which brings the **MTF** into the **MNF**, is the identity matrix, thus triangular, as we need. Therefore, with the transformation matrices shown in Proposition 3.2, we can always bring $\tilde{\Lambda}^{\tilde{w}}$ into the **EMNF**. Moreover, the claims that $(\bar{A}_1, \bar{B}_1^{\bar{w}})$ is controllable, $(\bar{C}_4, \bar{A}_4)$ is observable, $(\bar{A}_3, \bar{B}_3^{\bar{w}}, \bar{C}_3, \bar{D}_3^{\bar{w}})$ is prime follow from the corresponding results of Proposition 3.2. □

6.6. Proof of Theorem 4.1

Proof. By Theorem 3.6, for a given ODECS $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D^u)$, there exists an extended Morse transformation EM_{tran} such that $EM_{tran}(\Lambda^{uv})$ is in the **EMNF**. Therefore, the starting point of this proof is the **EMNF** given by (23). Since the system represented in the **EMNF** is already decoupled into four independent subsystems, we only need to transform each subsystem into its corresponding canonical form.

(i) We will prove that any controllable $\Lambda^{uv}_{n,m,s} = (A, B^u, B^v)$ can be transformed into the Brunovský canonical form with indices $(\epsilon_1, \dots, \epsilon_m)$ and $(\bar{\epsilon}_1, \dots, \bar{\epsilon}_s)$, then the transformation from $(\bar{A}_1, \bar{B}_1^u, \bar{B}_1^v)$ to $\left(\left[\begin{smallmatrix} A^{cu} & 0 \\ 0 & A^{cv} \end{smallmatrix}\right], \left[\begin{smallmatrix} B^{cu} \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ B^{cv} \end{smallmatrix}\right]\right)$ is straightforward to see. Since $\Lambda^{uv} = (A, B^u, B^v)$ is a control system without output, in view of the extended Morse equivalence of Definition 2.7, we just need to prove that there exist transformation matrices T_x, T_u, T_v, F_u, F_v, R such that the transformed system matrices

$$\left(T_x \left(A + B^u F_u + B^v \left(F_v + R F_u\right)\right) T_x^{-1}, T_x \left(B^u + B^v R\right) T_u^{-1}, T_x B^v T_v^{-1}\right)$$

are in the Brunovský canonical form (notice a triangular form of input transformation acting on $[B^u \ B^v]$). First, from the classical linear system theory (see, e.g., [11]), using only a state coordinates transformation and state feedback, i.e., choosing suitable T_x, F_v, F_u, and setting $T_u = I_m$, $T_v = I_s$, $R = 0$, we can transform Λ^{uv} into the following form:

$$\begin{cases} \dot{x}_i^j &= x_i^{j+1}, \quad 1 \le i \le m+s, \quad 1 \le j \le \kappa_i - 1, \\ \dot{x}_i^{\kappa_i} &= b_i^1 u_1 + \dots + b_i^m u_m + \bar{b}_i^1 v_1 + \dots + \bar{b}_i^s v_s, \quad 1 \le i \le m+s. \end{cases} \tag{38}$$

Moreover, without loss of generality, we assume $\operatorname{rank} B^w = m + s$ (if not, we can always permute the variables of u and v such that the first m_1 columns of B^u and the first s_1 columns of B^v are independent, where $m_1 = \operatorname{rank} B^u$ and $s_1 = \operatorname{rank} B^v$, then we will work with the matrices with these independent columns only, the remaining ones being zero by suitable transformations T_u and T_v). Thus the matrix $\Gamma = [\Gamma_u \ \Gamma_v]$, where $\Gamma_u = (b_i^l)$ and $\Gamma_v = (\bar{b}_i^{\bar{l}})$, where $1 \le i \le m+s$, $1 \le l \le m$

and $1 \leq \bar{l} \leq s$, is invertible. Then we suppose that the controllability indices κ_i satisfy

$$\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_{m+s} \geq 1.$$

Note that in the case of the Brunovský form for classical ODECS (with one kind of inputs), we could use $T_w = \Gamma$ as an input coordinates transformation matrix. However, Δ^{uv} has two kinds of inputs and the input coordinates transformation matrix should have a triangular form (see Remark 2.8(ii)). In order to have such an input coordinates transformation matrix, we implement the following procedure.

Step $i = 1$: two cases are possible: either for all $1 \leq j \leq s$, we have $\bar{b}_1^j = 0$ or there exists $1 \leq j \leq s$ such that $\bar{b}_1^j \neq 0$. In the first case, by the invertibility of Γ, there exists $1 \leq j \leq m$ such that $b_1^j \neq 0$. We assume $b_1^1 \neq 0$ (if not, we permute the u_j's), set $\ell_1 = 1$, $\epsilon_1 = \kappa_1$, and $\bar{\ell}_1 = 0$ and define

$$\begin{cases} \tilde{u}_1 = b_1^1 u_1 + \cdots + b_1^m u_m \\ \tilde{u}_j = u_j, \ 2 \leq j \leq m \end{cases}, \qquad \tilde{v}_j = v_j, \ 1 \leq j \leq s,$$

the system becomes (we delete "tildes" over u_j and v_j)

$$\begin{cases} \dot{x}_i^j &= x_i^{j+1}, \quad 1 \leq i \leq m+s, \quad 1 \leq j \leq \kappa_i - 1, \\ \dot{x}_1^{\epsilon_1} &= u_1, \\ \dot{x}_i^{\kappa_i} &= b_i^1 u_1 + \cdots + b_i^m u_m + \bar{b}_i^1 v_1 + \cdots + \bar{b}_i^s v_s, \quad 2 \leq i \leq m+s. \end{cases}$$

In the second case, assume $\bar{b}_1^1 \neq 0$ (if not, we permute the v_j's), set $\bar{\ell}_1 = 1$, $\bar{\epsilon}_1 = \kappa_1$, and $\ell_1 = 0$, and define

$$\begin{cases} \tilde{v}_1 = b_1^1 u_1 + \cdots + b_1^m u_m + \bar{b}_1^1 v_1 + \cdots + \bar{b}_1^m v_s, \\ \tilde{v}_i = v_i, \quad 2 \leq i \leq s, \end{cases}$$

and we get

$$\begin{cases} \dot{\tilde{x}}_1^{\kappa_1} &= \tilde{v}_1, \\ \dot{x}_i^{\kappa_i} &= \tilde{b}_i^1 u_1 + \cdots + \tilde{b}_i^m u_m + \tilde{\bar{b}}_i^1 \tilde{v}_1 + \tilde{\bar{b}}_i^2 \tilde{v}_2 + \cdots + \tilde{\bar{b}}_i^s \tilde{v}_s, \quad 2 \leq i \leq m+s. \end{cases}$$

Set

$$\begin{cases} \tilde{x}_1^j = x_1^j, \quad 1 \leq j \leq \bar{\epsilon}_1, \\ \tilde{x}_i^j = x_i^j - \tilde{\bar{b}}_i^1 x_1^{\kappa_1 - \kappa_i + j}, \quad 2 \leq i \leq m+s, \quad 1 \leq j \leq \kappa_i, \end{cases}$$

to get (we delete "tildes" over x_i, v_j, b_i and $\bar{b}_i$)

$$\begin{cases} \dot{x}_i^j &= x_i^{j+1}, \quad 1 \leq i \leq m+s, \quad 1 \leq j \leq \kappa_i - 1, \\ \dot{\tilde{x}}_1^{\bar{\epsilon}_1} &= v_1, \\ \dot{x}_i^{\kappa_i} &= b_i^1 u_1 + \cdots + b_i^m u_m + 0 + \bar{b}_i^2 v_2 + \cdots + \bar{b}_i^s v_s, \quad 2 \leq i \leq m+s. \end{cases}$$

Step $i = k+1$: Assume that after k steps, we have defined ℓ_k and ϵ_i, for $1 \leq i \leq \ell_k$, as well as $\bar{\ell}_k$ and $\bar{\epsilon}_i$ for $1 \leq i \leq \bar{\ell}_k$, such that $\ell_k + \bar{\ell}_k = k$, and the system reads (the term "0" is to indicate that

$v_1, \ldots, v_{\bar{\ell}_k}$ are missing)

$$\begin{cases} \dot{x}_i^j &= x_i^{j+1}, \quad 1 \le i \le \ell_k, \quad 1 \le j \le \epsilon_i - 1, \\ \dot{x}_i^{\epsilon_i} &= u_i, \qquad 1 \le i \le \ell_k, \\ \dot{\bar{x}}_i^j &= \bar{x}_i^{j+1}, \quad 1 \le i \le \bar{\ell}_k, \quad 1 \le j \le \bar{\epsilon}_i - 1, \\ \dot{\bar{x}}_1^{\bar{\epsilon}_i} &= v_i, \qquad 1 \le i \le \bar{\ell}_k, \\ \dot{x}_i^j &= x_i^{j+1}, \quad k+1 \le i \le m+s, \quad 1 \le j \le \kappa_i - 1, \\ \dot{x}_i^{\kappa_i} &= b_i^1 u_1 + \cdots + b_i^m u_m + 0 + \bar{b}_i^{\bar{\ell}_k+1} v_{\bar{\ell}_k+1} + \cdots + \bar{b}_i^s v_s, \quad k+1 \le i \le m+s. \end{cases}$$

Then two cases are possible, either for all $\bar{\ell}_k + 1 \le j \le s$, we have $\bar{b}_{k+1}^j = 0$ or there exists $\bar{\ell}_k + 1 \le j \le s$ such that $\bar{b}_{k+1}^j \ne 0$. In the first case, set $\ell_{k+1} = \ell_k + 1$, $\epsilon_{\ell_{k+1}} = \kappa_{k+1}$, $\bar{\ell}_{k+1} = \bar{\ell}_k$ and set

$$\begin{cases} \tilde{u}_j = b_{k+1}^1 u_1 + \cdots + b_{k+1}^m u_m, \quad j = \ell_{k+1}, \\ \tilde{u}_j = u_j, \quad \ell_{k+1} + 1 \le j \le m, \\ \tilde{v}_j = v_j, \quad 1 \le j \le s, \end{cases}$$

which is well-defined because, by controllability, at least one $b_{k+1}^j \ne 0$, for $j > \ell_k$. We get (we delete "tildes" over x_i, u_j and v_j)

$$\begin{cases} \dot{x}_i^j &= x_i^{j+1}, \quad 1 \le i \le \ell_{k+1}, \quad 1 \le j \le \epsilon_i - 1, \\ \dot{x}_i^{\epsilon_i} &= u_i, \qquad 1 \le i \le \ell_{k+1} \\ \dot{\bar{x}}_i^j &= \bar{x}_i^{j+1}, \quad 1 \le i \le \bar{\ell}_{k+1} = \bar{\ell}_k, \quad 1 \le j \le \bar{\epsilon}_i - 1, \\ \dot{\bar{x}}_1^{\bar{\epsilon}_i} &= v_i, \qquad 1 \le i \le \bar{\ell}_{k+1} = \bar{\ell}_k, \\ \dot{x}_i^j &= x_i^{j+1}, \quad k+2 \le i \le m+s, \quad 1 \le j \le \kappa_i - 1, \\ \dot{x}_i^{\kappa_i} &= b_i^1 u_1 + \cdots + b_i^m u_m + 0 + \bar{b}_i^{\bar{\ell}_k+1} v_{\bar{\ell}_k+1} + \cdots + \bar{b}_i^s v_s, \quad k+2 \le i \le m+s. \end{cases}$$

In the second case, assume $\bar{b}_{k+1}^{\bar{\ell}_k+1} \ne 0$ (if not, we permute the v_j's), set $\bar{\ell}_{k+1} = \bar{\ell}_k + 1$, $\bar{\epsilon}_{\bar{\ell}_{k+1}} = \kappa_{k+1}$, and $\ell_{k+1} = \ell_k$, and define

$$\begin{cases} \tilde{v}_j = b_{k+1}^1 u_1 + \cdots + b_{k+1}^m u_m + b_{k+1}^{\bar{\ell}_k+1} v_{\bar{\ell}_k+1} + \cdots + \bar{b}_{k+1}^s v_s, \quad j = \bar{\ell}_{k+1}, \\ \tilde{v}_j = v_j, \quad j \ne \ell_{k+1}, \end{cases}$$

we get

$$\begin{cases} \dot{\bar{x}}_{k+1}^{\kappa_{k+1}} &= \tilde{v}_{\bar{\ell}_{k+1}}, \\ \dot{x}_i^{\kappa_i} &= \tilde{b}_i^1 u_1 + \cdots + \tilde{b}_i^m u_m + \tilde{\bar{b}}_i^1 \tilde{v}_1 + \tilde{\bar{b}}_i^2 \tilde{v}_2 + \cdots + \tilde{\bar{b}}_i^s \tilde{v}_s, \quad k+1 \le i \le m+s. \end{cases}$$

Set

$$\begin{cases} \tilde{x}_i^j = x_i^j - \tilde{\bar{b}}_i^{\bar{\ell}_{k+1}} x_{k+1}^{\kappa_{k+1} - \kappa_i + j}, \quad k+2 \le i \le m+s, \quad 1 \le j \le \bar{\kappa}_i \\ \bar{x}_i^j = x_{k+1}^j, \quad i = \bar{\ell}_{k+1}, \quad 1 \le j \le \bar{\epsilon}_{\bar{\ell}_{k+1}}, \end{cases}$$

to get (we delete "tildes" over x_i, v_j, b_i, $\bar{b}_i$)

$$\begin{cases} \dot{x}_i^j &= x_i^{j+1}, \quad 1 \le i \le \ell_{k+1} = \ell_k, \quad 1 \le j \le \epsilon_i - 1, \\ \dot{x}_i^{\epsilon_i} &= u_i, \qquad 1 \le i \le \ell_{k+1} = \ell_k, \\ \dot{\bar{x}}_i^j &= \bar{x}_i^{j+1}, \quad 1 \le i \le \bar{\ell}_{k+1}, \quad 1 \le j \le \bar{\epsilon}_i - 1, \\ \dot{\bar{x}}_1^{\bar{\epsilon}_i} &= v_i, \qquad 1 \le i \le \bar{\ell}_{k+1}, \\ \dot{x}_i^j &= x_i^{j+1}, \quad k+2 \le i \le m+s, \quad 1 \le j \le \kappa_i - 1, \\ \dot{x}_i^{\kappa_i} &= b_i^1 u_1 + \cdots + b_i^m u_m + 0 + \bar{b}_i^{\bar{\ell}_{k+1}+1} v_{\bar{\ell}_{k+1}+1} + \cdots + \bar{b}_i^s v_s, \quad k+2 \le i \le m+s. \end{cases}$$

After $m+s$ steps, we have $\ell_{m+s} = m$ and $\bar{\ell}_{m+s} = s$ and we get the Brunovský canonical form of Λ^{uv} with indices $(\epsilon_1, \ldots, \epsilon_m)$ and $(\bar{\epsilon}_1, \ldots, \bar{\epsilon}_s)$:

$$\begin{cases} \dot{x}_i^j &= x_i^{j+1}, \quad 1 \le j \le \epsilon_i - 1, \quad 1 \le i \le \ell_{m+s} = m, \\ \dot{x}_i^{\epsilon_i} &= u_i, \quad 1 \le i \le \ell_{m+s} = m, \\ \dot{\bar{x}}_i^j &= \bar{x}_i^{j+1}, \quad 1 \le j \le \bar{\epsilon}_i - 1, \quad 1 \le i \le \bar{\ell}_{m+s} = s, \\ \dot{\bar{x}}_1^{\bar{\epsilon}_i} &= v_i, \quad 1 \le i \le \bar{\ell}_{m+s} = s. \end{cases}$$

(ii) The A^{nn}-matrix (corresponding to the uncontrollable and unobservable system) is $A^{nn} = \bar{A}_2$.

(iii) First, we can find a Morse transformation M_{tran}^1 with a triangular T_w such that

$$M_{tran}^1 \left(\begin{array}{c|c|c} \bar{A}_3 & \bar{B}_3^u & \bar{B}_3^v \\ \hline \bar{C}_3 & \bar{D}_3^u & \end{array} \right) = \left(\begin{array}{c|cc|c} A_p & B_p^u & 0 & B_p^v \\ \hline C_p & 0 & 0 & \\ 0 & 0 & I_\delta & \end{array} \right).$$

Since $(\bar{A}_3, \bar{B}_3^w, \bar{C}_3, \bar{D}_3^w)$ is prime, by Theorem 10 of [27], (A_p, B_p^w, C_p) enjoys the properties:

$$\mathcal{V}^*(A_p, B_p^w, C_p) = 0, \quad \mathcal{U}_w^*(A_p, B_p^w, C_p) = 0. \tag{39}$$

$$\mathcal{W}^*(A_p, B_p^w, C_p) = \mathbb{R}^{n_3}, \quad \mathcal{Y}^*(A_p, B_p^w, C_p) = \mathscr{Y}. \tag{40}$$

A little thought (or see Lemma 2 of [27]) and equation (39) give that $\begin{bmatrix} A_p & B_p^w \\ C_p & 0 \end{bmatrix}$ is of full column rank. Then by $\mathcal{V}^*(A_p, B_p^w, C_p) = (\mathcal{W}^*((A_p)^T, (C_p)^T, (B_p^w)^T))^\perp$ (see also the results of (50) below) and equation (40), we have $\begin{bmatrix} A_p & B_p^w \\ C_p & 0 \end{bmatrix}$ is of full row rank. Thus $\begin{bmatrix} A_p & B_p^w \\ C_p & 0 \end{bmatrix}$ is square and invertible.

Moreover, by item (i) of this proof, there exists a Morse transformation M_{tran}^2 with triangular T_w such that the pairs $(\hat{A}^{pu}, \hat{B}^{pu})$ and (A^{pv}, B^{pv}) below are in the Brunovský form with indices $(\sigma_1, \ldots, \sigma_c)$ and $(\bar{\sigma}_1, \ldots, \bar{\sigma}_d)$, respectively

$$M_{tran}^2 \left(\begin{array}{c|c|c} A_p & B_p^u & B_p^v \\ \hline C_p & 0 & \end{array} \right) = \left(\begin{array}{cc|c|c} \hat{A}^{pu} & 0 & \hat{B}^{pu} & 0 \\ 0 & A^{pv} & 0 & B^{pv} \\ \hline \hat{C}^u & C^v & 0 & \end{array} \right).$$

Then, according to the block-diagonal structure of $\hat{A}^{pu}$ and A^{pv}, the matrices $\hat{C}^u$ and C^v above have the form:

$$\hat{C}^u = \left[\hat{C}_1^u \;\middle|\; \hat{C}_2^u \;\middle|\; \cdots \;\middle|\; \hat{C}_c^u \right], \quad C^v = \left[C_1^v \;\middle|\; C_2^v \;\middle|\; \cdots \;\middle|\; C_d^v \right],$$

where $\hat{C}_i^u \in \mathbb{R}^{p_3 \times \sigma_i}$, $1 \le i \le c$ and $C_i^v \in \mathbb{R}^{p_3 \times \bar{\sigma}_i}$, $1 \le i \le d$.

Now the diagonal submatrices $(\hat{A}_i^{pu}, \hat{B}_i^{pu}, \hat{C}_i^u)$ of $(\hat{A}^{pu}, \hat{B}^{pu}, \hat{C}^u)$, for $1 \le i \le c$, and $(A_i^{pv}, B_i^{pv}, C_i^v)$ of (A^{pv}, B^{pv}, C^v), for $1 \le i \le d$, have to satisfy

$$\mathcal{W}^*(\hat{A}_i^{pu}, \hat{B}_i^{pu}, \hat{C}_i^u) = \mathbb{R}^{\sigma_i}, \quad \mathcal{W}^*(A_i^{pv}, B_i^{pv}, C_i^v) = \mathbb{R}^{\bar{\sigma}_i}, \tag{41}$$

since if not, equation (40) does not hold.

By a direct calculation, we have $\mathcal{W}_1(\hat{A}_i^{pu}, \hat{B}_i^{pu}, \hat{C}_i^u) = \operatorname{Im} \hat{B}_i^{pu}$ and $\mathcal{W}_1(A_i^{pv}, B_i^{pv}, C_i^v) = \operatorname{Im} B_i^{pv}$. Then the subspaces $\mathcal{W}_2(\hat{A}_i^{pu}, \hat{B}_i^{pu}, \hat{C}_i^u, 0)$ and $\mathcal{W}_2(A_i^{pv}, B_i^{pv}, C_i^v, 0)$ coincide with $\operatorname{Im} \hat{B}_i^{pu}$ and $\operatorname{Im} B_i^{pv}$, respectively, unless the last columns of $\hat{C}_i^u$ and C_i^v are zero vectors. By similar arguments, we can deduce that $\hat{C}_i^u$, $1 \le i \le c$ and C_i^v, $1 \le i \le d$ have the following form:

$$\hat{C}_i^u = \left[\begin{array}{c|c|c|c} \hat{c}_i^u & 0 & \cdots & 0 \end{array}\right], \quad C_i^v = \left[\begin{array}{c|c|c|c} c_i^v & 0 & \cdots & 0 \end{array}\right],$$

where $\hat{c}_i^u \in \mathbb{R}^{p_3}$ and $c_i^v \in \mathbb{R}^{p_3}$. Furthermore, since the columns of $\hat{A}_i^{pu}$ and A_i^{pv} corresponding to $\hat{c}_i^u$ and c_i^v are all zero, so by the inveritibility of $\left[\begin{smallmatrix} A_p & B_p^w \\ C_p & 0 \end{smallmatrix}\right]$, we see that the following matrix

$$T_y^{-1} = \left[\begin{array}{cccc|cccc} \hat{c}_1^u & \hat{c}_2^u & \dots & \hat{c}_c^u & c_1^v & c_2^v & \dots & c_d^v \end{array}\right]$$

is invertible. Finally, using T_y as an output coordinates transformation matrix, we get the following canonical form for C_p

$$T_y C_p = T_y \begin{bmatrix} \hat{C}^u & C^v \end{bmatrix} = \begin{bmatrix} \hat{C}^{pu} & 0 \\ 0 & C^{pv} \end{bmatrix}.$$

(iv) The proof of transforming $(\bar{A}_4^4, \bar{C}_2^4)$ into (A^o, C^o) is omitted since it is well-known in the linear control theory. □

7. Conclusion

In this paper, on one hand, for linear ODECSs, we modify and simplify the construction of the **MCF** given in [27] by proposing the Morse triangular form **MTF**. On the other hand, a bridge from the **MTF** of ODECSs to the **FBCF** of DACSs is constructed via the explicitation with driving variables procedure. It is shown that, after attaching a class of ODECSs with two kinds of inputs to a DACS, we can find connections between their geometric subspaces and canonical forms. Finally, an explicit algorithm for constructing transformations from the **MTF** into the **FBCF** is proposed via the explicitation procedure and an example is given to show how our results and algorithms can be applied to physical systems.

Appendix

Recall the following geometric subspaces for DACSs (see e.g. [30],[7]) of the form $\Delta^u : E\dot{x} = Hx + Lu$.

Definition 7.1. Consider a DACS $\Delta^u_{l,n,m} = (E, H, L)$. A subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is called $(H, E; \operatorname{Im} L)$-invariant if

$$H\mathscr{V} \subseteq E\mathscr{V} + \operatorname{Im} L.$$

A subspace $\mathscr{W} \subseteq \mathbb{R}^n$ is called restricted $(E, H; \operatorname{Im} L)$-invariant if

$$\mathscr{W} = E^{-1}(H\mathscr{V} + \operatorname{Im} L).$$

Definition 7.2. For a DACS $\Delta^u_{l,n,m} = (E, H, L)$, define the augmented Wong sequences as follows:

$$\mathscr{V}_0 = \mathbb{R}^n, \quad \mathscr{V}_{i+1} = H^{-1}(E\mathscr{V}_i + \operatorname{Im} L), \tag{42}$$

$$\mathscr{W}_0 = 0, \quad \mathscr{W}_{i+1} = E^{-1}(H\mathscr{W}_i + \operatorname{Im} L). \tag{43}$$

Additionally, define the sequence of subspaces $\hat{\mathscr{W}}_i$ as follows:

$$\hat{\mathscr{W}}_1 = \ker E, \quad \hat{\mathscr{W}}_{i+1} = E^{-1}(H\hat{\mathscr{W}}_i + \operatorname{Im} L). \tag{44}$$

Consider an ODECS $\Lambda^{uv}_{n,m,s,p} = (A, B^u, B^v, C, D)$ of the form

$$\Lambda^{uv} : \begin{cases} \dot{x} = Ax + B^u u + B^v v \\ y = Cx + D^u u. \end{cases}$$

The state, input and output space of Λ^{uv} will be denoted by $\mathscr{X}$, $\mathscr{U}_{uv}$ and $\mathscr{Y}$, respectively. The input subspaces of u and v will be denoted by $\mathscr{U}_u$ and $\mathscr{U}_v$, respectively. Thus we have $\mathscr{U}_{uv} = \mathscr{U}_u \oplus \mathscr{U}_v$. Recall that Λ^{uv} can be expressed as a classical ODECS $\Lambda^w_{n,m+s,p} = (A, B^w, C, D^w)$ of the form (2). The input space of Λ^w is denoted by $\mathscr{U}_w$, and, clearly, $\mathscr{U}_w = \mathscr{U}_{uv} = \mathscr{U}_u \oplus \mathscr{U}_v$. We now recall the invariant subspaces $\mathcal{V}$ and $\mathcal{W}$ defined in [26] and [27] for Λ^w (generalizing the classical invariant subspaces [2, 35, 36] given for $D^u = 0$).

Definition 7.3. For an ODECS $\Lambda^w_{n,m+s,p} = (A, B^w, C, D^w)$, a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called a null-output (A, B^w)-controlled invariant subspace if there exists $F \in \mathbb{R}^{(m+s)\times n}$ such that

$$(A + B^w F)\mathcal{V} \subseteq \mathcal{V} \quad \text{and} \quad (C + D^w F)\mathcal{V} = 0$$

and a subspace $\mathcal{U}_w \subseteq \mathbb{R}^{s+m}$ is called a null-output (A, B^w)-controlled invariant input subspace if

$$\mathcal{U}_w = (B^w)^{-1}\mathcal{V} \cap \ker D^w.$$

Denote by $\mathcal{V}^*$ (respectively $\mathcal{U}^*_w$) the largest null-output (A, B^w) controlled invariant subspace (respectively input subspace).

Correspondingly, a subspace $\mathcal{W} \subseteq \mathbb{R}^n$ is called an unknown-input (C, A)-conditioned invariant subspace if there exists $K \in \mathbb{R}^{n\times p}$ such that

$$(A + KC)\mathcal{W} + (B^w + KD^w)\mathscr{U}_w = \mathcal{W}$$

and a subspace $\mathcal{Y} \subseteq \mathbb{R}^p$ is called an unknown-input (C, A)-conditioned invariant output subspace if

$$\mathcal{Y} = C\mathcal{W} + D^w\mathscr{U}_w.$$

Denote by $\mathcal{W}^*$ (respectively $\mathcal{Y}^*$) the smallest unknown-input (C, A)-conditioned invariant subspace (respectively output subspace).

Lemma 7.4. [26] *Initialize $\mathcal{V}_0 = \mathscr{X} = \mathbb{R}^n$ and, for $i \in \mathbb{N}$, define inductively*

$$\mathcal{V}_{i+1} = \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]^{-1} \left(\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]\mathcal{V}_i + \operatorname{Im}\left[\begin{smallmatrix} B^w \\ D^w \end{smallmatrix}\right]\right) \tag{45}$$

and $\mathcal{U}_i \subseteq \mathscr{U}$ for $i \in \mathbb{N}$ are given by

$$\mathcal{U}_i = \left[\begin{smallmatrix} B^w \\ D^w \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} \mathcal{V}_i \\ 0 \end{smallmatrix}\right]. \tag{46}$$

Then $\mathcal{V}^* = \mathcal{V}_n$ *and* $\mathcal{U}_w^* = \mathcal{U}_n$.

Correspondingly, initialize $\mathcal{W}_0 = \{0\}$ *and, for* $i \in \mathbb{N}$, *define inductively*

$$\mathcal{W}_{i+1} = [\,A\;B^w\,]\left(\left[\begin{smallmatrix}\mathcal{W}_i\\ \mathscr{U}_w\end{smallmatrix}\right] \cap \ker[\,C\;D^w\,]\right) \tag{47}$$

and $\mathcal{Y}_i \subseteq \mathscr{Y}$ *for* $i \in \mathbb{N}$ *are given by*

$$\mathcal{Y}_i = [\,C\;D^w\,]\left[\begin{smallmatrix}\mathcal{W}_i\\ \mathscr{U}_w\end{smallmatrix}\right]. \tag{48}$$

Additionally, define a sequence $\hat{\mathcal{W}}_i$ *of subspaces as*

$$\hat{\mathcal{W}}_1 = \operatorname{Im} B^v, \quad \hat{\mathcal{W}}_{i+1} = [\,A\;B^w\,]\left(\left[\begin{smallmatrix}\hat{\mathcal{W}}_i\\ \mathscr{U}_w\end{smallmatrix}\right] \cap \ker[\,C\;D^w\,]\right). \tag{49}$$

Then $\mathcal{W}^* = \mathcal{W}_n = \hat{\mathcal{W}}_n$ *and* $\mathcal{Y}^* = \mathcal{Y}_n$.

Note that when considering the above defined invariant subspaces for the dual system $(\Lambda^w)^d$ of Λ^w, given by $(\Lambda^w)^d = (A^T, C^T, (B^w)^T, (D^w)^T)$, we have the following results [28],[27]:

$$\begin{aligned} \mathcal{V}^*(\Lambda^w) &= \left(\mathcal{W}^*((\Lambda^w)^d)\right)^\perp, \quad \mathcal{W}^*(\Lambda^w) = \left(\mathcal{V}^*((\Lambda^w)^d)\right)^\perp, \\ \mathcal{U}_w^*(\Lambda^w) &= \left(\mathcal{Y}^*((\Lambda^w)^d)\right)^\perp, \quad \mathcal{Y}^*(\Lambda^w) = \left(\mathcal{U}_w^*((\Lambda^w)^d)\right)^\perp. \end{aligned} \tag{50}$$

Acknowledgment. The first author of the paper is currently supported by Vidi-grant 639.032.733.

References

[1] U. M. Ascher and L. R. Petzold, *The numerical solution of delay-differential-algebraic equations of retarded and neutral type*, SIAM Journal on Numerical Analysis, 32 (1995), pp. 1635–1657.

[2] G. Basile and G. Marro, *Controlled and Conditioned Invariants in Linear System Theory*, Prentice Hall Englewood Cliffs, 1992.

[3] T. Beelen and P. Van Dooren, *An improved algorithm for the computation of Kronecker's canonical form of a singular pencil*, Linear Algebra and its Applications, 105 (1988), pp. 9–65.

[4] T. Berger, *Zero dynamics and stabilization for linear DAEs*, in Progress in Differential-Algebraic Equations, Springer, 2014, pp. 21–45.

[5] T. Berger, *Disturbance decoupling by behavioral feedback for linear differential–algebraic systems*, Automatica, 80 (2017), pp. 272–283.

[6] T. Berger, A. Ilchmann, and S. Trenn, *The quasi-weierstraß form for regular matrix pencils*, Linear Algebra and its Applications, 436 (2012), pp. 4052–4069.

[7] T. Berger and T. Reis, *Controllability of linear differential-algebraic systems–a survey*, in Surveys in Differential-Algebraic Equations I, Springer, 2013, pp. 1–61.

[8] T. Berger and T. Reis, *Regularization of linear time-invariant differential–algebraic systems*, Systems & Control Letters, 78 (2015), pp. 40–46.

[9] T. Berger and S. Trenn, *The quasi-Kronecker form for matrix pencils*, SIAM Journal on Matrix Analysis and Applications, 33 (2012), pp. 336–368.

[10] M. E. Bonilla and M. Malabre, *External reachability (reachability with pole assignment by PD feedback) for implicit descriptions*, Kybernetika, 29 (1993), pp. 499–510.

[11] P. Brunovskỳ, *A classification of linear controllable systems*, Kybernetika, 6 (1970), pp. 173–188.

[12] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols, *Feedback design for regularizing descriptor systems*, Linear Algebra and its Applications, 299 (1999), pp. 119–151.

[13] S. L. Campbell, *Singular linear systems of differential equations with delays*, Applicable Analysis, 11 (1980), pp. 129–136.

[14] Y. Chen, *Geometric Analysis of Differential-Algebraic Equations and Control Systems: Linear, Nonlinear and Linearizable*, PhD thesis, Normandie Université, INSA de Rouen, 2019.

[15] Y. Chen and W. Respondek, *Geometric analysis of differential-algebraic equations via linear control theory*, SIAM Journal on Control and Optimization, 59 (2021), pp. 103–130.

[16] L. Dai, *Singular Control Systems*, vol. 118, Springer, 1989.

[17] P. V. Dooren, *The computation of Kronecker's canonical form of a singular pencil*, Linear Algebra and its Applications, 27 (1979), pp. 103 – 140.

[18] E. Fridman, *Stability of linear descriptor systems with delay: a lyapunov-based approach*, Journal of Mathematical Analysis and Applications, 273 (2002), pp. 24–44.

[19] H. Glüsing-Lüerben, *Feedback canonical form for singular systems*, International Journal of Control, 52 (1990), pp. 347–376.

[20] U. Helmke and M. A. Shayman, *A canonical form for controllable singular systems*, Systems & Control letters, 12 (1989), pp. 111–122.

[21] L. Kronecker, *Algebraische Reduction der Schaaren bilinearer Formen*, Sitzungsber. Akad. Berlin, 1890.

[22] F. L. Lewis, *A survey of linear singular systems*, Circuits, Systems and Signal Processing, 5 (1986), pp. 3–36.

[23] F. L. Lewis, *A tutorial on the geometric analysis of linear time-invariant implicit systems*, Automatica, 28 (1992), pp. 119–137.

[24] J. J. Loiseau, K. Özçaldiran, M. Malabre, and N. Karcanias, *Feedback canonical forms of singular systems*, Kybernetika, 27 (1991), pp. 289–305.

[25] J. J. Loiseau and P. Zagalak, *On pole structure assignment in linear systems*, International Journal of Control, 82 (2009), pp. 1179–1192.

[26] B. Molinari, *A strong controllability and observability in linear multivariable control*, IEEE Transactions on Automatic Control, 21 (1976), pp. 761–764.

[27] B. P. Molinari, *Structural invariants of linear multivariable systems*, International Journal of Control, 28 (1978), pp. 493–510.

[28] A. Morse, *Structural invariants of linear multivariable systems*, SIAM Journal on Control, 11 (1973), pp. 446–465.

[29] A. Morse, *System invariants under feedback and cascade control*, in Mathematical Systems Theory, Springer, 1976, pp. 61–74.

[30] K. Özçaldiran, *A geometric characterization of the reachable and the controllable subspaces of descriptor systems*, Circuits, Systems and Signal Processing, 5 (1986), pp. 37–48.

[31] K. Ozcaldiran, *A complete classification of controllable singular systems*, in 29th IEEE Conference on Decision and Control, IEEE, 1990, pp. 3596–3597.

[32] S. Trenn, *A normal form for pure differential algebraic systems*, Linear Algebra and its Applications, 430 (2009), pp. 1070–1084.

[33] A. Varga, *Computation of Kronecker-like forms of a system pencil: Applications, algorithms and software*, in Proceedings of IEEE International Symposium on Computer-Aided Control System Design, IEEE, 1996, pp. 77–82.

[34] K. Weierstrass, *Zur Theorie der Bilinearen und Quadratischen Formen*, Monatsh. Akad. Wiss. Berlin, 1867.

[35] W. M. Wonham, *Linear Multivariable Control-A Geometric Approach*, Springer, 1974.

[36] W. M. Wonham and A. S. Morse, *Decoupling and pole assignment in linear multivariable systems: a geometric approach*, SIAM Journal on Control, 8 (1970), pp. 1–18.